DRAFT • DRAFT • DRAFT •
June 2015

Mathematics at Work 12

NOVA SCOTIA

CURRICULUM GUIDE

Mathematics at Work 12

Implementation Draft
June 2015

Website References

Website references contained within this document are provided solely as a convenience and do not constitute an endorsement by the Department of Education and Early Childhood Development of the content, policies, or products of the referenced website. The Department does not control the referenced websites and subsequent links, and is not responsible for the accuracy, legality, or content of those websites. Referenced website content may change without notice.

School boards and educators are required under the Department's *Public School Network Access and Use Policy* to preview and evaluate sites before recommending them for student use. If an outdated or inappropriate site is found, please report it to links@EDnet.ns.ca.

Mathematics at Work 12, Implementation Draft, June 2015

Prepared by the Department of Education and Early Childhood Development

Cataloguing-in-Publication Data

Acknowledgements

The Nova Scotia Department of Education and Early Childhood Education wishes to express its gratitude to the following organizations for granting permission to adapt their mathematics curriculum guide in the development of this guide.

The Western and Northern Canadian Protocol (WNCP) for Collaboration in Education

Prince Edward Island Department of Education and Early Childhood Development

Newfoundland and Labrador Department of Education

The New Brunswick High School Mathematics Curriculum Development Advisory Committee

We also gratefully acknowledge the contributions of the following individuals toward the development of the Nova Scotia Mathematics 10 curriculum.

Arlene Andreyck (Retired)
Cape Breton Victoria Regional School Board

Darryl Breen
Strait Regional School Board

Trudy Comeau
Tri-Country Regional School Board

Jennifer Courish
Chignecto Central Regional School Board

Bob Crane
Mi'kmaw Kina'matnewey

Darlene MacKeen Hudson
Chignecto Central Regional School Board

Erick Lee
Halifax Regional School Board

Linda Wheadon
Tri-County Regional School Board

David MacFarlane
Strait Regional School Board

Mark MacLeod
South Shore Regional School Board

Sonya O'Sullivan
Halifax Regional School Board

Mark Pettipas
Strait Regional School Board

Lynn Evans Phillips
South Shore Regional School Board

Marlene Urquhart
Cape-Breton Victoria Regional School Board

Tom Willis
Tri-County Regional School Board

Contents

Introduction

Background and Rationale

Mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society. It is essential that the mathematics curriculum reflect current research in mathematics instruction. To achieve this goal, Western and Northern Canadian Protocol's (WNCP) *The Common Curriculum Framework for Grades 10–12 Mathematics* (2008) has been adopted as the basis for the new mathematics curriculum in Nova Scotia.

The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan, and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and performance indicators agreed upon by the seven jurisdictions. The outcomes and performance indicators have been adapted for Nova Scotia. This document is based on both national and international research by the WNCP and the National Council of Teachers of Mathematics (NCTM).

There is an emphasis in the Nova Scotia curriculum on particular key concepts at each grade that will result in greater depth of understanding and, ultimately, stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

Purpose

This document provides sets of outcomes and performance indicators to be used as a mandated common base for defining mathematics curriculum expectations. This common base should result in consistent student outcomes in mathematics within the province of Nova Scotia. It should also enable easier transfer for students moving within the province or from any jurisdiction that has adopted the WNCP framework. This document is intended to clearly communicate to all education partners across the province the high expectations for students' mathematical learning.

Program Design and Components

Pathways

The *Common Curriculum Framework for Grades 10–12 Mathematics* (WNCP 2008), on which the Nova Scotia Mathematics 10–12 curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics* (WNCP 2006). In Nova Scotia, four pathways are available: Mathematics Essentials, Mathematics at Work, Mathematics, and Pre-calculus.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all four pathways are to provide prerequisite attitudes, knowledge, skills, and understandings for specific post-secondary programs or direct entry into the work force. All four pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour, and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of the Mathematics Essentials courses was designed in Nova Scotia to fill a specific need for Nova Scotia students. The content of each of the Mathematics at Work, Mathematics, and Pre-calculus pathways has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* (System Improvement Group, Alberta Education 2006) and on consultations with mathematics teachers.

MATHEMATICS ESSENTIALS (GRADUATION)

This pathway is designed to provide students with the development of the skills and understandings required in the workplace, as well as those required for everyday life at home and in the community. Students will become better equipped to deal with mathematics in the real world and will become more confident in their mathematical abilities.

MATHEMATICS AT WORK (GRADUATION)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, and statistics and probability.

MATHEMATICS (ACADEMIC)

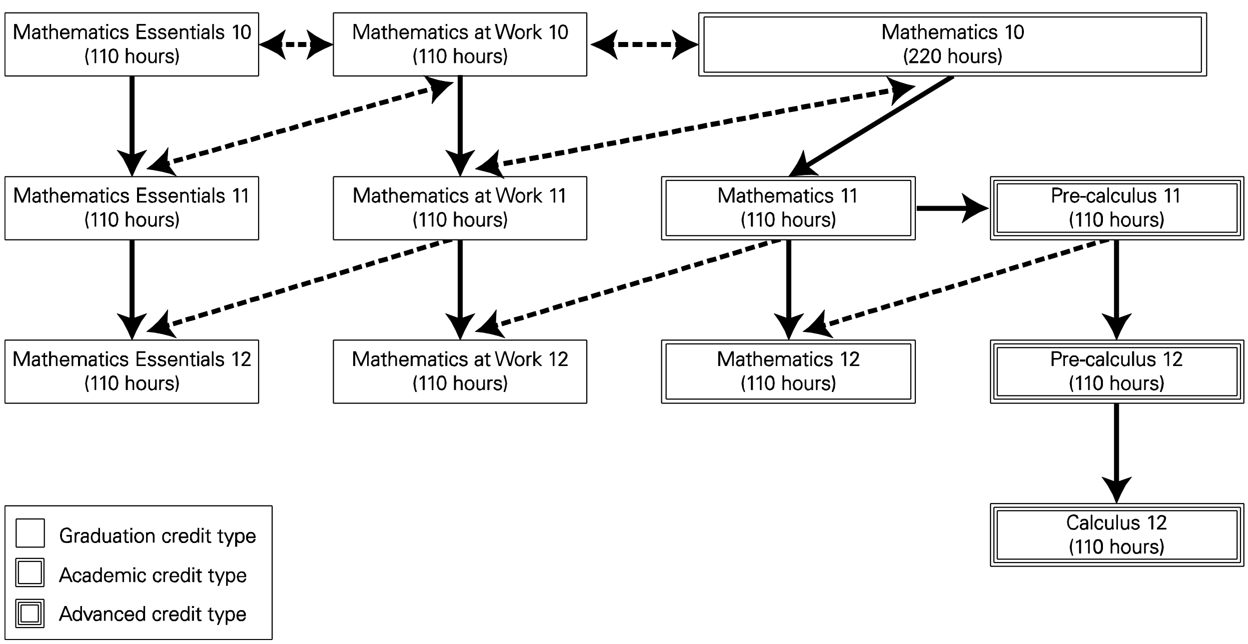
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that require an academic or pre-calculus mathematics credit. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, and statistics and probability. **Note:** After completion of Mathematics 11, students have the choice of an academic or pre-calculus pathway.

PRE-CALCULUS (ADVANCED)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, and permutations, combinations, and binomial theorem.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.



Instructional Focus

Each pathway in senior high mathematics pathways is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful.

Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems, and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning and assessment of learning.
-

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Assessment

Ongoing assessment for learning is essential to effective teaching and learning. Research has shown that assessment for learning (formative assessment) practices produce significant and often substantial learning gains, close achievement gaps, and build students' ability to learn new skills (Black & Wiliam 1998; OECD 2006). Student involvement in assessment promotes learning. Timely and effective teacher feedback and student self-assessment allow students to reflect on and articulate their understanding of mathematical concepts and ideas.

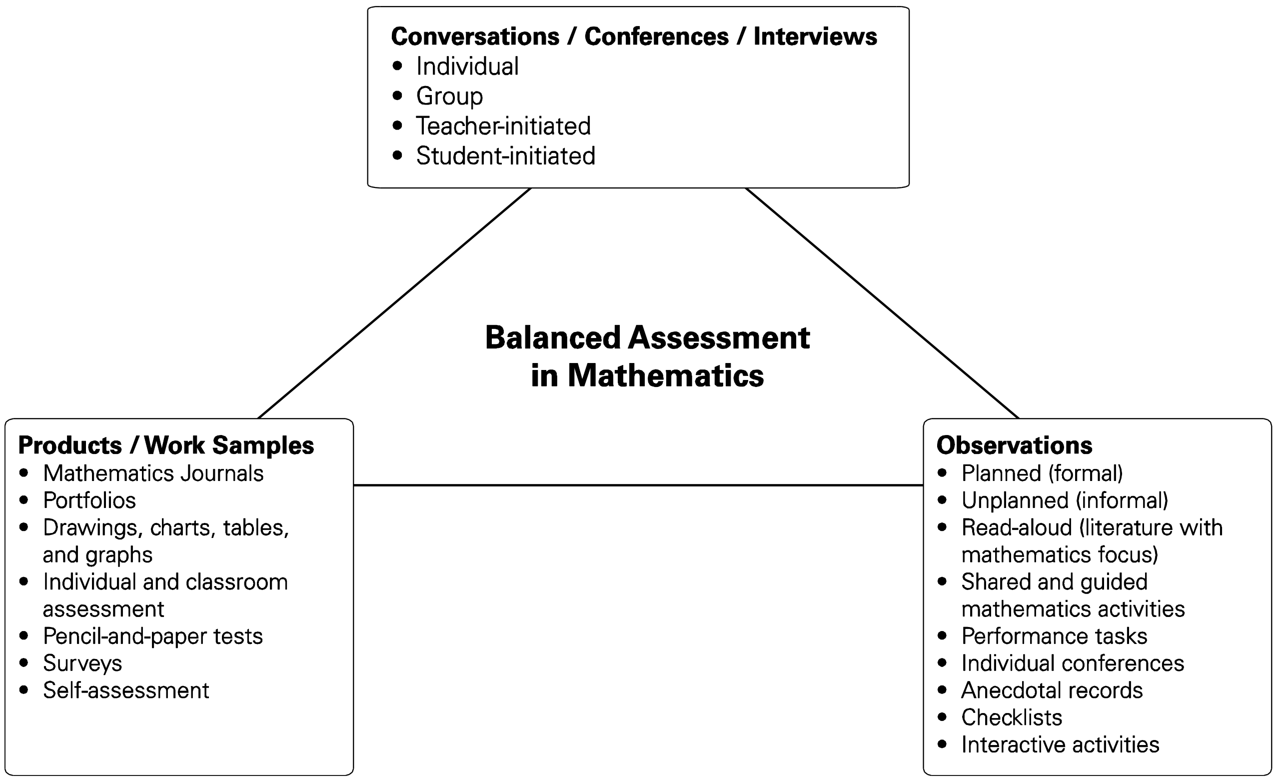
Assessment in the classroom includes

- providing clear goals, targets, and learning outcomes
- using exemplars, rubrics, and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning (Davies 2000)

Assessment for learning practices act as the scaffolding for learning, which only then can be measured through assessment of learning (summative assessment). Assessment of learning tracks student progress, informs instructional programming, and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning, and produce achievement gains.

Assessment of student learning should

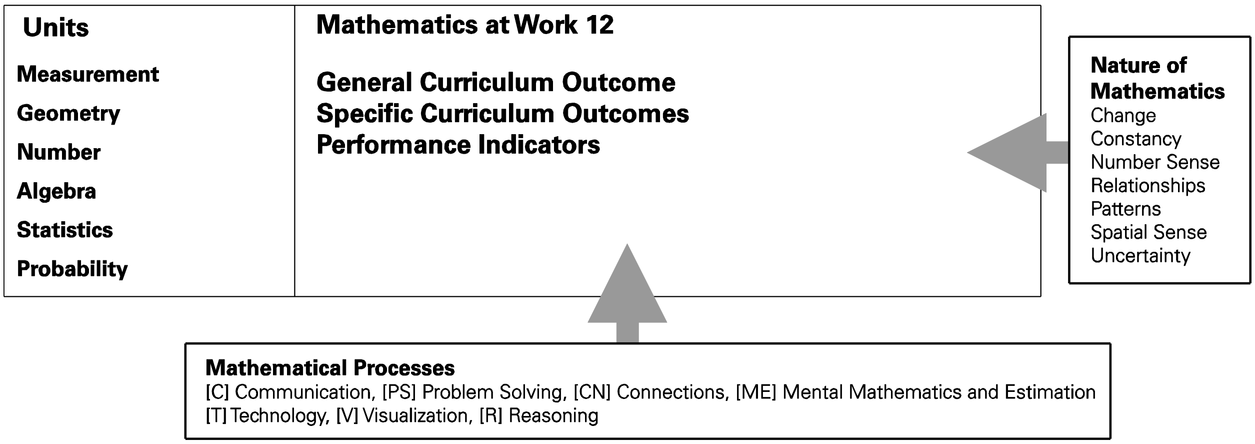
- align with curriculum outcomes
- clearly define criteria for success
- make explicit the expectations for students' performance
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction



Outcomes

Conceptual Framework for Mathematics 10–12

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



(Adapted with permission from Western and Northern Canadian Protocol, *The Common Curriculum Framework for K–9 Mathematics*, p. 5. All rights reserved.)

Structure of the Mathematics at Work 12 Curriculum

Units

Mathematics at Work 12 comprises five units:

- Measurement (M) (6–8 hours)
- Geometry (G) (35–37 hours)
- Number (N) (23–25 hours)
- Statistics (S) (12–14 hours)
- Probability (P) (10–12 hours)

Outcomes and Performance Indicators

The Nova Scotia curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes, and performance indicators.

General Curriculum Outcomes (GCOs)

General curriculum outcomes are overarching statements about what students are expected to learn in each strand/sub-strand. The GCO for each strand/sub-strand is the same throughout the pathway.

Measurement (M)

Students will be expected to develop spatial sense through direct and indirect measurement.

Geometry (G)

Students will be expected to develop spatial sense.

Number (N)

Students will be expected to develop number sense and critical-thinking skills.

Algebra (A)

Students will be expected to develop algebraic reasoning.

Statistics (S)

Students will be expected to develop statistical reasoning.

Probability (P)

Students will be expected to develop critical-thinking skills related to uncertainty.

Specific Curriculum Outcomes (SCOs) and Performance Indicators

Specific curriculum outcomes are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as expected for a given grade.

Performance indicators are samples of how students may demonstrate their performance of the goals of a specific curriculum outcome. The range of samples provided is meant to reflect the scope of the SCO. In the SCOs, the word **including** indicates that any ensuing items *must* be addressed to fully achieve the learning outcome. The phrase **such as** indicates that the ensuing items are provided for clarification only and are **not** requirements that must be addressed to fully achieve the learning outcome. The word **and** used in an outcome indicates that both ideas must be addressed to achieve the learning outcome, although not necessarily at the same time or in the same question.

Measurement (M)

M01 Students will be expected to demonstrate an understanding of the limitations of measuring instruments, including precision, accuracy, uncertainty, and tolerance, and to solve problems. [C, PS, R, T, V]

Performance Indicators

M01.01 Explain why, in a given context, a certain degree of precision is required.
M01.02 Explain why, in a given context, a certain degree of accuracy is required.
M01.03 Explain, using examples, the difference between precision and accuracy.
M01.04 Compare the degree of accuracy of two given instruments used to measure the same attribute.
M01.05 Relate the degree of accuracy to the uncertainty of a given measure.
M01.06 Analyze precision and accuracy in a contextual problem.
M01.07 Calculate maximum and minimum values, using a given degree of tolerance in context.
M01.08 Describe, using examples, the limitations of measuring instruments used in a specific trade or industry.
M01.09 Solve a problem that involves precision, accuracy, or tolerance

Geometry (G)

G01 Students will be expected to solve problems by using the sine law and cosine law, excluding the ambiguous case. [CN, PS, V]

Performance Indicators

G01.01 Identify and describe the use of the sine law and cosine law in construction, industrial, commercial, and artistic applications.
G01.02 Solve a problem using the sine law or cosine law when a diagram is given.

G02 Students will be expected to solve problems that involve triangles, quadrilaterals, and regular polygons. [C, CN, PS, V]

Performance Indicators

G02.01 Describe and illustrate properties of triangles, including isosceles and equilateral.
G02.02 Describe and illustrate properties of quadrilaterals in terms of angle measures, side lengths, diagonal lengths, and angles of intersection.

G02.03 Describe and illustrate properties of regular polygons.
G02.04 Explain, using examples, why a given property does or does not apply to certain polygons.
G02.05 Identify and explain an application of the properties of polygons in construction, industrial, commercial, domestic, and artistic contexts.
G02.06 Solve a contextual problem that involves the application of the properties of polygons.

G03 Students will be expected to demonstrate an understanding of transformations on a 2-D shape or a 3-D object, including translations, rotations, reflections, and dilations. [C, CN, R, T, V]

Performance Indicators

G03.01 Identify a single transformation that was performed, given the original 2-D shape or 3-D object and its image.
G03.02 Draw the image of a 2-D shape that results from a given single transformation.
G03.03 Draw the image of a 2-D shape that results from a given combination of successive transformations.
G03.04 Create, analyze, and describe designs, using translations, rotations, and reflections in all four quadrants of a coordinate grid.
G03.05 Identify and describe applications of transformations in construction, industrial, commercial, domestic, and artistic contexts.
G03.06 Explain the relationship between reflections and lines or planes of symmetry.
G03.07 Determine and explain whether a given image is a dilation of another given shape, using the concept of similarity.
G03.08 Draw, with or without technology, a dilation image for a given 2-D shape or 3-D object, and explain how the original 2-D shape or 3-D object and its image are proportional.
G03.09 Solve a contextual problem that involves transformations.

Number (N)

N01 Students will be expected to analyze puzzles and games that involve logical reasoning, using problem-solving strategies. [C, CN, PS, R]

Performance Indicators

(It is intended that this outcome be integrated throughout the course by using puzzles and games such as Sudoku, Mastermind, Nim, and logic puzzles.)

N01.01 Determine, explain, and verify a strategy to solve a puzzle or to win a game; for example,
- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backwards
- develop alternative approaches

N01.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.
N01.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

N02 Students will be expected to solve problems that involve the acquisition of a vehicle by buying, leasing, and leasing to buy. [C, CN, PS, R, T]

Performance Indicators

N02.01 Describe and explain various options for buying, leasing, and leasing to buy a vehicle.
N02.02 Solve, with or without technology, problems that involve the purchase, lease, or lease to purchase of a vehicle.
N02.03 Justify a decision related to buying, leasing, or leasing to buy a vehicle, based on factors such as personal finances, intended use, maintenance, warranties, mileage, and insurance.

N03 Students will be expected to critique the viability of small business options by considering expenses, sales, and profit or loss. [C, CN, R]

Performance Indicators

N03.01 Identify expenses in operating a small business.
N03.02 Identify feasible small-business options for a given community.
N03.03 Generate options that might improve the profitability of a small business.
N03.04 Determine the break-even point for a small business.
N03.05 Explain factors, such as seasonal variations and hours of operation, that might impact the profitability of a small business.

ALGEBRA (A)

A01 Students will be expected to demonstrate an understanding of linear relations by

- recognizing patterns and trends
- graphing
- creating tables of values
- writing equations
- interpolating and extrapolating
- solving problems

[CN, PS, R, T, V]

Performance Indicators

A01.01 Identify and describe the characteristics of a linear relation represented in a graph, table of values, number pattern, or equation.
A01.02 Sort a set of graphs, tables of values, number patterns, and/or equations into linear and non-linear relations.
A01.03 Write an equation for a given context, including direct or partial variation.
A01.04 Create a table of values for a given equation of a linear relation.
A01.05 Sketch the graph for a given table of values.
A01.06 Explain why the points should or should not be connected on the graph for a context.
A01.07 Create, with or without technology, a graph to represent a data set, including scatterplots.
A01.08 Describe the trends in the graph of a data set, including scatterplots.
A01.09 Sort a set of scatterplots according to the trends represented (linear, non-linear, or no trend).
A01.10 Solve a contextual problem that requires interpolation or extrapolation of information.
A01.11 Relate slope and rate of change to linear relations.
A01.12 Match given contexts with their corresponding graphs, and explain the reasoning.
A01.13 Solve a contextual problem that involves the application of a formula for a linear relation.

STATISTICS (S)

S01 Students will be expected to solve problems that involve measures of central tendency, including mean, median, mode, weighted mean, and trimmed mean. [C, CN, PS, R]

Performance Indicators

S01.01 Explain, using examples, the advantages and disadvantages of each measure of central tendency.
S01.02 Determine the mean, median, and mode for a set of data.
S01.03 Identify and correct errors in a calculation of a measure of central tendency.
S01.04 Identify the outlier(s) in a set of data.
S01.05 Explain the effect of outliers on mean, median, and mode.
S01.06 Calculate the trimmed mean for a set of data, and justify the removal of the outliers.
S01.07 Explain, using examples such as course marks, why some data in a set would be given a greater weighting in determining the mean.
S01.08 Calculate the mean of a set of numbers after allowing the data to have different weightings (weighted mean).
S01.09 Explain, using examples from print and other media, how measures of central tendency and outliers are used to provide different interpretations of data.
S01.10 Solve a contextual problem that involves measures of central tendency.

S02 Students will be expected to analyze and describe percentiles. [C, CN, PS, R]

Performance Indicators

S02.01 Explain, using examples, percentile ranks in a context.
S02.02 Explain decisions based on a given percentile rank.
S02.03 Explain, using examples, the difference between percent and percentile rank.
S02.04 Explain the relationship between median and percentile.
S02.05 Solve a contextual problem that involves percentiles.

PROBABILITY (P)

P01 Students will be expected to analyze and interpret problems that involve probability. [C, CN, PS, R]

Performance Indicators

P01.01 Describe and explain the applications of probability (e.g., medication, warranties, insurance, lotteries, weather prediction, 100-year flood, failure of a design, failure of a product, vehicle recalls, approximation of area).
P01.02 Calculate the probability of an event based on a data set.
P01.03 Express a given probability as a fraction, decimal, and percent and in a statement.
P01.04 Explain the difference between odds and probability.
P01.05 Determine the probability of an event, given the odds for or against.
P01.06 Explain, using examples, how decisions may be based on a combination of theoretical probability calculations, experimental results, and subjective judgements.
P01.07 Solve a contextual problem that involves a given probability.

Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to
- communicate in order to learn and express their understanding of mathematics (Communication [C])
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines (Connections [CN])
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation [ME])
- develop and apply new mathematical knowledge through problem solving (Problem Solving [PS])
- develop mathematical reasoning (Reasoning [R])
- select and use technologies as tools for learning and solving problems (Technology [T])
- develop visualization skills to assist in processing information, making connections, and solving problems (Visualization [V])

The Nova Scotia curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning. The key to these process standards is presented in a box, as shown below, with each specific outcome within the units.

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics.

Students also need to communicate their learning using mathematical terminology. Communication can help students make connections between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, written and symbolic—of mathematical ideas. Students must communicate *daily* about their mathematics learning. This enables them to reflect, to validate, and to clarify their thinking and provides teachers with insight into students' interpretations of mathematical meanings and ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts.

When students encounter new situations and respond to questions of the type, How would you ... ? or How could you ... ?, the problem-solving approach is being modeled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

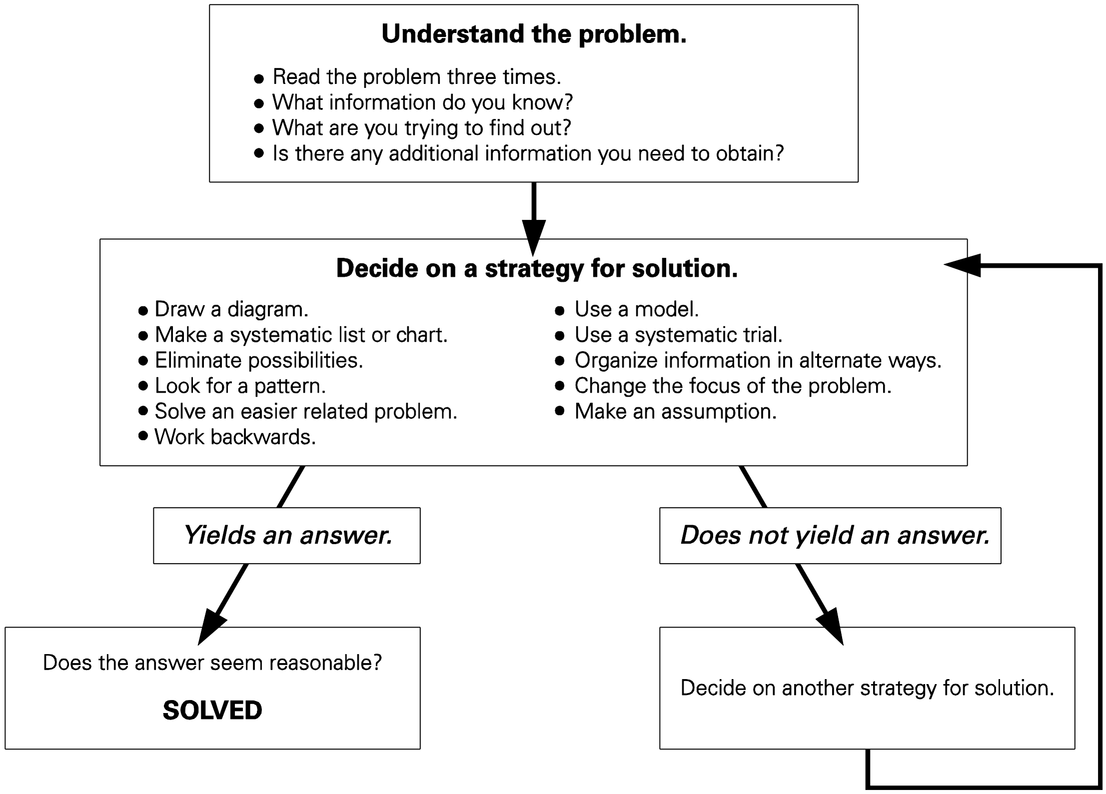
Both conceptual understanding and student engagement are fundamental in molding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive- and deductive-reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for, and engage in finding, a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

A possible flow chart to share with students is as follows:



Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to one another or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding. … *Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*." (Caine and Caine 1991, 5).

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates learning. Concepts and skills should also be connected to everyday situations and other curricular areas. For example, when developing literacy skills students learn to make text-to-world, text-to-text, and text-to-self connections. Students can also make connections to make mathematics come alive through math-to-world, math-to-math, and math-to-self connections.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external aids.

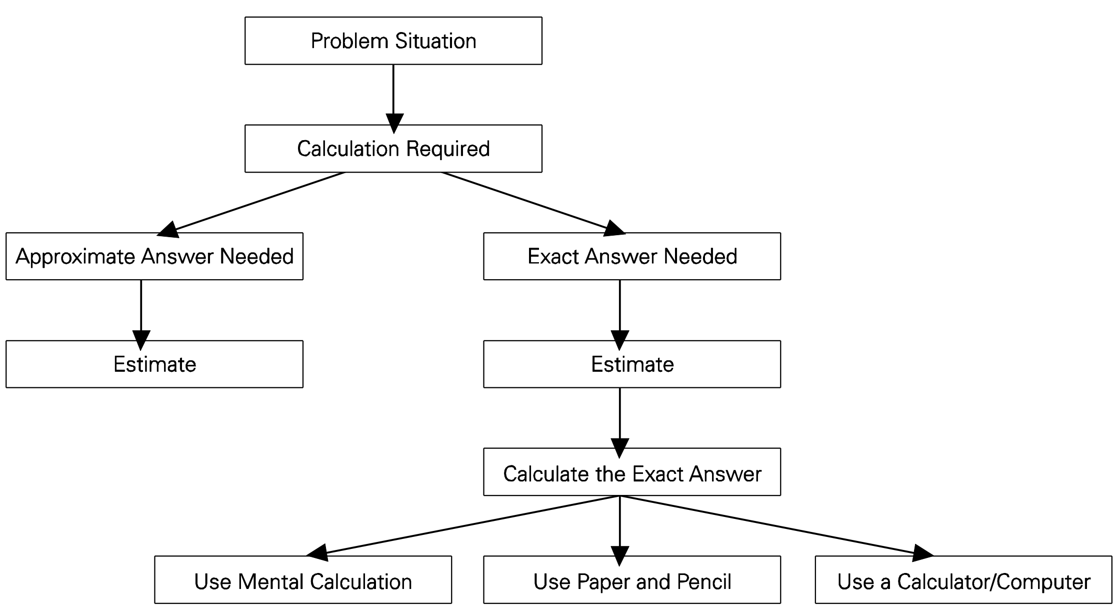
Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math.” (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers, and are more able to use multiple approaches to problem solving.” (Rubenstein 2001) Mental mathematics “provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers.” (Hope et al. 1988, v)

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when, and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process as illustrated below.



The skill of estimation requires a sound knowledge of mental mathematics. Both are necessary to many everyday experiences, and students should be provided with frequent opportunities to practise these skills.

Technology [T]

Technology can be effectively used to contribute to and support the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators, computers, and other technologies can be used to

- explore and represent mathematical relationships and patterns in a variety of ways
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of foundational concepts
- develop personal procedures for mathematical operations
- simulate situations
- develop number and spatial sense
- generate and test inductive conjectures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world." (Armstrong 1993, 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and to know several estimation strategies. (Shaw and Cliatt 1989, 150)

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, Why do you believe that's true/correct? or What would happen if ...

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true.

Mathematics reasoning involves informal thinking, conjecturing, and validating—these help students understand that mathematics makes sense. Students are encouraged to justify, in a variety of ways, their solutions, thinking processes, and hypotheses. In fact, good reasoning is as important as finding correct answers. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics, and these are woven throughout this document. These components include change, constancy, number sense, relationships, patterns, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen 1990, 184).

Students need to learn that new concepts of mathematics as well as changes to previously learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers, and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms **stability**, **conservation**, **equilibrium**, **steady state** and **symmetry** (AAAS–Benchmarks 1993, 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.
- Lines with constant slope.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy. (British Columbia Ministry of Education, 2000, 146) Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities, and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Curriculum Document Format

This guide presents the mathematics curriculum so that a teacher may readily view the scope of the outcomes that students are expected to achieve during that year. Teachers are encouraged, however, to examine what comes before and what follows, to better understand how students' learning at a particular grade level is part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes.

When a specific curriculum outcome is introduced, it is followed by the mathematical processes and performance indicators for that outcome. A scope and sequence is then provided, which relates the SCO to previous and next grade SCOs. Also for each SCO, there are background information, assessment strategies, suggested instructional strategies, and suggested models and manipulatives, mathematical vocabulary, and resource notes. For each section, the guiding questions should be used to help with unit and lesson preparation.

Assessment Strategies

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcome and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO

Mathematical Processes		
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections
[ME] Mental Mathematics and Estimation		
[T] Technology	**[V]** Visualization	**[R]** Reasoning

Performance Indicators

Describes observable indicators of whether students have achieved the specific outcome.

Scope and Sequence

Previous grade or course SCOs	Current grade SCO	Following grade or course SCOs

Background

Describes the "big ideas" to be learned and how they relate to work in previous grade and work in subsequent courses.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Sample tasks that can be used to determine students' prior knowledge.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Some suggestions for specific activities and questions that can be used for both instruction and assessment

FOLLOW-UP ON ASSESSMENT

Planning for Instruction

Choosing Instructional Strategies

Suggested strategies for planning daily lessons.

SUGGESTED LEARNING TASKS

Suggestions for general approaches and strategies suggested for teaching this outcome.

SUGGESTED MODELS AND MANIPULATIVES

MATHEMATICAL LANGUAGE

Teacher and student mathematical language associated with the respective outcome.

Resources/Notes

Contexts for Learning and Teaching

Beliefs about Students and Mathematics Learning

“Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.” (National Council of Teachers of Mathematics 2000, 20).

- The Nova Scotia mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning that have grown out of research and practice. These beliefs include the following:
- Mathematics learning is an active and constructive process.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.
- Learners are individuals who bring a wide range of prior knowledge and experiences and who learn via various styles and at different rates.
- Leaning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best constructed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals, and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics. The learning environment should value, respect, and address all students’ experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals of Mathematics Education

The main goals of mathematics education are to prepare students to
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- become mathematically literate adults, using mathematics to contribute to society
- commit themselves to lifelong learning

Students who have met these goals will

- gain understanding and appreciation of the contributions of mathematics as a science, a philosophy, and an art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding. Students should be encouraged to

- take risks
- think and reflect independently
- share and communicate mathematical understanding
- solve problems in individual and group projects
- pursue greater understanding of mathematics
- appreciate the value of mathematics throughout history

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for student success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations, and engage in reflective practices.

To experience success, students must be taught to set achievable goals and assess their progress as they work toward these goals. Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Engaging All Learners

*"*No matter how engagement is defined or which dimension is considered, research confirms this truism of education: *The more engaged you are, the more you will learn."* (Hume 2011, 6)

Student engagement is at the core of learning. Engagement in learning occurs when students are provided with opportunities to become more invested in their learning. This is critical for teachers to take into account when planning and implementing instruction. Effective instruction engages, embraces, and supports all learners through a range of learning experiences that are both age and developmentally appropriate.

This curriculum is designed to provide learning opportunities that are equitable, accessible, and inclusive of the many facets of diversity represented in today's classrooms. When teachers know their students as individual learners and as individual people, their students are more likely to be motivated to learn, persist in challenging situations, and apply reflective practices.

Supportive Learning Environments

A supportive and positive learning environment has a profound effect on students' learning. Students need to feel physically, socially, emotionally, and culturally safe in order to take risks with their learning. In classrooms where students feel a sense of belonging, see their teachers' passion for learning and teaching, are encouraged to actively participate, and are challenged appropriately, they are more likely to be successful.

Teachers recognize that not all students progress at the same pace nor are they equally positioned in terms of their prior knowledge of particular concepts, skills, and learning outcomes. Teachers are able to create more equitable access to learning when

- instruction and assessment are flexible and offer multiple means of representation
- students have options to engage in learning through multiple ways
- students can express their knowledge, skills, and understanding in multiple ways

(Hall, Meyer, and Rose 2012)

In a supportive learning environment, teachers plan learning experiences that support *each* student's ability to achieve curriculum outcomes. Teachers use a variety of effective instructional approaches that help students to succeed, such as

- providing a range of learning opportunities that build on individual strengths and prior knowledge
- providing all students with equitable access to appropriate learning strategies, resources, and technology
- involving students in the creation of criteria for assessment and evaluation
- engaging and challenging students through inquiry-based practices
- verbalizing their own thinking to model comprehension strategies and new learning
- balancing individual, small-group, and whole-class learning experiences
- scaffolding instruction and assignments as needed and giving frequent and meaningful descriptive feedback throughout the learning process
- integrating "blended learning" opportunities by including an online environment that extends learning beyond the physical classroom
- encouraging students to take time and to persevere, when appropriate, in order to achieve a particular learning outcome

Multiple Ways of Learning

"Advances in neuroscience and education research over the past 40 years have reshaped our understanding of the learning brain. One of the clearest and most important revelations stemming from brain research is that there is no such thing as a 'regular student.'" (Hall, Meyer, and Rose 2012, 2) Teachers who know their students well are aware of students' individual learning differences and use this understanding to inform instruction and assessment decisions.

The ways in which students make sense of and demonstrate learning vary widely. Individual students tend to have a natural inclination toward one or a few learning styles. Teachers are often able to detect learning strengths and styles through observation and through conversation with students. Teachers can also get a sense of learning styles through an awareness of students' personal interests and talents. Instruction and assessment practices that are designed to account for multiple learning styles create greater opportunities for all students to succeed.

While multiple learning styles are addressed in the classroom, the three most commonly identified are:

- auditory (such as listening to teacher-modelled think-aloud strategies or participating in peer discussion)
- kinesthetic (such as examining artifacts or problem-solving using tools or manipulatives)
- visual (such as reading print and visual texts or viewing video clips)

For additional information, refer to *Frames of Mind: The Theory of Multiple Intelligences* (Gardner 2007) and *How to Differentiate Instruction in Mixed-Ability Classrooms* (Tomlinson 2001).

A Gender-Inclusive Curriculum and Classroom

It is important that the curriculum and classroom climate respect the experiences and values of all students and that learning resources and instructional practices are not gender-biased. Teachers promote gender equity and inclusion in their classrooms when they

- articulate equally high expectations for all students
- provide equal opportunity for input and response from all students
- model gender-fair language, inclusive practices, and respectful listening in their interactions with students
- identify and openly address societal biases with respect to gender and sexual identity

Valuing Diversity: Teaching with Cultural Proficiency

"Instruction that is embedded in socially meaningful contexts, and tasks that are meaningful and relevant to the lives of students, will engage students in high-level problem-solving and reasoning and enhance students' engagement (Frankenstein 1995; Gutstein 2003; Ladson-Billings 1997; Tate 1995)." (Herzig 2005)

Teachers appreciate that students have diverse life and cultural experiences and that individual students bring different prior knowledge to their learning. Teachers can build upon their knowledge of their students as individuals, value their prior experiences, and respond by using a variety of culturally-proficient instruction and assessment practices in order to make learning more engaging, relevant, and accessible for all students. For additional information, refer to *Racial Equity Policy* (Nova Scotia Department of Education 2002) and *Racial Equity / Cultural Proficiency Framework* (Nova Scotia Department of Education 2011).

Students with Language, Communication, and Learning Challenges

Today's classrooms include students who have diverse language backgrounds, abilities, levels of development, and learning challenges. By observing and interacting with students and by conversing with students and/or their families, teachers gain deeper insights into the student as a learner. Teachers can use this awareness to identify and respond to areas where students may need additional support to achieve their learning goals. For students who are experiencing difficulties, it is important that teachers distinguish between those students for whom curriculum content is challenging and those for whom language-based factors are at the root of apparent academic difficulties. Students who are learning English as an additional language may require individual support, particularly in language-based subject areas, while they become more proficient in their English language skills. Teachers understand that many students who appear to be disengaged may be experiencing difficult life or family circumstances, mental health challenges, or low self-esteem, resulting in a loss of confidence that affects their engagement in learning. A caring, supportive teacher demonstrates belief in the students' abilities to learn and uses the students' strengths to create small successes that help nurture engagement in learning and provide a sense of hope.

Students who Demonstrate Exceptional Talents and Giftedness

Modern conceptions of giftedness recognize diversity, multiple forms of giftedness, and inclusivity. Some talents are easily observable in the classroom because they are already well developed and students have opportunities to express them in the curricular and extracurricular activities commonly offered in schools. Other talents only develop if students are exposed to many and various domains and hands-on experiences. Twenty-first century learning supports the thinking that most students are more engaged when learning activities are problem-centred, inquiry-based, and open-ended. Talented and gifted students usually thrive when such learning activities are present. Learning experiences may be enriched by offering a range of activities and resources that require increased cognitive demand and higher-level thinking with different degrees of complexity and abstraction. Teachers can provide further challenges and enhance learning by adjusting the pace of instruction and the breadth and depth of concepts being explored. For additional information, refer to *Gifted Education and Talent Development* (Nova Scotia Department of Education 2010).

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in business education, career education, literacy, music, physical education, science, social studies, technology education, and visual arts.

Measurement
6–8 hours

GCO: Students will be expected to develop spatial sense through direct and indirect measurement.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SC0 M01 Students will be expected to demonstrate an understanding of the limitations of measuring instruments, including precision, accuracy, uncertainty, tolerance, and to solve problems. [C, PS, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

M01.01 Explain why, in a given context, a certain degree of precision is required.
M01.02 Explain why, in a given context, a certain degree of accuracy is required.
M01.03 Explain, using examples, the difference between precision and accuracy.
M01.04 Compare the degree of accuracy of two given instruments used to measure the same attribute.
M01.05 Relate the degree of accuracy to the uncertainty of a given measure.
M01.06 Analyze precision and accuracy in a contextual problem.
M01.07 Calculate maximum and minimum values, using a given degree of tolerance in context.
M01.08 Describe, using examples, the limitations of measuring instruments used in a specific trade or industry.
M01.09 Solve a problem that involves precision, accuracy, or tolerance.

Scope and Sequence

Mathematics at Work 11	**Mathematics at Work 12**
M01 Students will be expected to solve problems that involve SI and imperial units in surface area measurements and verify the solutions. **M02** Students will be expected to solve problems that involve SI and imperial units in volume and capacity measurements.	**M01** Students will be expected to demonstrate an understanding of the limitations of measuring instruments, including precision, accuracy, uncertainty, tolerance, and to solve problems.

Background

This outcome is introduced discretely and then integrated throughout the course.

In Mathematics at Work 10 and Mathematics at Work 11, students explored the SI and imperial systems of measurement. They used various measuring devices to calculate measurements in SI and imperial units. Students worked with units for length, area, volume, capacity, mass, and temperature. They also worked with unit conversions within and between both systems. This will be the students' first experience with the terms **precision, accuracy,** and **tolerance** in the context of a mathematics class. When measuring something, it is important that both accuracy and precision are taken into account. They will explore these concepts as they apply to a variety of situations.

Accuracy is the degree of closeness of a measured value to the true value and depends on how carefully the measuring device is used. It is related to the amount of error made in a measurement. The closer the measured value is to the true value, the more accurate the measurement value is considered to be.

Any measuring device must be used correctly. If a tape measure, for example, is not placed correctly on the item to be measured, then no matter how small the scale divisions, the answer will be **inaccurate**. Accuracy of a measurement depends on both how well the measuring tool has been calibrated and on the skill of the person using the measuring tool (e.g., a photocopied ruler may not have an accurate scale).

Precision is the degree of exactness to which a measurement is expressed. Both the type of tool chosen and the scale division of the tool used will impact the precision of a measurement. For example, to measure the width of a barn, a measuring tape of appropriate length would be a better tool than a metre stick even when both the tape and ruler have the same scale divisions. If there were two tapes to choose from, the precision will vary depending on how fine the scale divisions are on the measuring tapes. If a tape measure is marked off in 10 cm intervals, measurements will be far less precise than with a tape measure marked off in 1 mm intervals.

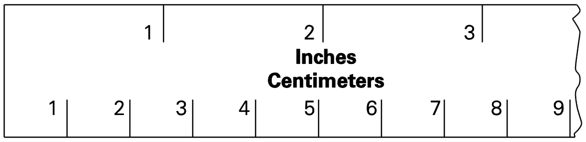
The measuring instrument used will limit the degree of accuracy and precision. Some common measuring errors include the use of incorrect units or reading the measuring device incorrectly. The length of a picture frame may be measured with precision to the nearest millimetre at 15.3 cm, but if the actual measure was 16.3 cm, the measurement is not accurate. The length of this picture frame may be measured accurately, using a centimetre measuring tool, to 16 cm, but the measurement is not as precise as when using a measuring tool marked off in 1 mm intervals.

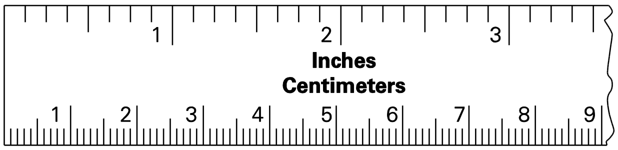
Applying the concepts of precision and accuracy to real-world applications helps students think critically about measurements in everyday life. The measurements the contractor makes must be accurate and the tools he uses must be in appropriate units to give him precise measurements for the task. When buying baseboard, the contractor’s measuring tape might be in feet. When installing baseboards, he needs a tape measure marked off in sixteenths of an inch. The degree of accuracy is determined by the skill with which the person does the measurement. The degree of precision is determined by the tools the person chooses to use when doing the measurement. For example, if a contractor is buying baseboards for a room, the measurements need to be accurate. However, when installing the baseboards, the measurements need to be precise and the carpenter must still measure with accuracy. Students will analyze several similar situations in which accuracy and precision are important.

The use of the measuring instrument depends on the situation. Different types of measuring devices may be more suitable for making certain measurements, which may lead to varying degrees of accuracy and precision. A ruler or tape measure is useful for accurately and precisely measuring most linear dimensions. However, a caliper may be more appropriate for identifying and measuring diameters. To distinguish between a standard three-inch pipe and a standard four-inch pipe, a plumber may use a tape measure. A millwright constructing a non-standard-size pipe may have to use a micrometer to measure its diameter. Understanding how the choice of measurement tools affects the accuracy and precision of a measurement will help students make informed choices when measuring.

There is **a degree of uncertainty** when measuring due to the limitations of measuring devices. The **margin of error** of a measurement, if not stated, is half the precision of the measuring device. In other words, true measurement falls within a halfway measure of that point and the next highest and lowest decimal measure. For example, when a measurement is written as 0.01, the implication is that the true measurement is between 0.005 and 0.015. When a measurement is written as 12 the implication is that precision is to the one’s place and the true value is between 11.5 and 12.5. The margin of error is ± 0.5. However, if the measurement is written as 12.0, then the implication is that precision is to the hundredths’ place so the measurement is more precise and the true measurement falls between 11.95 and 12.05. The margin of error is ± 0.05.

Consider which tool below can provide a more precise measurement:





There is a greater degree of uncertainty when devices have less precision. The top ruler can measure to the nearest centimetre with certainty. The margin of error is 0.5 cm, so any measurement in the range of 0.5 cm below the actual measurement and 0.5 cm above the actual measurement would be acceptable. Using the bottom ruler, one-tenth of a centimetre can be measured with certainty, with a margin of error of one-twentieth of a centimetre.

No measurement can be perfectly accurate, as there will always be some degree of error, regardless of the precision. This degree of or allowance for error is called the **tolerance** of the measurement. The **range of tolerance** is the greatest range of variation that can be allowed. Sometimes an amount of tolerance that is acceptable is given. For example, when machining pistons, actual measurements cannot range more than 1 mm from the stated measure. However, when knitting mittens, actual measurements up to 1–2 cm more or less of the intended measurement are of no consequence. Depending on the situation, different levels of variation can be tolerated.

This tolerance acknowledges that measurements need to only be acceptable for the job they are intended to do. This allows for minimum and maximum allowable measurements. This is students' first exposure to the ± symbol. They are expected to calculate the maximum and minimum values given the acceptable tolerance.

Students should be provided with several daily-life examples of tolerance relative to measurement. For example, most speedometers have tolerances of ± 10%, mainly due to variations in tire diameter. Sources of error due to tire diameter variations are wear, temperature, pressure, vehicle load, and minimal tire size. Students may be interested to know that vehicle manufacturers usually calibrate speedometers to read high by an amount equal to the average error, to ensure that their speedometers never indicate a lower speed than the actual speed of the vehicle.

Baking provides another useful example of the relevance of levels of precision and range of tolerance**.** Most recipes tolerate considerable variation in measurements. For example, when baking a cake there is a lot of variation in the weight of an egg. The amount of butter can also vary considerably. If a cake recipe calls for 150 g butter, an amount from 145 g to 155 g of butter is acceptable. The measurement tolerance for butter is 5 g. If the piece of butter placed on a scale measures 152 g, this would fall within the range of tolerance, and it is not necessary to shave a little butter from the piece. Assuming the scale is accurate and the margin of error is 0.5, a measure of 152 g indicates that the butter weighs between 151.5 g and 152.5 g, which is more precision than the cook needs. On the other hand, the recipe also calls for 10 g cinnamon. This requirement has smaller error tolerance. No cook will be happy with a reading of 12 g because this could mean as much as 25% more cinnamon than is called for. The cook will remove a little cinnamon from the scale, until it displays 10 g, accepting a range of tolerance of from

9.5–10.5 g. In this example, the scale is more precise than required to weigh the butter, but the one-gram precision is required to weigh the cinnamon.

Students can also be presented with a tolerance, required to calculate an acceptable range of values, and then asked to comment on the accuracy or precision of a given measurement.

It is important that students be given the opportunity to apply their understanding of the meanings of **accuracy**, **precision**, and **tolerance** to a wide variety of contextual problems similar to the examples given above.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Calculate the volume of concrete needed for a sidewalk block that measures 5 ft. × 5 ft. × 9 in.
- Identify objects in the room that would be approximately
  (a) 1 cm long
  (b) 2 m long
  (c) 3 ft. long
- What fraction of 1 m^2 is a sticky note that is 10 cm × 10 cm?
- Approximate the area of the classroom floor. Measure the dimensions of the floor and calculate the area.
- Choose an object in the classroom such as a paper box or some other container. Approximate the volume of the object you selected. Measure the dimensions of the container and calculate the volume.
- From a display of cans, select one, approximate the volume, measure the dimensions, and calculate the volume.
- Convert the following:
  (a) ____ g = 153 mg
  (b) 30 L = ____ mL
  (c) ____ g = 0.43 kg
  (d) $\frac{1}{2}$ cup = ____ Tbsp.
  (e) 3 000 000 cm^3 = ____ m^3
  (f) 1 cubic ft. = ____ cubic in.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the maximum and minimum allowable measurements, based on the given tolerances.

  (a) $34'' \pm \frac{1}{8}''$ (b) 3 kg ± 5 g

  (c) 1.2699” ± 0.0009”

  (d) 2.000” ± 0.002”

  (e) 10.203 mm ± 0.024 mm

  (f) 64.86 mm ± 0.03 mm
- A machine part can measure 44.3 ± 0.3 mm. Determine the acceptable maximum and minimum dimensions.
- The mass of a Canadian toonie produced by the Royal Canadian Mint is 6.92 g with a tolerance of ±0.01 g. Otherwise, they are rejected for circulation. What are the maximum and minimum allowable masses for a toonie?
- Determine whether accuracy and/or precision is important in each of the situations given. Justify your answer.
  - A contractor develops material lists to provide reasonable estimates for jobs.
  - A seamstress hems a pair of pants.
  - A house painter determines how much paint is needed.
- Given a piece of lumber labelled as being 2.0 m, what is the

  (a) implied tolerance

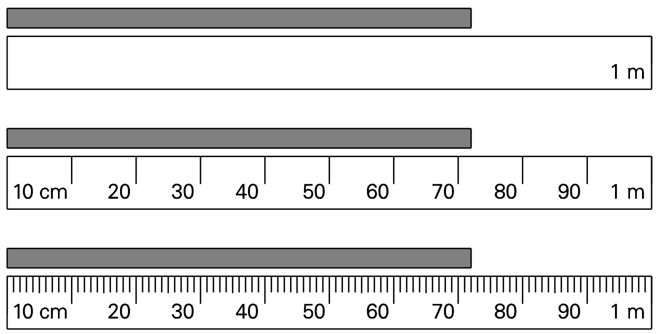
  (b) longest the piece of lumber could be and still satisfy the implied error tolerance

  (c) shortest the piece of lumber could be and still satisfy the implied error tolerance

  (d) tolerance range

  (**Note:** This might work better with a true piece of lumber and with three different measurements specified. Have students measure the lumber themselves.)
- Two students use a metre stick marked to centimetres to measure the width of a lab table. One records 84.0 cm being as precise as possible. The other student records 80 cm being as precise as possible. How could they get different answers, and who is right?

- Using the three different metre sticks shown below,
  (a) What is the measure of the ribbon in each case?
  (b) Which metre stick would give the highest degree of precision?
  (c) Does it matter which metre stick is used to measure the length of a nail? Why or why not?
  (d) Does it matter which metre stick is used to measure the approximate width of a room? Why or why not?



- *Journal Entries:*
  (a) Consider the statement, Measure twice; cut once. In your entry, discuss the implications for accuracy and precision and how this statement relates to different contexts (e.g., cooking, sewing, carpentry, electrical, and plumbing).
  (b) When estimating materials for a job, 10–15% is often added to the estimate. Why is this a common practice?
- A dairy farmer prepares 15 mL of antibiotic in a syringe marked in 1-mL increments. The tolerance is ± 0.5 mL. What are the maximum and minimum allowable measurements?

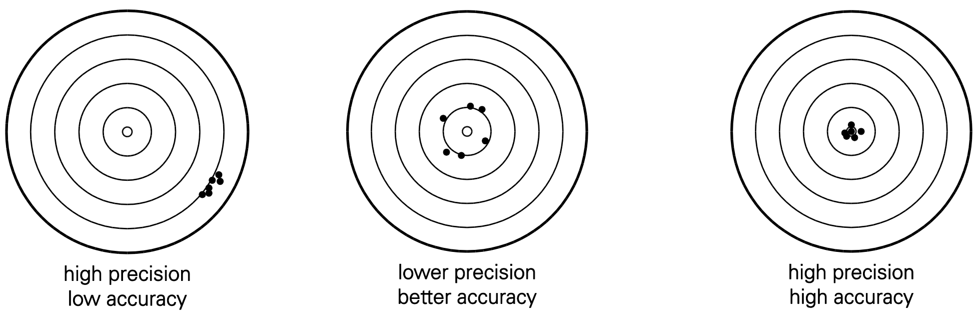
Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Arrange with the trades teachers in the school a visit to the shop to examine instruments used for measuring and the specific jobs for which they are used.
- Have students measure items and discuss their answers in terms of accuracy and precision.
- Discuss with students examples of measurements that are
  - accurate but not precise
  - precise but not accurate
- Provide a wide range of measuring instruments (or ask students to bring them in)—calipers, metre stick, clear 15-cm ruler, measuring tape etc.—and have students determine which instrument is best for measuring various objects. Conversely, have students bring in various objects to be measured and have them select the best measurement tool to measure that object.

- To demonstrate accuracy, have each student carefully measure the length of some object in the classroom (about 1 m long) and record their measurements. Students should do this individually, without input from their peers, and record their lengths on small pieces of paper, submitted anonymously. In the next class, draw a number line and mark each measurement with a solid dot above the corresponding position on the scale (i.e., a dot plot). Students will see that some of them measured more accurately than others.
- Present various situations and ask students to determine whether accuracy or precision is important.
  - A contractor develops material lists to provide reasonable estimates for jobs.
  - A seamstress hems a pair of pants.
  - A house painter determines how much paint is needed.
- Discuss several cases to illustrate why it is important to be accurate and/or precise. Examples follow.
  - A person being treated for diabetes requires 2.5 units of insulin to account for the number of carbohydrates. He draws up 2.8 units of insulin to take.
  - A carpenter cuts a piece of crown moulding that is required to be 22.5°. Due to the tool being used, she can only cut to the nearest degree. When completed, the angle is noted to be 22°.
- Precision and accuracy can be explained with reference to a dartboard / bull's eye as shown below:



- Have students interview various tradespeople, asking questions such as the following:
  - What tools do you use to measure?
  - Why do you use different measuring tools? Is there one that you prefer?
  - Which tools do you need to use to be accurate?
  - Which tools do you need to use to be precise?
  - What factors do you consider when purchasing tools?
- Using Think-Pair-Share, give individual students time to think about the question below. Students then pair up with a partner to discuss their ideas. After pairs discuss, students share their ideas in a small-group or whole-class discussion.
  - A carpenter's tape typically shows $\frac{1}{32}$ divisions within the first foot. Beyond the first foot, the smallest division is $\frac{1}{16}$. Why is this the case?
- Discuss with students the significance of the tolerance in situations such as the following:
  - The dimensions of a door has a tolerance of 2 mm.
  - A thermometer has a tolerance of 0.1°C.

- Ensure that all students understand what an expression like 8 cm ± 0.5 cm means.
- Discuss why it is important for tradespeople to be familiar with the idea of tolerances. Time and materials will not be wasted in trying to make items that are dimensionally exact, but that function according to the requirements of their use.

SUGGESTED MODELS AND MANIPULATIVES

- calipers
- metre and yard sticks
- rulers and measuring tapes
- string

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- accuracy
- degree of uncertainty
- implied precision
- margin of error
- precision
- tolerance
- tolerance range
- uncertainty

Resources/Notes

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 1: Measurement and Probability
    - > Sections 1.1 and 1.4
    - > Skill Check
    - > Test Yourself
  - Chapter 5: Properties of Figures
    - > Section 5.1
  - Chapter 7: Trigonometry
    - > Section 7.1

Geometry
35–37 hours

GCO: Students will be expected to develop spatial sense.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO G01 Students will be expected to solve problems by using the sine law and cosine law, excluding the ambiguous case. [CN, PS, V]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

G01.01 Identify and describe the use of the sine law and cosine law in construction, industrial, commercial, and artistic applications.

G01.02 Solve a problem using the sine law or cosine law when a diagram is given.

Scope and Sequence

Mathematics at Work 11	Mathematics at Work 12
G01 Students will be expected to solve problems that involve two and three right triangles.	**G01** Students will be expected to solve problems by using the sine law and cosine law, excluding the ambiguous case.

Background

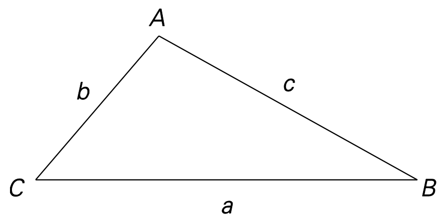
In Mathematics at Work 10, students used the three primary trigonometric ratios (sine, cosine, and tangent) and the Pythagorean theorem to determine the side lengths and angle measures in right triangles. This was extended in Mathematics at Work 11 to include problems involving two and three right triangles. Students also used the terminology **angle of elevation**, **angle of depression**, and **angle of inclination**. A review of correctly labelling sides in relation to opposite angles may be needed at this time.

Students will now solve oblique (non-right angled) triangles using **the sine law** (also commonly referred to as the law of sines) and **the cosine law** (also commonly referred to as the law of cosines) and apply these in various problem-solving situations. Students should become familiar with the use of the sine law before the cosine law is introduced.

Students are not expected to derive these laws, but should understand when each law is appropriate. Students should only be given problems that have unique solutions, excluding the ambiguous case of the sine law.

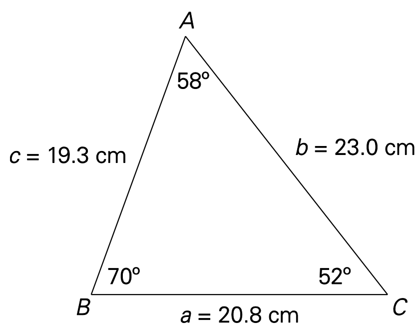
Diagrams are to be provided for all problems. Students should also be encouraged to redraw diagrams with both the given and unknown information indicated.

For ΔABC, *a*, *b*, and *c* represent the measures of the sides opposite $\angle A$, $\angle B$, and $\angle C$ respectively.



In a triangle, the ratio of the length of any side to the sine of the opposite angle is a constant. This ratio can be written $\frac{\text{length of any side}}{\sin(\text{opposite angle})}$ or $\frac{a}{\sin A}$.

Example:



$\frac{a}{\sin A} = \frac{20.8}{\sin 58^\circ} \approx 24.5$ $\frac{b}{\sin B} = \frac{23.0}{\sin 70^\circ} \approx 24.5$ $\frac{c}{\sin \text{C}} = \frac{19.3}{\sin 52^\circ} \approx 24.5$

The **sine law** describes this relationship between the sides and angles in any triangle and can be expressed as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

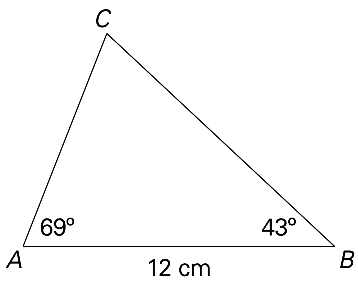
The **sine law** can be used when

- two angles and any side are known

  or

- two sides and an angle opposite to one of the given sides are known (not including any ambiguous case)

Students will need to consider what information is needed to solve problems using this law. They should understand that to use the sine law they may have to first determine the measure of the third angle. Consider an example such as the following:



Students can use the property that the sum of the angles in a triangle is 180°. Therefore, the measure of $\angle C$ is 68°. Students can then proceed to use the sine law to find the length of side *b*.

Since the law of sines involves a ratio of the sine of an angle to the length of its opposite side, it will not work if

- no angle of the triangle is known

  or
- one angle and its opposite side are not known

Students will also apply the cosine law to determine unknown lengths and angle measures in triangles.

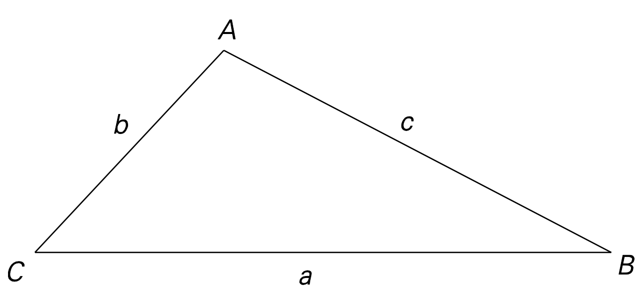
The **cosine law** (also commonly referred to as the law of cosines), **describes** the relationship between the cosine of an angle and the lengths of the three sides of any triangle. It can be used to solve a problem modelled by a triangle when

- two sides and the **included** (or contained) angle are known

  or
- all three sides are known

The cosine law may be written in two forms, depending on whether students are solving for a missing side or angle.

For $\triangle ABC$,



- When solving for an unknown side of the triangle, the cosine law is expressed as
  - $c^2 = a^2 + b^2 - 2ab\cos C$, when solving for side c
  - $a^2 = b^2 + c^2 - 2bc\cos A$, when solving for side a
  - $b^2 = a^2 + c^2 - 2ac\cos B$, when solving for side b

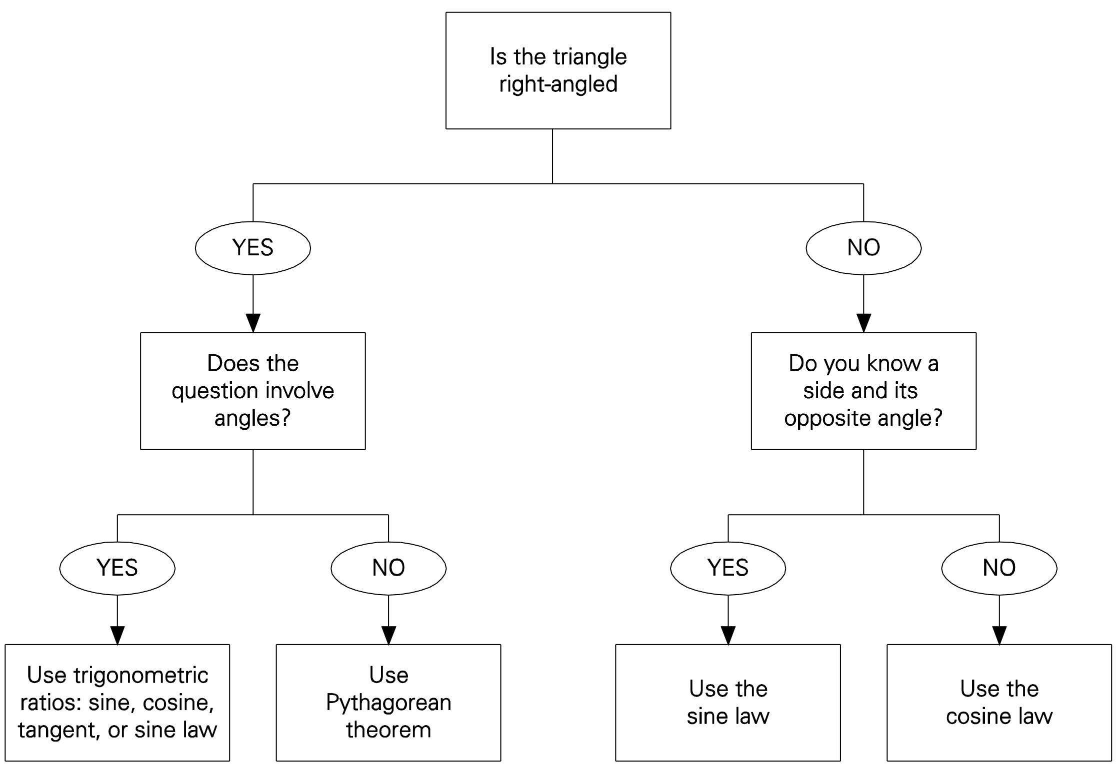
When solving for an unknown angle of the triangle, the cosine law is expressed as

- $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, when solving for $\angle C$.
- $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, when solving for $\angle A$.
- $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$, when solving for $\angle B$.

It is important that students learn the patterns that exist in the cosine formula. Once the pattern is understood, they can write the cosine law for any labelled triangle using any variables.

If students know two sides and a **non-included angle** (an angle that does not lie between the two given sides), they can use the cosine law in conjunction with the sine law to find the other side. As an alternative, they could apply the sine law twice. Students should be exposed to numerous examples to find the method that works best for them.

When solving triangles, a graphic organizer such as the one below can guide students as they decide on the most efficient method to use when solving for an unknown angle and/or side.



It is important that students develop an appreciation for the applicability of the sine law and cosine law to real-life situations. This may be developed by using examples that are based on construction, industrial, commercial, and artistic applications.

Another application that could be discussed is aircraft design and flight. For example, the aircraft engineer must calculate the plane's velocity as well as the air velocity in order to make it as aerodynamic as possible. The sine law can be used to find the angle that the plane must travel to compensate for wind velocity. The cosine law can be used to determine the magnitude of the resultant ground speed of the aircraft along the chosen bearing direction. Although not expected to complete such a problem, students could be exposed to a variety of similar applications.

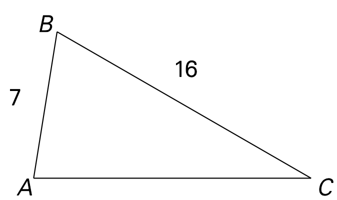
Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

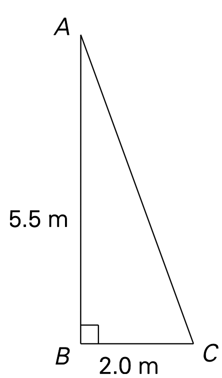
- Explain the error that was made (or errors) in calculating $\angle B$.



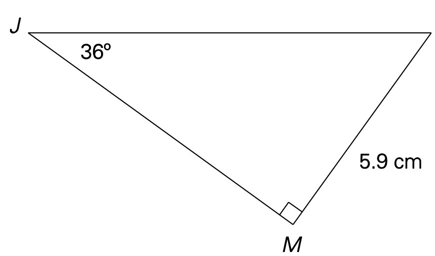
$$\cos B = \frac{7}{16}$$
$$\angle B = \cos^{-1}\left(\frac{7}{16}\right)$$
$$\angle B = 64^\circ$$

- Evaluate each trigonometric ratio. Express your answer to three decimal places.
  (a) $\sin 47^\circ$
  (b) $\cos 11^\circ$

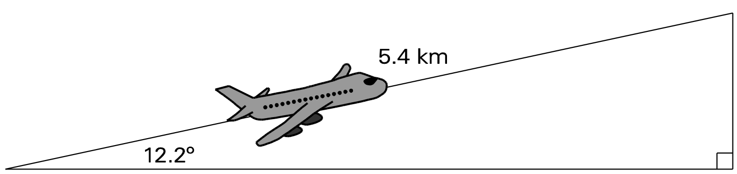
- Determine the measure of $\angle A$. Express your answer to the nearest degree.



- Determine the length of the hypotenuse.



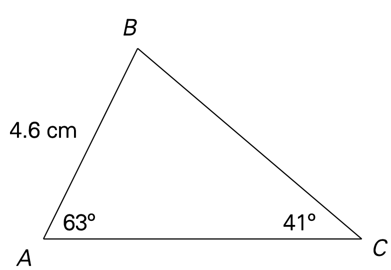
- Identify situations where trigonometric ratios are used to determine measurement of angles and lengths indirectly (e.g., determining the height of a flagpole or a tree from the ground).
- A pilot starts her takeoff and climbs steadily at an angle of 12.2°. Determine the horizontal distance the plane has travelled when it has climbed 5.4 km along its flight path. Express your answer to the nearest tenth of a kilometre.



WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

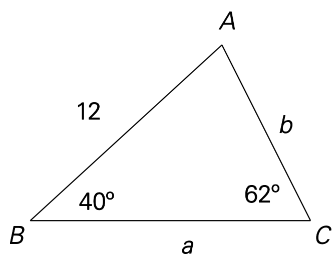
- You are asked to find the length of *AC* (or *b*) in the following diagram.



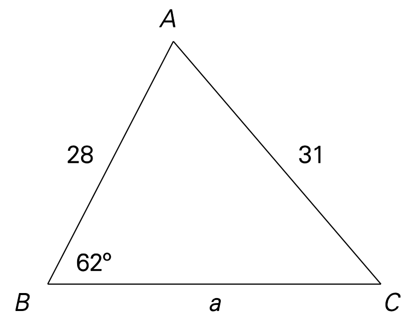
  (a) What information are you given in the triangle?
  (b) What other piece of information do you need to know in order to find the length of the unknown measure *b*?
  (c) How would you determine the unknown measure?
- Answer the following questions:
  (a) Why does the sine law have three ratios in its equation?
  (b) How can you tell which ratios to use?
  (c) Given any triangle, what is the minimum amount of information needed about a triangle's side lengths and angle measures to determine all of the missing sides and angles? What possible combinations will work?

- Solve for the indicated sides and angles in each triangle below. Round all side lengths to one decimal place and angle measures to the nearest degree.

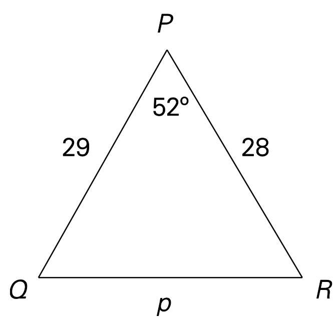
(a)



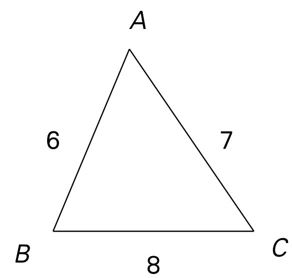
(b)



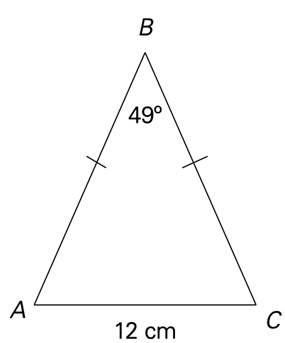
(c)



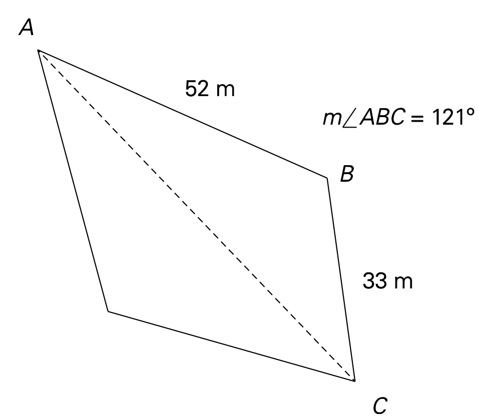
(d) Find the measure of $\angle C$.



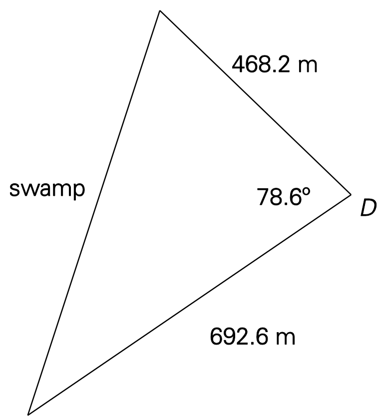
- Find the missing side lengths in the isosceles triangle below.



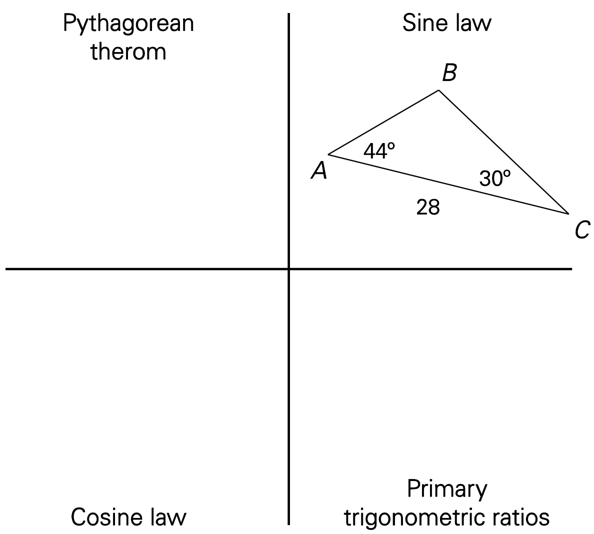
- The diagram to the right shows the dimensions and shape of a piece of land that is to be split between two siblings along the diagonal. To clearly mark this division between the properties, they want to install a fence. What length of fencing would be needed?



- A surveyor needs to find the length of a swampy area near a lake. The surveyor sets up her transit at point *D*. She measures the distance to one end of the swamp to be 468.2 m and the distance to the other end of the swamp to be 692.6 m. The angle of sight between the two ends of the swamp is 78.6°. Find the length of the swamp.



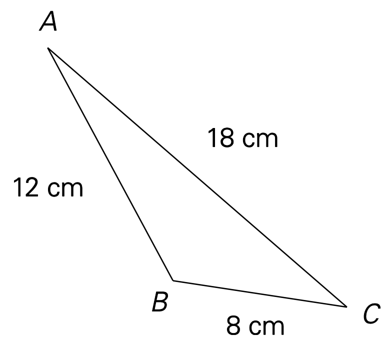
- What quantities must be known in a triangle before the sine law can be used? The cosine law?
- Obtain a set of cards or questions from your teacher. Each card or question will contain a triangle with information regarding the lengths of the sides and angles. Sort these cards or questions according to a specific strategy you would use to find the missing side lengths or angle measures. (Pythagorean theorem, sine law, cosine law, primary trigonometric ratios)



- Research a real-world problem that makes use of the sine law, the cosine law, or both the sine and the cosine laws. Share the problem with the class and discuss the method for solving.

- Find the error that was made (or errors) in calculating $\angle C$ and provide the correct solution.

(a)



$\cos\angle C = \dfrac{a^2+b^2-c^2}{2ab}$

$\cos\angle C = \dfrac{8^2+12^2-18^2}{2(8)(18)}$ Step 1

$\cos\angle C = \dfrac{16+144-324}{288}$ Step 2

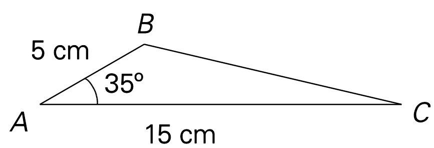
$\cos\angle C = \dfrac{-164}{288}$ Step 3

$\cos\angle C = -0.569$ Step 4

$\cos\angle C = \cos^{-1}(-0.569)$ Step 5

$\cos\angle C = 125°$ Step 6

(b)



$a^2 = b^2+c^2-2bc\cos A$

$a^2 = 15^2+5^2-2(15)(5)(\cos 35°)$ Step 1

$a^2 = 225+25-150(\cos 35°)$ Step 2

$a^2 = 100(\cos 35°)$ Step 3

$a^2 = 100(0.819)$ Step 4

$a^2 = 81.9$ Step 5

$a = 9.1$ Step 6

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

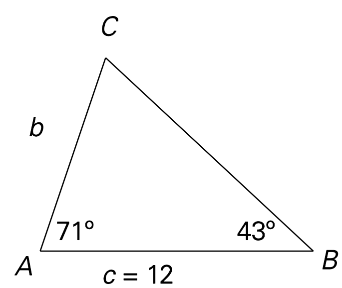
- Start with a review of trigonometric ratios and the Pythagorean theorem. Give students an oblique triangle to solve and allow them to reach the conclusion that these methods cannot be used, and another method must be used. This allows for the introduction of the sine law, followed by the cosine law.
- When introducing the sine and cosine laws, provide students with diagrams illustrating different contextual situations and have them determine which law should be applied to solve a problem.
- The Geometer’s Sketch Pad (Key Curriculum Press 2015), GeoGebra (International GeoGebra Institute 2015), or other graphing software can be used to investigate the sine and cosine laws.
- Use observation to assess student understanding by providing students with several practice problems that use the sine law and/or the law of cosines. As teachers observe students working through the problems, ask them the following questions:
  - What is the unknown? Is it an angle or a side?
  - How can you complete the calculations?
  - Does your conclusion answer the question asked?

- In groups, have students create a sine law application question and a cosine law application question on chart paper to give to another group to solve.
- Students can use what they know about right triangles to explore the sine law.
  - Have them calculate $\sin A$, $\sin B$, and $\sin C$ for any right triangle. Repeat this procedure for an oblique triangle where all side and angle measures are known. Complete a chart similar to the one shown:

Triangle	a	b	c	$\angle A$	$\angle B$	$\angle C$	$\frac{a}{\sin A}$	$\frac{b}{\sin B}$	$\frac{c}{\sin C}$
Right									
Oblique									

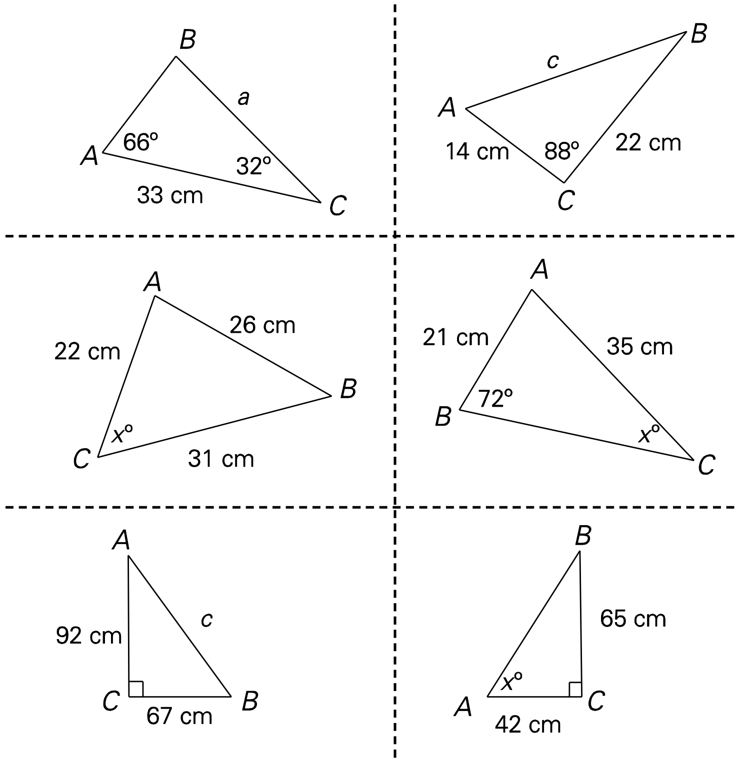
  - Ask students the following questions:
    - > What do you notice about the ratios calculated?
    - > Is this also valid if you use the reciprocal of the ratios?
- Before applying the sine law to determine unknown lengths and angle measures in triangles, ask students to consider what information is needed to solve problems using this law. Remind students that since the law of sines involves a ratio of the sine of an angle to the length of its opposite side, it will not work if no angle of the triangle is known or if one angle and its opposite side are not known.
- When solving triangles, encourage students to consider the following questions:
  - What is the given information?
  - What am I solving for?
  - With the given information, should I use the sine law or the cosine law? Is there a choice?
  - Which form of the law do I use to solve for an unknown side?
  - Which form do I use to solve for an unknown angle?
- When working with the sine law, students sometimes incorrectly identify side and opposite angle pairs. To avoid this error, encourage them to use arrows on the diagram when identifying the angle and its opposite side. They could also encounter problems when they multiply to solve the ratios. When solving $4\sin A = 12\sin 30^\circ$, for example, students may incorrectly write $4\sin A = \sin 360^\circ$ as their next step. The use of brackets may clarify the situation for many students and thus teachers should encourage students to write $4(\sin A) = 12(\sin 30^\circ)$.

  Another common student error occurs when students try to solve a triangle given two angles and an included side, mistakenly thinking that there is not enough information to use the sine law. Consider an example such as the triangle to the right.



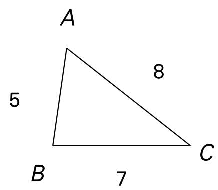
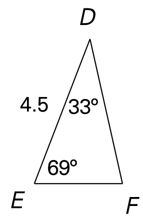
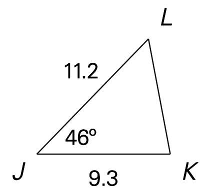
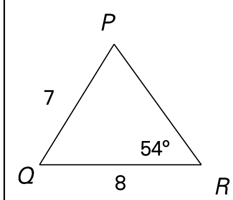
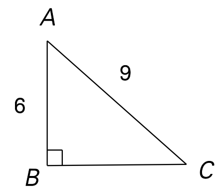
  Students can use the property that the sum of the angles in a triangle is 180°. The measure of $\angle C$ therefore is 66°. They can then proceed to use the sine law to find the length of side *AC*.

- In the activity, Four Corners, students have to think about which method they would use to solve a triangle. Post four signs, one in each corner, labelled sine law, cosine law, Pythagorean theorem, and trigonometric ratios. Provide each student with one triangle. Instruct the students to make a decision as to which method they would use to find the missing angle or side and to stand in the corner where it is labelled. Once students are all placed, ask them to discuss why their triangle(s) would be best solved using that particular method. Sample triangles are given below:



- Encourage students to check the reasonableness of their answer. For example, since $\angle C$ is a little smaller than $\angle A$, we expect the length of side *AB* to be a little shorter than the length of side *CB*. Students should also consider asking questions such as, Is the shortest side opposite the smallest angle? Is the longest side opposite the largest angle?
- Have students consider why the cosine law is the only option to find the unknown angle if three sides are known or if two sides and the included angle are known. Common errors encountered by students using the cosine law may include
  - using the formula to determine a side when the missing measure is an angle
  - not applying the order of operations correctly
- When working with the cosine law, students sometimes incorrectly apply the order of operations. When asked to simplify $a^2 = 365 - 360\cos 70^\circ$, for example, students often write $a^2 = 5\cos 70^\circ$. To avoid this error, teachers should emphasize that multiplication is to be completed before subtraction.

- Ask students to create, or create with students, a graphic organizer. When solving triangles the organizer can guide students as they choose the most efficient method to use in solving an unknown angle and/or side. A sample table organizer is shown below.

				
(3 side measures known)	(angle, side, angle measures known)	(side, angle, side measures known)	(side, side, angle measures known)	(right angle and any two other measures known)
cosine law	sine law	cosine law	sine law or cosine law	Pythagorean theorem and/or trigonometric ratios or sine law

- Invite guest speakers who use trigonometry in their work (e.g., engineers, military personnel, architects) or professions that use navigation, such as forestry, air traffic, or aerospace technicians and engineers.
- Draw a triangle with an obtuse angle to show that the cosine of an angle between 90° and 180° is a negative number. Ensure that students are aware of this—without going into the explanation of the why—so that they would not be surprised when they get a negative number.
- Have students investigate how the length of the metre was established. The length of a metre was initially determined to be 1/10 000 000 of the distance from the equator to the North Pole. In order to determine this distance accurately, French surveyors used triangulation to measure the length of the meridian arc from Dunkirk to Barcelona.
- *Enrichment Activity:* Have students investigate questions involving bearing readings.

SUGGESTED MODELS AND MANIPULATIVES

- metre and yard sticks
- protractors and rulers

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- acute triangle
- cosine law
- included (contained) angle
- oblique triangle
- obtuse triangle
- non-included angle
- sine law

Resources/Notes

Digital

- GeoGebra (International GeoGebra Institute 2015): www.geogebra.org
- The Geometer's Sketchpad (Key Curriculum 2015; NSSBB #: 50474, 50475, 51453)

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 7: Trigonometry
    - Sections 7.1, 7.2, and 7.3
    - Skill Check
    - Test Yourself
    - Chapter Project

SCO G02 Students will be expected to solve problems that involve triangles, quadrilaterals, and regular polygons. [C, CN, PS, V]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

G02.01 Describe and illustrate properties of triangles, including isosceles and equilateral.
G02.02 Describe and illustrate properties of quadrilaterals in terms of angle measures, side lengths, diagonal lengths, and angles of intersection.
G02.03 Describe and illustrate properties of regular polygons.
G02.04 Explain, using examples, why a given property does or does not apply to certain polygons.
G02.05 Identify and explain an application of the properties of polygons in construction, industrial, commercial, domestic, and artistic contexts.
G02.06 Solve a contextual problem that involves the application of the properties of polygons.

Scope and Sequence

Mathematics at Work 11	**Mathematics at Work 12**
G01 Students will be expected to solve problems that involve two and three right triangles.	**G02** Students will be expected to solve problems that involve triangles, quadrilaterals, and regular polygons.

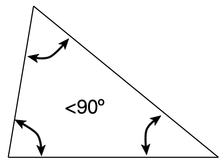
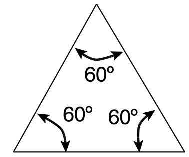
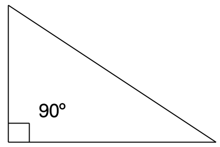
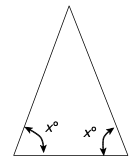
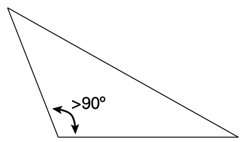
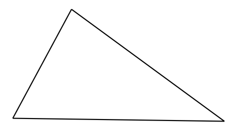
Background

Students will solve problems through exploration of the properties of polygons. This outcome provides students with a good opportunity to explore patterns of geometric shapes, and to practise logical thinking.

In Mathematics 9, students worked with line symmetry. Students worked with acute, right, obtuse, straight, and reflex angles in Mathematics at Work 10 (G06). They will now further explore these topics and apply properties of triangles, quadrilaterals, and regular polygons in contextual situations.

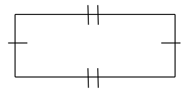
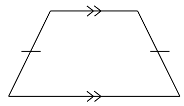
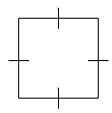
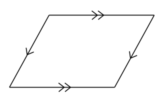
Students have had limited recent exposure to the terminology of geometry, and therefore, a review will be necessary before properties are explored. Students will have worked extensively with right triangles.

In this unit, students will investigate the classification of triangles based on side lengths and angle measurements. Triangles can be classified by their angles as acute, obtuse, or right, and by their sides as scalene, isosceles, or equilateral.

Classification by Angles		Classification by Sides	
Acute Triangle All angles are less than 90°		*Equilateral Triangle* Three congruent sides and three congruent angles that are always 60°	
Right Triangle Has a right angle (90°)		*Isosceles Triangle* Two congruent sides and two congruent angles	
Obtuse Triangle Has an angle more than 90°		*Scalene Triangle* No congruent sides and no congruent angles	

- Students will explore and be able to describe and illustrate the following triangle properties.
  - The largest angle is always opposite the largest side.
  - The smallest angle is always opposite the smallest side.
  - The sum of the angles of a triangle is always 180°.
  - All angles in an equilateral triangle are equal to 60°.
  - All sides in an equilateral triangle are equal length.
  - An isosceles triangle has two equal angles that are acute. The third angle could be acute, right, or obtuse.
  - In an isosceles triangle, the angles opposite to the equal sides are equal.
  - The exterior angle of a triangle is equal to the sum of interior angles across from it.
  - The sum of the lengths of any two sides of a triangle must be greater than the third side.
  - A scalene triangle has no equal angles.
  - A scalene triangle has no equal side lengths.

Once triangle properties are established, students will explore the properties of quadrilaterals and regular polygons. In this unit, the quadrilaterals that students work with are parallelograms, rectangles, squares, and isosceles trapezoids. A review of these types of quadrilaterals should be done before exploring their properties.

Rectangle	
Isosceles Trapezoid	
Square	
Parallelogram	

Quadrilateral	**Properties**				
	Angles congruent	Sides congruent	Sides parallel	Diagonal bisect angle is 90°	Diagonal lengths are congruent
square	all = 90°	all	opposites	yes	yes
rectangle	all = 90°	opposites	opposites	no	no
parallelogram	opposites congruent	opposites	opposites	no	no
isosceles trapezoids	no	no	one pair of opposites	no	no

In Mathematics at Work 10, students worked with parallel lines with a focus on angle relationships that occur with parallel lines cut by a transversal (G05).

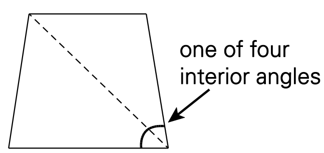
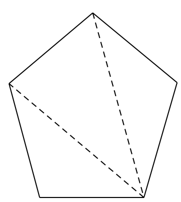
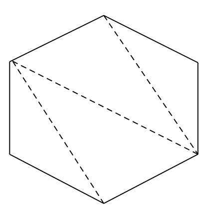
Students will explore and be able to describe and illustrate the following quadrilateral properties.

- The sum of the angles in a quadrilateral is 360˚.
- The opposite angles of a parallelogram are congruent.
- Squares and rectangles have four right angles.
- The two angles opposite the equal sides of an isosceles trapezoid are congruent.
- Both pairs of opposite sides of a rectangle are parallel.
- Both pairs of opposite sides of a parallelogram are congruent.
- The diagonals of a rectangle are congruent.
- The diagonals of a square are congruent.
- The diagonals of an isosceles trapezoid are congruent.
- The diagonals of a parallelogram are not congruent. The longest diagonal will be opposite the largest angle.
- The diagonals of all quadrilaterals studied will intersect at the midpoint of the diagonals.
- The diagonals of a square will intersect to form right angles.
- The sum of the angles where the diagonals of a quadrilateral intersect will be 360˚.

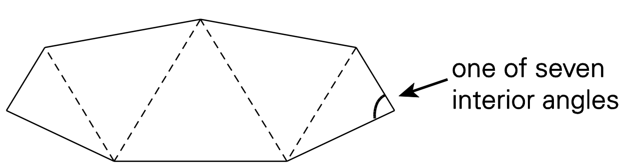
While students described and compared the sides and angles of regular and irregular polygons in Mathematics 6, they will only work with **regular polygons** here. **Regular polygons** have sides and angles that are equal. Equilateral triangles and squares are members of this group.

They will explore the sum of the interior angles of a regular polygon and determine the measure of each angle. As students explore regular polygons with increasing numbers of sides, they will discover that the sum of the **interior angles** of any polygon can be determined as 180° × the number of triangles formed by joining the vertices of the polygon. To discover this relationship, students can separate each polygon into triangles by drawing diagonals. For example, a quadrilateral can be divided into 2 triangles, so the sum of the **interior angles** is 2 × 180° = 360°.

A table such as the following could be used to help students with their investigation. They should note that, when dividing the polygon into triangles, each vertex of a triangle must be a vertex of the original polygon.

Number of sides	Diagram	Number of triangles formed	Sum of the angles
4		2	360˚
5		3	540˚
6		4	720˚

Since a seven-sided figure (heptagon) can be divided into five triangles, the sum of the **interior angles** will be 5 × 180° = 900°.



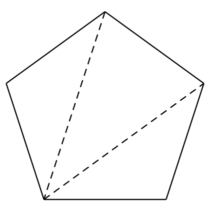
Heptagon

Through this investigation, students should learn that the sum of the angles increases by 180˚ as the number of sides increases by one. They should also observe that the number of triangles formed is always two less than the number of sides in the polygon. The formula for the sum of the measures of the interior angles of a polygon could then be used.

$S = 180(n - 2)$, where S represents the sum of the interior angles and n is the number of sides of the polygon.

The sum of the **interior angles** of a **regular polygon** can be determined by the method shown above. For regular polygons, each **interior angle** can also be determined by dividing the sum of the interior angles, by the number of sides or angles.

For example, a regular pentagon can be divided into three triangles so the sum of the interior angles will be 180°(5 – 2) = 540°, and the measure of each interior angle will be 540° ÷ 5 = 108°. This can also be shown by combining the statements $\frac{180°(5-2)}{5} = 108°$.



Regular Pentagon

If a polygon is regular, the formula that can be used to determine the measure of each interior angle is $M = \frac{180°(n-2)}{n}$ where M is the measure of the angle and n is the number of sides of the regular polygon.

When working with polygons, it is important for students to distinguish between regular polygons and irregular polygons. The formula $S = 180(n - 2)$, applies to all polygons, whereas the formula $M = \frac{180°(n-2)}{n}$ only applies to regular polygons.

Another property of some regular polygons is the ability to tessellate. A tessellation is created when a shape is repeated over and over again covering a plane without any gaps or overlaps.

Students should be given the opportunity to work with concrete materials to observe how the shapes either fit together, leave gaps open, or overlap each other. The focus here is on determining if a given polygon will tessellate. Students are not expected to work with polygon combinations or to create tessellations. Pattern blocks may be useful for this exploration. (Students should understand that equilateral triangles, squares, and regular hexagons will tessellate.

Students will also explore the number of diagonals in a regular polygon by drawing the diagonals from each vertex and counting the total number. It is not necessary to introduce students to the formula for determining the number of diagonals in a regular polygon.

Students will explore line symmetry in triangles, quadrilaterals, and regular polygons. A 2-D shape has **line symmetry** if one half of the shape is a reflection of the other half. The reflection occurs across a line. The **line of symmetry**, or **line of reflection**, can be horizontal, vertical, or oblique and may or may not be part of the diagram itself. To help explain line symmetry, students should view examples and non-examples.

Students will investigate the number of lines of symmetry that exist in various 2-D shapes, including triangles, quadrilaterals, and regular polygons. They should also explore symmetry in letters, pictures, logos, etc. In examining shapes to be classified based on the number of lines of symmetry, it is important to include shapes that are asymmetrical. Students can complete shapes and designs with line symmetry using square tiles or pattern blocks, folded paper, transparent mirrors, grid paper, or technology tools, such as a drawing program or dynamic geometry software.

Relating a line of symmetry to a line of reflection should enable students to complete a figure, describe the completed shape, and describe the reflection.

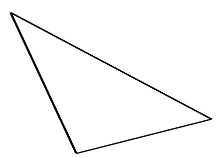
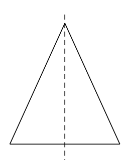
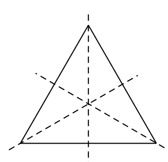
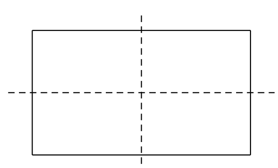
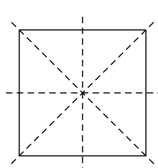
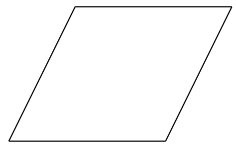
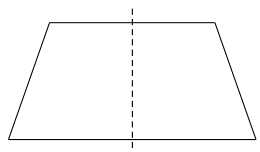
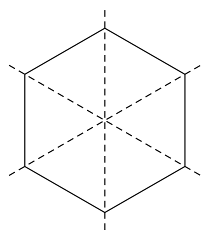
Figure Name	Diagram	Number of lines of symmetry
Scalene Triangle		0
Isosceles Triangle		1
Equilateral Triangle		3
Rectangle		2
Square		4

Figure Name	**Diagram**	**Number of lines of symmetry**
Parallelogram		0
Isosceles Trapezoid		1
Regular Hexagon		6

Students should conclude that the number of lines of symmetry in a regular polygon (which includes the equilateral triangle and the square) is equal to the number of vertices.

There are many real-life applications using the properties of polygons. Students should explore flooring tiles, cutting construction materials, quilting, squaring frames on buildings, and building design. They should explore different types of art such as paintings, jewellery, quilts, tiles, murals, and cultural artwork. Students could analyze pictures, logos, flags, signs, playing cards, and kaleidoscopes. Computer-assisted software could be useful when experimenting with properties of polygons in designs or photos.

In all cases, an effort should be made to contextualize problems using domestic, trades, and occupational examples. Students should answer questions such as,

- How do you know the frame is square?
- How can you locate the centre of a room?
- Will the plans for a roof truss for a house work?

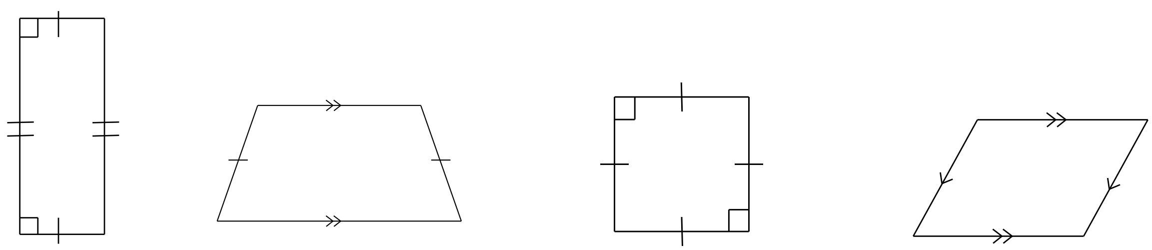
Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Name each of these polygons.



- Draw an angle with each of the following measures:

  (a) 90° (b) 55° (c) 180° (d) 125°
- Using geo-strips, construct quadrilaterals that meet each of the criteria given. Name each quadrilateral.

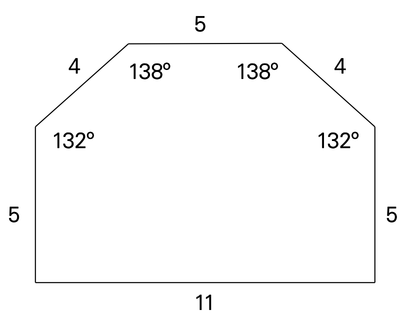
  (a) One or two sets of parallel sides

  (b) At least one right angle

  (c) Congruent sides

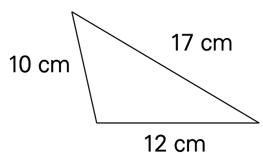
  (d) Parallel sides and right angles
- The diagram shown below represents the top view of a patio.

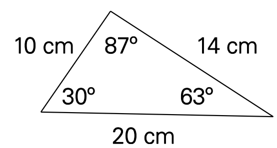
  Given the angle measures and side lengths (in centimetres), reproduce the drawing using a scale factor of 2.



WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

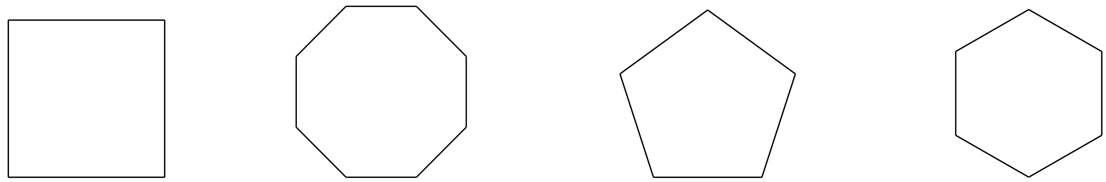
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Compare and contrast the following quadrilaterals:
  (a) A rectangle and a square
  (b) An isosceles trapezoid and a parallelogram
  (c) A rectangle and a parallelogram
  (d) A quadrilateral and an isosceles trapezoid
- Determine whether each of the following triangles is possible:
  (a)
  

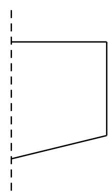
  (b)
  

- Sketch three triangles that have perimeters of 24 cm. Explain why these triangles are possible using triangle properties.
- What is the measure for each interior angle in a regular pentagon?
- What is the sum of the interior angles of a hexagon?
- If the sum of the interior angle of a regular polygon equals 1440°, how many sides does the polygon have?
- Explain why the sum of the interior angles of any polygon can be determined as 180° × the number of triangles formed by joining the vertices of the polygon.
- Explain why $M = \frac{180^\circ(n-2)}{n}$ cannot be used to determine the measure of the interior angle of a parallelogram.
- Determine the sum of the interior angles and the measure of each interior angle of the following polygons.
  (a) 7-sided polygon
  (b) 10-sided polygon
  (c) 12-sided polygon

- Determine if each of the following **regular polygons** will tessellate.

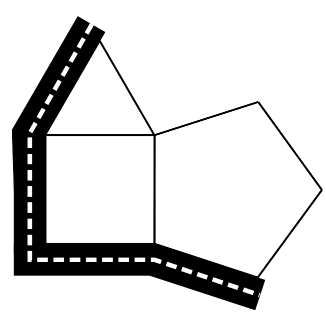


- The diagram below shows one half of a polygon. The dotted line is a line of symmetry. Draw and identify the type of polygon.



- Draw each of the following, if possible.
  (a) A quadrilateral with exactly one line of symmetry
  (b) A quadrilateral with exactly two lines of symmetry
  (c) A quadrilateral with exactly three lines of symmetry
  (d) A quadrilateral with exactly four lines of symmetry
- Do rectangles and parallelograms have the same number of lines of symmetry? Explain why or why not.
- Explain why a rectangle has two lines of symmetry, while a square has four lines of symmetry.
- A local contractor was hired to build a shed 6 m wide by 8 m long. The four walls of the shed did not look even (plumb) to the homeowner. How could the homeowner check that the walls were "square"?
- **Enrichment Question:** A surveyor is laying out a new subdivision, divided into four lots. Three of the lots are shown in the diagram.

  (a) If a fourth lot is added that borders the pentagon and the triangle, what will be the angle of the new lot at the corner shared with the other three lots?
  (b) What are possible shapes of the fourth lot?



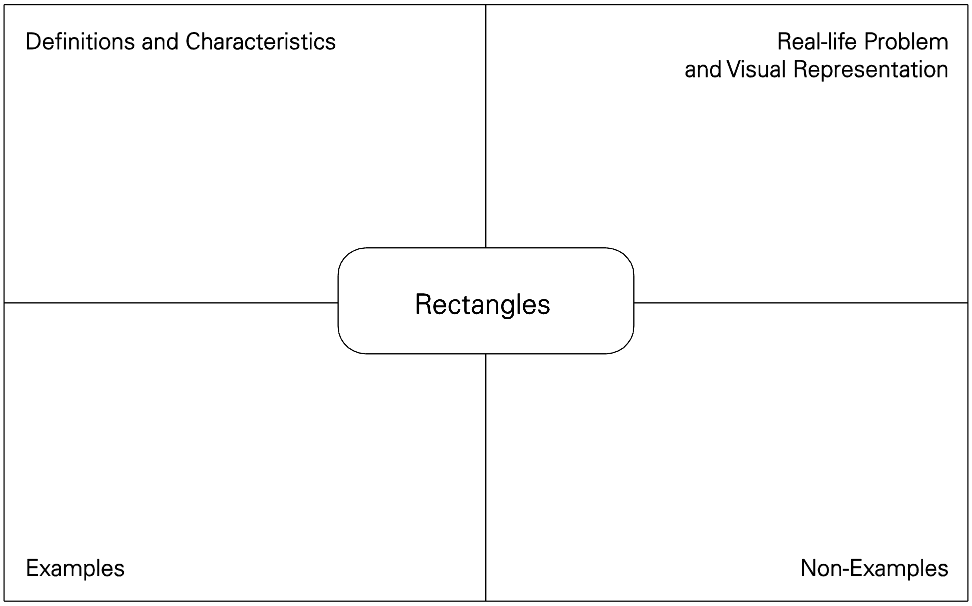
Planning for Instruction

SUGGESTED LEARNING TASKS

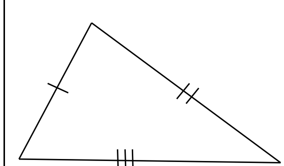
Consider the following sample instructional strategies when planning lessons.

- Since students have not had any recent exposure to 2-D geometry, review the definitions and characteristics of basic geometric shapes, as well as skills such as naming line segments and angles.
- For those students who struggle with remembering the names for each triangle, a strategy such as the following may help. Put the three triangle names in alphabetical order. Then apply the 3-2-1 rule: the first triangle (equilateral) has three equal sides, the second (isosceles) has two equal sides, and the third (scalene) has one, or no equal sides.
- Students could go on a “math walk” throughout the building or neighbourhood to take photographs of polygons. They should write a description of the polygons in each photo, including their properties and their purpose. Ask them to consider if the polygons are functional, visually appealing, or both.
- Students should have the opportunity to construct their own understanding of the various properties through explorations, both manually and through the use of computer software such as The Geometer’s Sketchpad (Key Curriculum Press 2015) and GeoGebra (International GeoGebra Institute 2015). Once exploration has taken place, there are several websites that summarize these properties and allow students to confirm their conjectures. Some examples are as follows:
  - “Geometry and Trig Reference Area,” *CoolMath.com:* www.coolmath.com/reference/geometry-trigonometry-reference.html
  - “Triangles Side and Angles: Interior and Exterior Angles and Sides,” *Math Warehouse:* www.mathwarehouse.com/geometry/triangles
  - “Interactive Quadrilaterals,” Mat*hIsFun.com:* www.mathsisfun.com/geometry/quadrilaterals-interactive.html.
- Have students work together to create a word wall for the unit. Prepare strips of card stock or recipe cards for the word wall items. Students could first independently skim and scan sections of the text for unfamiliar words and symbols and record their words one per card. They should then work in small groups to compare cards and compile a master collection to be posted in the classroom. Alternatively, this could be used as a unit project where terms are added to the wall when introduced.
- Present students with triangles that are labelled with side lengths. Ask them to explain if the triangles are possible.

- Provide students with a template of the Frayer model and ask them to complete the sections individually or as a group to consolidate their understanding of rectangles. This same activity can be repeated for any figure.



- Use materials such as a mirror, mira, or tracing paper to help show the concept of symmetry.
- In small groups and given a variety of regular polygons made of different colours of card stock, ask students to cut them into triangles using the vertices as end points. Once the number of triangles “contained” in each is discovered, ask students to determine the sum of the interior angles. Students may reassemble and mount the exploded polygon on paper above their indication of the angle measure.
- Present students with several regular polygons and ask them to explain which polygon will have the greatest number of diagonals.
- In groups of two, one student uses rubber bands to create half of a polygon on a geo-board. Using another rubber band, the second student will complete the other half of the polygon. Students should discuss if the polygon is regular and if there is another type of polygon that can be formed.
- Students could play “I Have ... Who Has” for properties of triangles, quadrilaterals, and regular polygons. Provide them with a loop card as shown below. Choose a student to start and read the “Who has ...” part of the card aloud. A student will respond “I have ...” answering with the correct polygon described by the properties. The student continues to read, “Who has ...” This will continue until all of the class has read their cards. This could be scaffolded by having pairs of students holding a card or cards for the activity.

I have … a scalene triangle 
Who has … a four-sided figure with only one pair of parallel sides?

- Students could fold paper cut-out versions of polygons to discover lines of symmetry. They should trace each successful fold with a different colour to determine the number of lines of symmetry. Another approach is to have students fold a sheet of paper in half and cut out a shape of their choosing. When they open the paper, the fold line will be a line of symmetry. Alternatively, miras could be used. If the shape is symmetrical where the mira has been placed, the image of one side of the shape will fall right on top of the other side of the shape.

Suggested Models and Manipulatives

- compass
- geo-boards
- grid paper
- miras
- mirrors
- pattern blocks
- protractors
- rulers and measuring tapes
- string
- tracing paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- acute angle
- diagonal
- equilateral triangle
- hexagon
- isosceles trapezoid
- isosceles triangle
- line of symmetry
- line symmetry
- obtuse angle
- octagon
- parallelogram
- pentagon
- quadrilateral
- rectangle
- regular polygon
- scalene triangle
- square
- tessellate

Resources/Notes

Digital

- "Geometry and Trig Reference Area," *CoolMath.com* (CoolMath.com LLC 2015): www.coolmath.com/reference/geometry-trigonometry-reference.html
- "Interactive Quadrilaterals," *MathIsFun.com* (MathIsFun.com 2015): www.mathsisfun.com/geometry/quadrilaterals-interactive.html
- "Triangles Side and Angles: Interior and Exterior Angles and Sides," *Math Warehouse* (Math Warehouse 2015): www.mathwarehouse.com/geometry/triangles
- GeoGebra (International GeoGebra Institute 2015): www.geogebra.org
- The Geometer's Sketchpad (Key Curriculum 2015; NSSBB #: 50474, 50475, 51453)

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 5: Properties of Figures
    - > Sections 5.1, 5.2, and 5.3
    - > Skill Check
    - > Test Yourself
    - > Chapter Project
    - > Games and Puzzles

SCO G03 Students will be expected to demonstrate an understanding of transformations on a 2-D shape or a 3-D object, including translations, rotations, reflections, and dilations. [C, CN, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

G03.01 Identify a single transformation that was performed, given the original 2-D shape or 3-D object and its image.
G03.02 Draw the image of a 2-D shape that results from a given single transformation.
G03.03 Draw the image of a 2-D shape that results from a given combination of successive transformations.
G03.04 Create, analyze, and describe designs, using translations, rotations, and reflections in all four quadrants of a coordinate grid.
G03.05 Identify and describe applications of transformations in construction, industrial, commercial, domestic, and artistic contexts.
G03.06 Explain the relationship between reflections and lines or planes of symmetry.
G03.07 Determine and explain whether a given image is a dilation of another given shape, using the concept of similarity.
G03.08 Draw, with or without technology, a dilation image for a given 2-D shape or 3-D object, and explain how the original 2-D shape or 3-D object and its image are proportional.
G03.09 Solve a contextual problem that involves transformations.

Scope and Sequence

Mathematics at Work 11	Mathematics at Work 12
G02 Students will be expected to solve problems that involve scale. **G03** Students will be expected to model and draw 3-D objects and their views. **G04** Students will be expected to draw and describe exploded views, component parts, and scale diagrams of simple 3-D objects.	**G03** Students will be expected to demonstrate an understanding of transformations on a 2-D shape or a 3-D object, including translations, rotations, reflections, and dilations.

Background

Students will explore transformations of two-dimensional shapes and three-dimensional objects. They will discover that a **transformation** is a process whereby a set of points change. The change can involve location, size, or both. They will first work with transformations independently and then progress to a combination of transformations in the Cartesian plane as well as other mediums. An emphasis should be the use of the formal language of transformations, such as **dilation**, **translation**, **reflection**, and **rotation**, instead of *stretches*, *slides*, *flips,* and *turns*.

In previous grades, students have performed and identified transformations of 2-D shapes and 3-D objects. In Mathematics 9 they worked with reductions and enlargements of 2-D shapes and properties of similar polygons. Students were introduced to the concept of similarity in Mathematics at Work 10 (G02). In Mathematics at Work 11, students drew reduction and enlargements of shapes given a scale factor and worked with orthographic and isometric drawings of 3-D objects (G02 and G03).

Students will be introduced to the notation used when applying transformations. The image is the final shape and/or position of the pre-image after transformation(s). The image of point A, for example, after a transformation of any type is labelled point A'.

When describing transformations, students should be able to recognize a given transformation as a dilation, a reflection, a translation, a rotation, or some combination of these. Students will work through a variety of examples where they follow a set of instructions involving successive transformations (for example, a translation followed by a reflection). It is important for them to identify the order in which these transformations occur. A different order of the same transformations may lead to a different result.

When given a shape or object and its image, students should be able to describe

- a dilation, by using the given scale factor
- a translation, using words and notation describing the translation
- a reflection, by determining the location of the line of reflection
- a rotation, using degree or fraction-of-turn measures, both clockwise and counter-clockwise, and identify the location of the centre of a rotation (A **centre of rotation** may be located on the shape, such as a vertex of the original image, or off the shape.)

When investigating properties of transformations, consider the concept of congruency that was developed informally in previous grades. In discussing the properties of transformations, students should consider if the transformation of the image

- has side lengths and angle measures the same as the given image
- is similar to or congruent to the given image
- has the same orientation as the given image
- appears to have remained stationary with respect to the given image

The criteria required for similarity in shapes (i.e., corresponding angles are congruent and corresponding sides are proportional) should be reviewed. Introduce students to the concept of dilation as a transformation in which an object is enlarged or reduced by a constant factor. **Dilations** are transformations in which a polygon is enlarged or reduced by a scale factor around a given centre point. This will be their first exposure to the term **dilation**.

Providing students with an opportunity to explore real-world examples of scale diagrams, they should discuss the concept of a scale factor, be able to determine the scale factor, and use it to create enlargements and reductions. Students will also draw dilations of 2-D shapes and 3-D objects, with or without technology. They could use isometric dot paper when drawing their dilations of objects.

Students should be aware of the effect of the magnitude of a scale factor. They should be asking themselves questions such as, What happens when a scale factor is greater than one? Less than one? A common student error occurs when students interchange the numerator and denominator while calculating scale factor. Understanding that for an enlargement the scale factor is greater than one, and for a reduction the scale factor is less than one, should help students avoid making that mistake.

Like units are necessary when finding a scale factor. Students could be asked to find the actual size, using the scale, or to convert the scale provided in one form to a different form.

Example:

- If the scale is given as the ratio 1:50, how many metres does 7.5 cm represent?

To solve this problem, students should understand that they could multiply 50 by 7.5 cm or they may set up the proportion: $\frac{1}{50}=\frac{7.5}{x}$. They should initially determine that x = 375. Since 7.5 is in centimetres, then 375 is also in centimetres, and when converted equals 3.75 m.

Unlike dilations, transformations such as translations, reflections, and rotations do not change the size of the figure. Introduce students to the concept of a translation as a transformation that slides an image in a straight line without changing its size or orientation.

Students were introduced to the concepts of plotting points on a Cartesian plane in Mathematics 7. Most translations that students will be working with will be on a Cartesian plane but should not be limited to this medium. For example, quilts, tiles, and stone pathways are also possible mediums.

Students will be able to
- perform horizontal, vertical, or successive translations, given an image
- draw an image given the coordinates and perform horizontal, vertical, or successive translations
- identify the translations that have occurred given a 2-D shape or a 3-D object and their images

The initial focus should be on shapes such as rectangles and triangles before more complex shapes are introduced.

Students should recognize situations whereby translations are used to solve problems in everyday life.

Example:

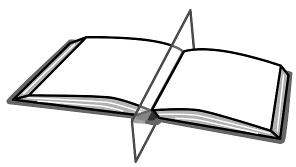
- Rearranging furniture in a given room.
- Playing games, such as Battleship, checkers, chess, and Tetris.
- Creating ceramic tile patterns.
- Creating 3-D drawing.

Providing students with an opportunity to explore transformations in the context of everyday experiences is important. Examples could include the following:
- *Construction:* signs, high-rise buildings, woodworking, tiles
- *Industrial:* blue prints, bricklaying
- *Commercial:* jewellery, logos
- *Domestic:* quilts, playing cards

- *Artistic:* paintings, murals, kaleidoscopes, synchronized sports

In Mathematics 9, students were introduced to lines of symmetry. A figure has **line symmetry** when there is a line that divides it into two reflected parts. Shapes can have no line of symmetry or multiple lines of symmetry. These lines can exist in any orientation (vertical, horizontal, slanted). Computer-assisted software can be useful when experimenting with symmetry of designs or photos.

A **reflection** is a transformation in which the object is shown as its mirror image in a line of reflection. Corresponding points on both sides of that line are the same distance away from the line. The line of symmetry, or line of reflection, can be horizontal, vertical, or oblique, and may or may not be part of the diagram itself. Students should be able to make the distinction that 2-D shapes have an axis of symmetry, while 3-D objects have a plane of symmetry. The figure below illustrates a 3-D plane of symmetry.



When working with reflections, students will complete the following:
- Given a line of reflection, produce a reflection of an image.
- Given a line of reflection, produce a reflection image given the coordinates of an object.
- Given a 2-D shape or a 3-D object, identify the reflection(s) that have occurred.
- Given a 2-D shape or a 3-D object, identify the transformations that have occurred.

Provide students with the opportunity to recognize where reflections are used in everyday life. Examples could include
- the reflection of the word AMBULANCE written on the front of an ambulance
- lines of reflection in different types of materials, such as signs, tile patterns, patio stones, jewellery, crystals, quilts, playing cards, paintings, murals, kaleidoscopes
- symmetrical patterns in clothing or woodworking projects

In Mathematics 9, students were introduced to **rotation symmetry**. A figure has rotational symmetry if it can be turned about its centre so that it fits in its original outline. The **angle of rotation** is the minimum angle required to turn a figure onto itself.

A **rotation** is a transformation that moves an object around a fixed point, called the **centre of rotation**. The majority of the rotations that students will be working with are on a Cartesian plane, but examples should not be limited to this medium. Students could, for example, use tracing paper, computer software, or dot paper.

Students will initially work with rotations of 90˚, 180˚, and 270˚ both clockwise and counter-clockwise. Explore various methods for completing rotations such as rotational rule or compass and protractor. When working with rotations, students will complete the following tasks:
- Perform a specified rotation or a combination of successive transformations given any 2-D shape.
- Draw an image, given the coordinates, and perform a specified rotation or a combination of successive transformations.

- Identify the transformation(s) that have occurred given a 2-D shape or a 3-D object and their images.

Many designs use rotational properties. Examples include the following:
- Ferris wheels
- fans
- ceramic tiles
- quilting patterns
- creating a 3-D drawing
- logos

By looking at design samples, students should be able to identify the centre of rotation as well as the rotational angle.

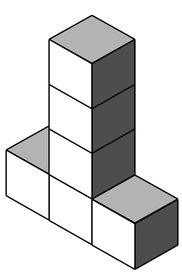
Assessment, Teaching, and Learning

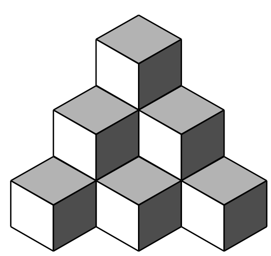
Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

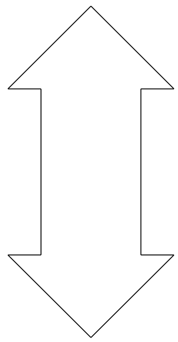
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Draw a coordinate grid with an x-axis and a y-axis that range from –10 to 10. Label the origin and the Quadrants I to IV. Plot the following points: (–5, 3), (3, –5), (4, 9), (0, 7), (–4, 0).
- Draw each object on isometric dot paper.

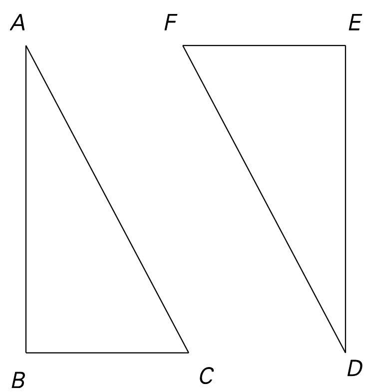




- Enlarge the size of the shape by a scale factor of three.



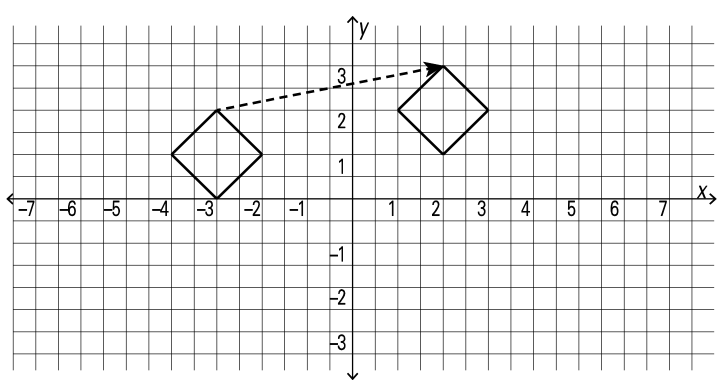
- Given that $\triangle ABC \cong \triangle DEF$, list the corresponding angles and corresponding sides for $\triangle ABC$ and $\triangle DEF$.



WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

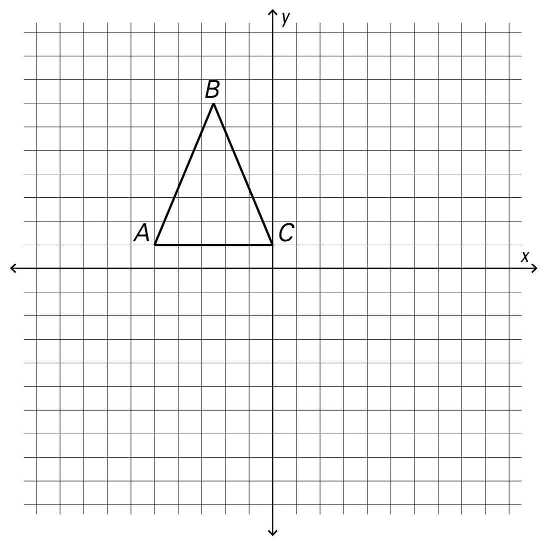
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Describe the translation shown in the diagram.

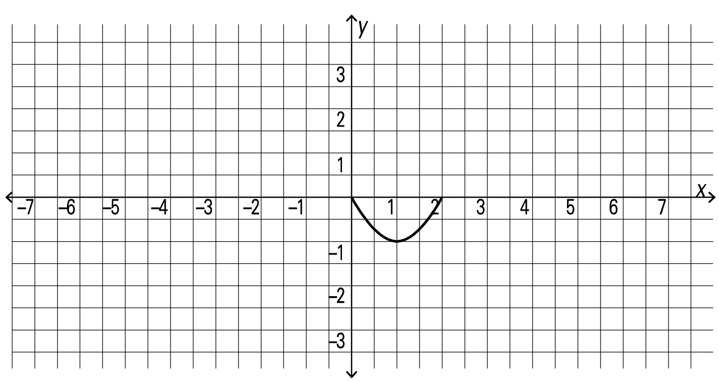


- Point A is at (1, 2) on a coordinate grid. What are the coordinates of point A after it is translated
  - (a) four units left
  - (b) six units up
  - (c) seven units right and two units down
  - (d) three units left and nine units down
- Use the diagram to complete the translations indicated below:

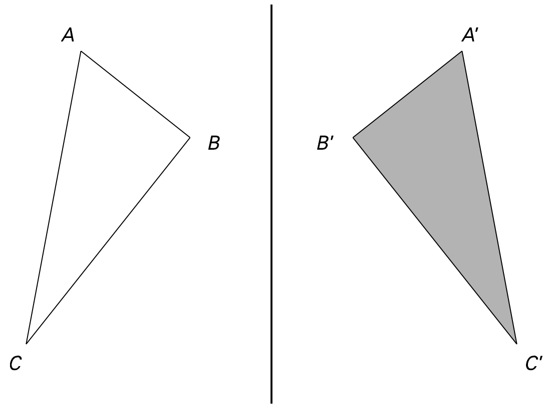
  Translate $\triangle ABC$
  - (a) three units left and two units up
  - (b) four units right and one unit down



- Plot the point (1, 1) on a coordinate grid. If this point is translated two units left and four units up three successive times, what will be the new coordinates of the point?
- Reflect the given shape in the line of reflection at $x = -1$.



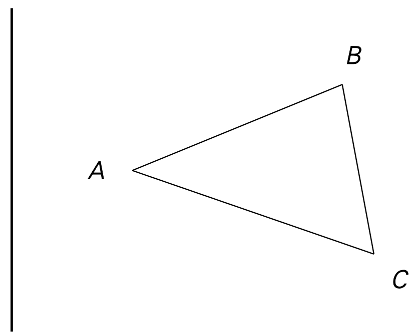
- Use the diagram to answer the questions that follow:



(a) Identify the type of transformation.

(b) What is the relationship between ΔABC and $\Delta A'B'C'$?

- Construct a reflection of the following image:



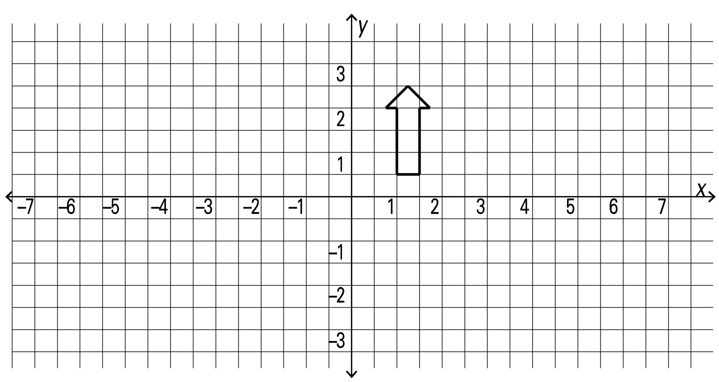
- Rotate each of the following points according to the instructions.

(a) (4, 6); 90° counter-clockwise about the origin

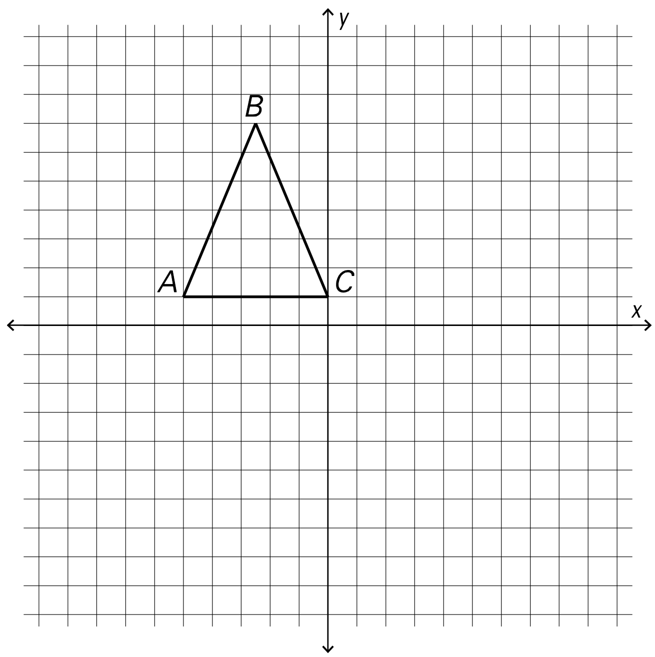
(b) (3, –1); 180° about the origin

(c) (–1, –3); 90° clockwise about the centre of rotation (0, 1)

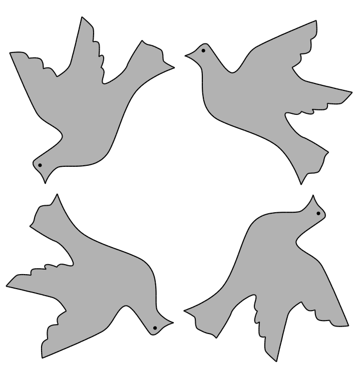
- Rotate the given shape 90° clockwise about the origin.



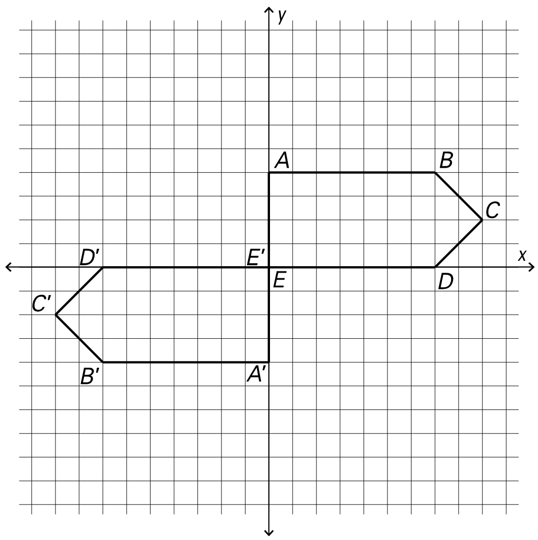
- On a coordinate grid, transform each point as indicated.
  (a) (3, 1); rotate 90° counter-clockwise about the origin, and reflect over the *x*-axis
  (b) (2, –1); rotate 180° about the origin, reflect in the line $x = 1$, and translate two units down.
- Transform the shape as instructed using a method of your choice.



  (a) Rotate 180˚ about (0, 0)
  (b) Rotate 90˚ clockwise about (0, 1)
  (c) Rotate 45˚ counter clockwise about the origin
  (d) Rotate 270˚ counter clockwise about (1, 3) and translate three units down.
- Identify the center of rotation and the rotational angle in a design such as the following. How can you check the accuracy of their location?

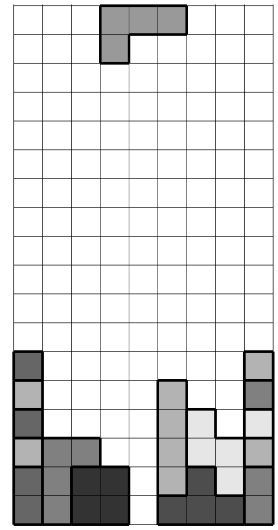


- Analyze the following diagram and identify the transformations that occurred. (There is more than one answer.)

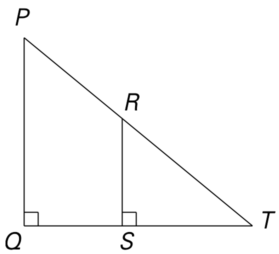


- Plot these points on a coordinate grid: *A*(–1, 3), *B*(2, 4), *C*(3, –2), *D*(–4, –5). Join the points to draw polygon *ABCD*.
  - (a) Translate the polygon five units right and three units up. Write the coordinates of each vertex of the image polygon $L'M'N'O'$
  - (b) Reflect the image polygon $A'B'C'D'$ in the *x*-axis. Write the coordinates of each vertex of the image *polygon* $A"B"C"D"$.
- Plot these points on a coordinate grid: *L*(–5, 8), *M*(0, 8), *N*(0, 5), *O*(–5, 5). Join the points to draw the polygon *LMNO*.
  - (a) Describe the polygon you have sketched.
  - (b) Dilate the polygon *LMNO* using a scale factor of 2.5 and centre point of (0, 0).
  - (c) Write the coordinates of the image polygon $L'M'N'O'$.
  - (d) Rotate the image polygon $L'M'N'O'$ 90° clockwise around the centre point and label the resulting coordinates.
- Why do you think it is necessary to build a smaller model of a boat before building the actual boat?
- Design a logo that includes geometric shapes. Consider the following points to help with your design:
  - (a) Decide on the dimensions of an enlargement of the logo that would fit on a banner or billboard.
  - (b) Determine the scale factor.
  - (c) Create a business card using the logo by repeating the process for a reduction.

- The computer game Tetris is a tiling game that uses seven tiles, each composed of four squares. The playing screen is a 9 × 20 grid of squares. During the game, tiles fall from the top of the screen and must be translated or rotated to fill complete rows at the bottom of the screen. Each time a row is filled, it disappears. Given the following image, describe the transformations that must occur to fill three complete rows.



- Answer the questions below based on the diagram shown.



  (a) Which triangles are similar?

  (b) Measure the sides and determine the ratios of

    (i) $\frac{PQ}{QT}$ and $\frac{RS}{ST}$

    (ii) $\frac{PQ}{PT}$ and $\frac{RS}{RT}$

    (iii) $\frac{QT}{PT}$ and $\frac{ST}{RT}$

  (c) Is *DTPQ* a dilation of *DTRS*? Explain your reasoning.

- Find four flags that demonstrate line symmetry.

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Provide students with a multitude of 2-D and 3-D shapes and have them perform transformations prior to identifying transformations on a given object.
- Look at selected work by M. C. Escher and have students identify the transformations used in his artwork. (www.mcescher.com/Gallery/gallery-symmetry.htm)
- Create a grid on the floor with masking tape or use a 100-square mat (Learning Carpet). Use rope or coloured tape to place the axes. One student chooses a spot. Another student directs the first student to translate the position. This activity could progress to more than three students on the grid holding ribbon to form a polygon. A different student directs them (as vertices) to “walk through” various transformations.
- Ask students to create a design involving translations, reflections, rotations, and/or dilation. Have them exchange with a classmate who will then describe the transformations.

- Students could use computer-assisted software, such as Google Sketchup, Microsoft Publisher, or Microsoft Paint, or cell phone tools or apps to experiment with transformations of designs or photos.
- Students could find a picture they would like to enlarge or reduce. They can then dilate photos by either using grid paper or using technology.
- Each student could be given a section of a photo, sketch, map, etc., to enlarge to create a class mural.
- Provide students with a series of shapes on a Cartesian plane and ask them to perform dilations, using scale factors greater than one and less than one. They should explain how the shapes and images are proportional.
- Have students visit an interactive website relating to transformations, such as Math Open Reference (www.mathopenref.com/dilate.html). Ask the following questions to promote discussion.
  - What happens to the image when the scale factor is 2, 1, and 0.5?
  - What will happen to the image when the shape is changed?
  - Predict what will happen to the image if you change one point to create a triangle.
- Ask students to create a design using translations and then describe how the translations were used in their design.
- Block-sliding puzzles are available as an app for mobile devices. As students play this game, ask them how it relates to the transformations they have studied in this unit. They should try to move to the exit in as few translations as possible. Puzzles include Move-It!, Unblock Me, Sliding Block Puzzle, Shift It, Slide It, and Cohesian.
- Transformations provide a good opportunity to introduce a cross-curricular project or activity. This might include another subject area, such as art or technology education.
- Have students describe how rotations are used to create the design. Software applications such as Photoshop, Microsoft Publisher, AutoCAD (AutoDesk Inc. 2015), Sketchup (Google 2015), and cell phone tools have built-in rotation capabilities. Students can use a digital image to explore the rotational tools of the software. They should then investigate the role of transformations in the creation of designs. Students could also create their own design using a combination of successive transformations.

Suggested Models and Manipulatives

- dot paper
- grid paper
- isometric dot paper
- miras
- protractors
- rulers and measuring tapes
- string
- tracing paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- centre of rotation
- dilation
- image
- line of reflection
- reflection
- rotation
- scale factor
- successive transformations
- transformation
- translation

Resources/Notes

Digital

- "Dilation of a Polygon," Math Open Reference (Math Open Reference 2009): http://mathopenref.com/dilate.html
- "Picture Gallery 'Symmetry,'" The M.C. Escher Company BV (M.C. Escher 2015): www.mcescher.com/Gallery/gallery-symmetry.htm
- AutoCAD (Autodesk 2015): www.autodesk.com/education/free-software/autocad
- Sketchup (Google 2015): http://sketchup.com
- The Geometer's Sketchpad (Key Curriculum 2015; NSSBB #: 50474, 50475, 51453)

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 6: Transformations
    - Sections 6.1, 6.2, 6.3, and 6.4
    - Skill Check
    - Test Yourself
    - Chapter Project
    - Games and Puzzles

Number
23–25 hours

GCO: Students will be expected to develop number sense and critical-thinking skills.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO N01 Students will be expected to analyze puzzles and games that involve logical reasoning, using problem-solving strategies. [C, CN, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

(It is intended that this outcome be integrated throughout the course by using puzzles and games such as Sudoku, Mastermind, Nim, and logic puzzles.)

N01.01 Determine, explain, and verify a strategy to solve a puzzle or to win a game; for example,

- guess and check
- look for a pattern
- make a systematic list
- draw or model
- eliminate possibilities
- simplify the original problem
- work backwards
- develop alternative approaches

N01.02 Identify and correct errors in a solution to a puzzle or in a strategy for winning a game.

N01.03 Create a variation on a puzzle or a game, and describe a strategy for solving the puzzle or winning the game.

Scope and Sequence

Mathematics at Work 11	Mathematics at Work 12
N01 Students will be expected to analyze puzzles and games that involve numerical reasoning, using problem-solving strategies.	**N01** Students will be expected to analyze puzzles and games that involve logical reasoning, using problem-solving strategies.

Background

It is intended that this outcome not be taught in isolation but that it be integrated throughout the course. Timing and integration of this outcome should be included in planning. Students can be exposed to three or four games at different times, whether it be at the beginning or end of each unit, or a set "game day." Students could engage in a game when they are finished other work. As students work through the different games and puzzles, they will begin to develop effective strategies for solving the puzzle or game.

Students have experience with discussing strategies used to solve a puzzle or win a game. In Mathematics at Work 10, students focused on playing and analyzing games and puzzles that involve spatial reasoning. Mathematics at Work 11 focused on students' numerical reasoning to solve puzzles and play games. In Mathematics at Work 12, the focus is on playing and analyzing games and puzzles that involve logical reasoning.

It is not enough for students to only do the puzzle or play the game. They must be given a variety of opportunities to analyze the puzzles they solve and the games they play. The goal is to develop their problem-solving abilities by using a variety of strategies and to be able to apply these skills to other contexts in mathematics. Students also need the opportunity to create a variation of a puzzle or a game. They may have ideas for games or puzzles that would challenge their classmates.

Students need time to play and enjoy each game before analysis begins. They can then discuss the game, determine the winning strategies, and explain these strategies through demonstration, oral explanation, or in writing.

Games also provide opportunities for building self-concept, enhancing reasoning and decision making, and developing positive attitudes towards mathematics through reducing the fear of failure and error. In comparison to more formal activities, greater learning can occur through games due to increased interactions among students, opportunities to explore intuitive ideas, and problem-solving strategies.

Students' thinking often becomes apparent through the actions and decisions they make during a game, so teachers have the opportunity to formatively assess learning. The emphasis of this outcome should not be on the successful completion of any puzzle or game but rather on analyzing logical reasoning and problem-solving strategies.

Problem-solving strategies will vary depending on the puzzle or game. Some students will use strategies such as working backwards, looking for a pattern, using guess and check, or by eliminating possibilities, while others will plot their moves by trying to anticipate their opponents' moves. As students play games and analyze strategies, they explore mathematical ideas and compare different strategies for efficiency.

A variety of puzzles and games, such as board games, online puzzles and games, and pencil-and-paper games should be used. It would be beneficial for teachers to share games and puzzles that promote the intent of this outcome. It is best to always try games in advance as instructions to games are not always clear.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Recall a game that you have played that requires strategies involving numerical reasoning.
- Recall a game that you have played that requires strategies involving spatial reasoning.
- Play a game of tic-tac-toe. What strategy do you use?

- Create a variation of a puzzle or a game you have played, and describe a strategy for solving the puzzle or winning the game.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Zookeeper George was in charge of feeding all of the animals in the morning. He had a regular schedule that he followed every day. Can you figure it out from the clues?
  (a) The giraffes were fed before the zebras but after the monkeys.
  (b) The bears were fed 15 minutes after the monkeys.
  (c) The lions were fed after the zebras.

	6:30 a.m.	6:45 a.m.	7:00 a.m.	7:15 a.m.	7:30 a.m.
Bears					
Giraffes					
Lions					
Monkeys					
Zebras					

Answers: 6:30 a.m. monkeys; 6:45 a.m. bears; 7:00 a.m. giraffes; 7:15 a.m. zebra; 7:30 a.m. lions

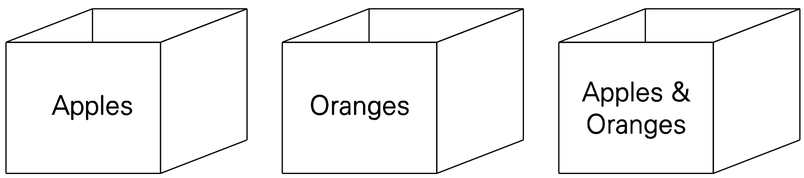
(Source: PuzzlesParadise, “Feeding Time ~ a logic problem,” www.puzzlersparadise.com/puzzles/feedingtime.html)

- Amanda, Kesia, Alex, and Sarah each have different coloured cars. One car is red, one is blue, one is white, and the other is black. Who owns which car?
  (a) Clue 1: Amanda’s car is not red or white.
  (b) Clue 2: Kesia’s car is not blue or white.
  (c) Clue 3: Alex’s car is not black or blue.
  (d) Clue 4: Sarah’s car is red.

  An organized list similar to the following might be helpful in solving the puzzle. Place an X in a box to eliminate a possibility and a √ in a box when the answer is known.

	Amanda	Kesia	Alex	Sarah
red				
blue				
white				
black				

- A man wishes to cross the river with a wolf, a goat, and some hay. He has a small boat, but unfortunately, he can only take one thing across at a time. What is worse, if he leaves the wolf and the goat alone together, the wolf will eat the goat, and if he leaves the goat with the hay, the goat will eat the hay. How does he do it?
- You have three boxes of fruit. One contains just apples, one contains just oranges, and one contains both apples and oranges. Each box is labelled as shown:



However, it is known that none of the boxes are correctly labelled. How can you label the boxes correctly if you are only allowed to take and look at just one piece of fruit from one of the boxes?

- Four friends met on Saturday morning for breakfast. Each friend ordered a different drink and breakfast meal during their visit and, when it was time to leave, each got a different drink to go.

  In your group, using the organized list below (or one similar), complete the table to determine the first name of the friend, the drink and meal each ordered for breakfast, and the drink each ordered to go.

  Clues:

  (a) Brenda had waffles but not an espresso.

  (b) The friend who ordered the pancakes also ordered decaf coffee to go, but did not have cranberry juice.

  (c) The woman who ordered the omelet had water to drink, but she was not Amy.

  (d) The two friends who ordered juice were Emily and the friend who ordered an egg sandwich.

  (e) The friend who had a cappuccino did not order orange juice.

  (f) Melony ordered a hot tea to go.

	cranberry juice	milk	orange juice	water	egg sandwich	omelet	pancakes	waffles	cappuccino	decaf coffee	espresso	hot tea
Amy												
Brenda												
Emily												
Melony												
cappuccino												
decaf coffee												
espresso												
hot tea												
egg sandwich												
omelet												
pancakes												
waffles												

Note: As a differentiated task, provide groups of students with one of the friend's solution. There are many logical reasoning puzzle books available that include similar but simpler or more complex puzzles, such as this example. Work through one example with the class before asking them to complete one on their own.

- Your friend is having trouble starting the sudoku puzzle to the right. Find three of the missing numbers, and explain the strategy you used to help him get started.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

- Sanjai placed the circled numbers in the sudoku below. Identify and correct his errors.

4		1	2	9			7	5
2			3			8		
	7			8				6
			1		3		6	2
1		5				4		3
7	3	7	6		8			
6	4			2			3	
		7			1			4
8	9			6	5	1		7

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Use puzzles or games as warm-up activities throughout the term.
- Have students create their own brainteaser to share with the class.
- There may be situations where students are able to play the game and solve problems but are unable to determine a winning strategy. By participating with the group and thinking through the strategies out loud, the group can hear the reasoning for selected moves. Ask the groups' opinions about moves in the game and facilitate discussions around each of the other players' moves and strategies.
- The following are some tips for using games in the mathematics class:
  - Use games for specific purposes.
  - Keep the number of players in each group to two to four, so that turns come around quickly.
  - Communicate to students the purpose of the game.
  - Engage students in post-game discussions.
  - When students feel that they have obtained a strategy for the solution of a puzzle or game, ask them to share that strategy with the class.
- As students work through the games and puzzles, keep a checklist of the games and puzzles each of them is working on.
- There are many websites that include logical puzzles and games. Check out the sites prior to giving the links to students to ensure that the links are live and appropriate for the level of difficulty and the outcome being addressed.
- Invite students to bring in their own games and puzzles involving logical reasoning. This may involve bringing in board games from home or searching the Internet to find a game or puzzle that interests them. Students can introduce this game/puzzle by providing information such as the following:
  - What is the puzzle or game, and where did you find it?
  - Describe the puzzle or game. Why did you select it?
  - Describe the objective of the game and the rules of play.
  - What strategy would you use to solve the puzzle or play the game?
- As students play games or solve puzzles, ask probing questions and listen to their responses. Record the different strategies and use these strategies to begin a class discussion. The following are some possible discussion starters.
  - Thumbs up if you liked the game, thumbs sideways if it was okay, and thumbs down if you did not like it. What did you like about it? Why?
  - What did you notice while playing the game?
  - Did you make any choices while playing?
  - Did anyone figure out a way to quickly find a solution?

  Note: If pressed for time, use exit cards and then debrief the following day.

- Students could work in groups where each member has been exposed to a different game or puzzle. Ask students to
  - explain the rules of the game in their own words to the other group members
  - give a brief demonstration of how the game is played
  - give advice, if necessary, to other students trying to solve the puzzle or play the game
- Logic puzzles are often based on statements that contain clues to the solution of the problem. These clues may be positive or negative (i.e., they may tell you part of the answer or tell you what the answer is not). Discuss with students that it is just as important to know what can be eliminated as it is to find the actual correct answer. To keep track of these possibilities (and non-possibilities), students could use organizers such as tables or lists.
- Puzzles and games stations are a good way to organize activities. Consider the following tips when creating the stations.
  - Some stations may require multiple games, while other stations may involve one game that requires more time to play.
  - Divide students into small groups. At regular intervals, have students rotate to the next station.
  - As students play a game, pose questions about the strategies they are using.
  - Engage students in a post-game discussion.
- While observing students working together to solve puzzles, focus on questions such as,
  - How did your group get started?
  - Were there any challenges?
  - Which clue did you find most helpful?
  - Were there any clues that did not help you solve the puzzle?
  - What might you do differently next time?
- Have students develop a game for classmates to play and/or use a known game. Students can then change a rule or parameter and explain how it affects the outcome of the game.
- Have students switch partners periodically to provide opportunities for new strategies to be shared.
- Students can work through puzzles or games individually or with a partner. They could record their progress in a table, such as the one shown below.

Puzzle	Solved?	Strategy	Comments/Hints

- In the game, High or Low?, students guess whether the next card is higher or lower than the last one dealt. They could also be asked to determine the exact probability that the next card will be higher or lower or the same suit.

  Divide the class into small groups, giving each one no more than 10 regular playing cards. Each group should have a different set of cards and probabilities to figure out. Students can use a graphic organizer, such as the following one, to keep track of the play.

Card	Probability *High*?	Probability *Low*?	Probability *Equal*?	Your Guess?	Correct Guess?

Once the activity is complete, use reflection questions such as the following:

- How many of your group's guesses were right? How do you think your group did overall?
- Was this game easier or more difficult than you thought before you started? Explain.
- How would the game change if you put each card back into the deck and shuffled every time before dealing a new card?

Students could complete an exit card for this activity.

How could we change this game to make it easier? More challenging?

- To create a game, students could use the rules of an existing game, but use different materials or add extra materials. They could also use the idea for a game and change the rules. Another option is to use a board game and add mathematics tasks to it. Rather than writing tasks directly onto the boards, students can place coloured stickers on certain spaces and make up colour-coded cards with questions. A game such as Snakes and Ladders, for example, can be modified to Operation Snakes and Ladders. The board can be used with two dice. On each turn, to determine the number of spaces to move, the player has the option of multiplying, dividing, adding, or subtracting the two numbers, with a maximum answer of twenty.

  The following guiding questions could be used to help students evaluate their games.

  - Can the game be completed in a short time?
  - Is there an element of chance built in?
  - Are there strategies that can be developed to improve the likelihood of winning?

Suggested Models and Manipulatives

- puzzles
- various board games and puzzles (A short list is provided below to illustrate the types of games and puzzles that are primarily spatial, numerical, or logical.)

Spatial (Mathematics at Work 10)	**Numerical (Mathematics at Work 11)**	**Logical (Mathematics at Work 12)**
Othello tangrams pentominoes Free Flow Blokus	KenKen Kakuro cribbage	sudoku chess checkers backgammon Mastermind Clue Game of Life

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- conjecture
- elimination
- logic
- logical reasoning
- strategy
- systematic list

Resources

Digital

There are numerous games and puzzles available on the Internet. What follows are just a few suggestions of spatial games and puzzles, available for free online. Many can also be acted out, done on paper, or done with models. (**Note:** It is expected that the teacher confirm the validity of the site prior to directing students to it.)

- "Feeding Time ~ a logic problem," *PuzzlersParadise* (Shelly Hazard 2005): www.puzzlersparadise.com/puzzles/feedingtime.html
- "Logic Games," *Coolmath-Games.*com (Coolmath.com 2015): www.coolmath-games.com/1-logic-games-01.html
- "Logic Mazes," *Robert Abbott* (Abbott 2011): www.logicmazes.com (Various types of logic mazes.)
- "Nim," *Jill Britton* (Britton 2015): http://britton.disted.camosun.bc.ca/nim.htm (Nim is an ancient game that can be played online, on paper, or using sticks. The winner is the player to not pick up the last stick [Common Nim] or to pick up the last stick [Straight Nim].)
- "Mancala Game," *Math Playground* (MathPlayground.com 2014): www.mathplayground.com/mancala.html (Mancala is a logic game that develops critical thinking skills.)

- *"The Nim Number Game," Jefferson Lab* (Jefferson Lab 2015): http://education.jlab.org/nim/index.html
- *MathIsFun.com* (MathIsFun.com 2014)
  - "Tic-Tac-Toe": www.mathsisfun.com/games/tic-tac-toe.html
  - "Dots and Boxes Game": www.mathsisfun.com/games/dots-and-boxes.html
  - "Four in a Line": www.mathsisfun.com/games/connect4.html
- Various other games that can be found online are as follows:
  - Blockus
  - Rushhour
  - chess
  - Blockers
  - Towers of Hanoi
  - board games
  - Free Flow

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 1: Measurement and Probability
    - > Sections 1.2 and 1.4
    - > Games and Puzzles
  - Chapter 2: Working with Data
    - > Sections 2.1 and 2.2
    - > Games and Puzzles
  - Chapter 3: Linear Relationships
    - > Section 3.3
    - > Games and Puzzles
  - Chapter 4: Real-Life Decisions
    - > Sections 4.1, 4.2, and 4.3
    - > Games and Puzzles
  - Chapter 5: Properties of Figures
    - > Section 5.2
    - > Games and Puzzles
  - Chapter 6: Transformations
    - > Section 6.4
    - > Games and Puzzles
  - Chapter 7: Trigonometry
    - > Section 7.2
    - > Games and Puzzles

SCO N02 Students will be expected to solve problems that involve the acquisition of a vehicle by buying, leasing, and leasing to buy. [C, CN, PS, R, T]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

N02.01 Describe and explain various options for buying, leasing, and leasing to buy a vehicle.
N02.02 Solve, with or without technology, problems that involve the purchase, lease, or lease to purchase of a vehicle.
N02.03 Justify a decision related to buying, leasing, or leasing to buy a vehicle, based on factors such as personal finances, intended use, maintenance, warranties, mileage, and insurance.

Scope and Sequence

Mathematics at Work 11	**Mathematics at Work 12**
N02 Students will be expected to solve problems that involve personal budgets. **N03** Students will be expected to demonstrate an understanding of compound interest.	**N02** Students will be expected to solve problems that involve the acquisition of a vehicle by buying, leasing, and leasing to buy.

Background

Many students have already made or soon will be making real-life financial decisions, including purchasing or leasing a vehicle. Given the variety of options available when acquiring a vehicle, it is important to consider many factors when making this decision.

When deciding whether to buy or lease a car, there are a number of things that must be considered. Some of these considerations are listed below.

Buying Option

- The greatest benefit of buying a car is that it may actually be owned one day. Implied in this benefit is that a person will one day be free of car payments. The car can be sold at any time, and is not locked into any type of fixed ownership period.
- When buying a car, the insurance premiums on the policy are typically lower than for leasing.
- By owning a car, there are no mileage restrictions that typically exist when leasing.
- On the downside, the monthly payment is usually higher on a purchased car than on a leased car. Additionally, dealers usually require a reasonable down payment, so the initial out-of-pocket cost is higher when buying a car.
- Presumably, as a car loan is paid down, there is an ability to build equity in the vehicle. Unfortunately, however, this is not always the case. When purchasing a car, the payments reflect

the whole cost of the car, usually amortized over a four- to six-year period. Depreciation can take a significant toll on the value of your car, especially in the first couple of years, so the car will lose much of its original value in a short amount of time.

- Carefully check out (or have a qualified mechanic check out) the condition of a used vehicle.

Leasing Option

- Perhaps the greatest benefit of leasing a car is the lower out-of-pocket costs when acquiring and maintaining the car. Leases require little or no down payment, and there are no upfront sales tax payments. Additionally, monthly payments are usually lower, and there is the potential to have a new car every few years.
- With a lease, the car is actually being rented for a fixed number of months (typically 36 to 48 months). Therefore, only the use (or depreciation), of the car is paid for during that period, and the full depreciation cost of the vehicle is not absorbed.
- By leasing a car, there will always be a car payment to make. As long as a car is leased, it is never really owned.
- Leasing also provides an alternative when buying a car is not an option, due to not having the required down payment or having a higher monthly payment.
- For business owners, leasing a car may offer tax advantages if the vehicle is used for business purposes.
- Depending on the type of lease, when the lease term is up, the keys are turned over to the car dealership and another vehicle can be leased, or the remaining value of the vehicle can be financed and the owner goes from making lease payments to loan payments, thereby leasing to own.
- Leased vehicles have a mileage allowance; any kilometres over this allowance will be charged at a fixed rate. The cost of extra kilometres driven is calculated when the vehicle is returned. If the car is driven a great deal during the year, this can cost a significant amount of money.
- Insurers usually charge higher coverage costs for leased vehicles. However, depending on the driver’s age, driving record, and place of residence, that additional cost may be nominal.

When exploring the option of buying a vehicle, students will consider options such as

- paying the full purchase price plus sales tax
- making a down payment and taking out a loan for the remainder
- taking advantage of special offers from dealerships, such as reduced interest rates or clearance prices on particular models
- buying a new or used vehicle

When considering the possibility of leasing a vehicle, it is important to consider things such as the

- lease term
- lease rate
- security deposit
- kilometre allowance
- delivery charge
- option to purchase

When considering leasing to buy, factors to think about are the

- **residual** value (the estimated value of the car at the end of the lease, determined by the car dealership when the lease is signed)
- maintenance of an older vehicle

Students worked with ratios, percents, and decimals in Mathematics 7 (N03), Mathematics 8 (N03, N04, N05), and Mathematics at Work 10 (A01). They solved problems involving personal budgets and explored the concept of compound interest in Mathematics at Work 11 (N03). Students will now use these skills as they analyze and solve problems that involve acquiring a vehicle through buying, leasing, or leasing to buy.

Students will examine the advantages and disadvantages of buying, leasing, and leasing to buy a vehicle. They will identify the benefits and disadvantages of each and use this information, combined with their own personal factors, such as affordability, to make an informed decision.

Students will solve problems that involve calculating the total cost of purchasing a vehicle after taxes and fees have been applied. Using technology, students will determine the monthly payment based on a given interest rate. The financing interest rate is a compound rate and cannot be calculated in the same manner as the tax rate. For this reason, technology should be used to calculate a monthly payment. It is important to understand that a down payment or discount is applied after taxes. It is important to explore the payment schedules available when purchasing a vehicle (i.e., monthly or biweekly) and assess how this affects payments. Biweekly payments are made more frequently, but are less than half of a monthly payment.

To solve problems that involve leasing a vehicle, calculate

- the total monthly payment after taxes are applied
- the first monthly payment, which may include delivery or licensing fees

When determining the total cost of a leased vehicle over the lease term, consider the monthly lease payments and the cost of financing the residual value. If choosing to purchase the vehicle at the end of the lease, students will be expected to calculate the monthly payment using a given residual value and interest rate. The total cost of purchasing a vehicle should be compared with the total cost of leasing a vehicle to decide on the most economical option to acquire a vehicle. They will calculate monthly and annual fixed costs of owning a vehicle when including fees, such as warranty, insurance, and license fees. Students could also explore the relationship between the type of car and the fixed costs of owning that car.

The variable costs of owning a vehicle, including cost of fuel and maintenance, will be considered. Students will calculate annual maintenance costs, given a maintenance schedule with recommended repairs. Discuss the ongoing variable costs associated with maintaining a vehicle after the warranty has expired, particularly major repairs (e.g., transmission replacement) and then consider whether it is more financially viable to continue maintaining an older vehicle without monthly payments or opting to purchase a new vehicle with warranty. Ideas generated through class discussion could be recorded in a chart such as the following:

Old Vehicle		New Vehicle	
Advantages	Disadvantages	Advantages	Disadvantages

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

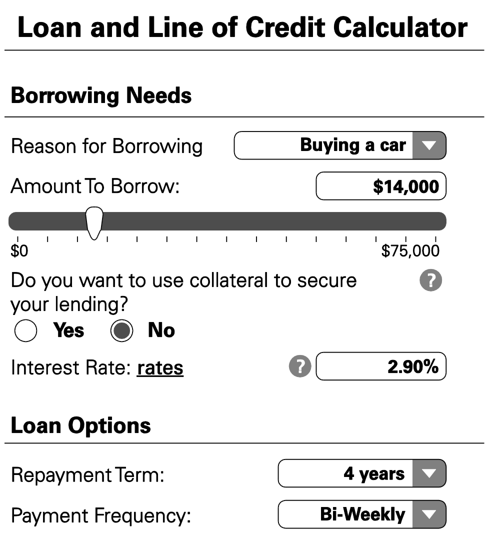
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- You need four new tires. Which company has the better deal?
  (a) Regular price: \$120. Sale price: 30% off the regular price.
  (b) Regular price: \$120. Sale price: Buy one; get a second one for half price.
  (c) Regular price: \$120. Sale price: Buy three, get one free.
- Round the following to the nearest tenth of a percent and then write as a decimal.
  (a) 56.37%
  (b) 0.072%
- Express five years in terms of months.
- Mentally determine the tax on a purchase of \$2000 (tax rate is 15 %).
  (a) The formula for calculating simple interest is $I = Prt$. P represents the principal, r represents the rate of interest per year, and t represents the term of the investment in years. Determine how much simple interest is earned on \$5000 at 2%/year for 10 years.
  (b) The final amount for compound interest can be calculated using the formula, $FV = P(1 + i)^n$.
  FV represents final amount or future value
  P represents the principal
  i represents the interest earned per compounding period
  n represents the total number of compounding periods of the investment
  Determine how much compound interest is earned on \$5000 at 2%/year for 10 years.
  (Interest earned = $FV – P$)
  (c) Which is the better investment and by how much?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Insurance premiums are higher for young adult males than for young adult females. Do you agree or disagree with this practice? Justify your response.
- You are purchasing a new vehicle. You are deciding between taking a full loan for the purchase cost or spending the $3000 in your savings account as a down payment. The purchase price is $18,000 (taxes included) and the car loan will be for five years at an annual interest rate of 3.9%. What will be your decision? Include calculations in your response. What financial and personal factors should be considered?
- Using an online loan calculator or a graphing calculator, input purchase price, annual interest rate, and the number of payments to calculate the monthly loan payment.



- You are purchasing a new vehicle, which you plan to drive 2000 km a month. You are given the option to purchase a "Freedom Driving" warranty package that costs $750 (plus tax), which will cover all maintenance for two years. The dealer provides you with the following maintenance schedule:

Maintenance	Schedule	Cost (plus tax)
Oil change	5000 km	$50
Tire rotation	5000 km	$25
Air filter	once yearly	$45

- Will you purchase the optional warranty package? Use the following calculations to justify your decision.
  (a) Calculate the cost of oil changes over a two-year period.
  (b) Calculate the cost of tire rotations over a two-year period.
  (c) Calculate the cost of replacing the air filter over a two-year period.

- Determine whether a hybrid vehicle is worth the extra cost.
  - (a) Find the base price and city fuel economy of a hybrid vehicle and compare them with a similar gas model (e.g., Toyota Camry vs. Camry Hybrid).
  - (b) Using the current cost of gasoline, determine the number of kilometres that would need to be driven to save the equivalent of the difference of the base model prices (excluding taxes).
  - (c) Be prepared to present your findings to the class using a method of their choice.
- François is trying to decide whether to buy a hybrid car. He researches a hybrid car and records his findings:

Car	**Cost**	**Fuel Economy (city)**	**Distance Driven/Year**
Gas	$25,995	9.2 L/100 km	22 000 km
Hybrid	$30,995	5.1 L/100 km	22 000 km

- Complete the following table to determine which car would be the most economical option over a five-year period.

Calculation (five-year period)	**Gas**	**Hybrid**
Number of kilometres		
Number of litres of fuel		
Cost of fuel at $1.35/L		

- Zack has just bought a car, and expects that he will drive about 3000 km per month. The car's fuel consumption is 5.5 L/100 km. The table shows common maintenance and repair costs.

Maintenance/Repair	**Schedule**	**Cost**
Change oil and filter	Every 3 months	$35
Rotate tires	Every 3 months	$25
Replace air filter	Every 12 months	$30
Replace windshield wipers	Every 6 months	$40
Replace front brakes	Every 50 000 km	$400
Replace back brakes	Every 100 000 km	$400
Replace tires	Every 2 years	$1200
Replace timing belt	Every 150 000 km	$1050

  - (a) How much will Zack spend on gas each month? (Assume that gas is $1.20/L.)
  - (b) How much will Zack spend on maintenance in the first year?
  - (c) Zack intends to keep the car for six years. How much can he expect to spend over that time in repairs and maintenance?
- Sunil is buying a new van for $25,000 at a car dealership.
  - (a) What is the cost of the van, including HST?
  - (b) The dealership is offering 0.9% financing for up to 60 months. Sunil decides to finance the van for 60 months. What will it cost, in total, for the van?
  - (c) Sunil has $5000 for a down payment. How much money will she have to finance?
  - (d) What will Sunil's monthly payment be?

- Zuri wants to lease a new car that is priced at $15,500.
  - (a) His monthly payment is $254.17 plus HST. What will be his total monthly payment?
  - (b) Zuri has to pay a license fee of $155 plus the first month's payment. How much must he pay before he can take the car?
  - (c) Zuri leases the car for 36 months. At the end of the lease, he decides to buy the car. The annual interest rate is 3%, and he will take out the loan for two years. How much will Zuri pay in total for the car if the residual value of the car is $10,000?
  - (d) If Zuri had bought the car initially, he could have had a yearly interest rate of 0.9% for five years. How much would he have saved if he had bought the car initially?
- Jane and Aleisha want to buy a used car that they see on a dealer's lot. They have $2500 for a down payment. The dealer provides them with the following information about a possible purchase:
  - the payments are $365 per month for 36 months
  - annual insurance premium is $755
  - two-year/100 000 km extended warranty is available for $800
  - (a) Calculate their monthly fixed cost.
  - (b) If Jane and Aleisha keep their car for two more years after it is paid off, calculate their monthly fixed cost during this two-year period.
  - (c) They decide to buy the extended warranty. How much money, in total, will Jane and Aleisha have paid on the car during the first five years?

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Ensure that students are able to explain the advantages and disadvantages of buying, leasing, and leasing-to-own a vehicle.
- Discuss with the class the various fixed and variable costs that are associated with operating a vehicle.
- Ask students to collect information regarding the acquisition of a vehicle.
  - Identify a new or used car that you would like to own.
  - Determine if you would purchase, lease, or lease to buy this vehicle.
  - Using a free online insurance quote website, determine the insurance premium that you would have to pay to drive this vehicle.
  - Compare your insurance premium with a partner. What factors account for the differences in insurance costs?

  Follow-up discussion should focus on any differences noted.
- Discuss buying a used car both from a car dealership and a personal sale. Students must understand the importance of knowing the warranty, if any, that comes with the purchase of a used vehicle. It is also valuable to learn the past history of the car and to have the car personally inspected.

- Discuss available options when purchasing a new car. Brainstorm a list of “must have” and “nice to have” options. Ask students to find a newspaper ad advertising a car they would like to own. They should then visit the dealer website and determine the total cost and monthly payment of the vehicle, including the options they have determined to be a “must have.” Ask them to compare the advertised price with the quote they have generated on the website.
- Many financial institutions have online loan calculators or download an app to a smart phone.
- Ask a local financial institution loan officer, insurance representative, and/or car dealership representative to speak to the class about the advantages and disadvantages of leasing and buying a vehicle, depending on the individual’s situation.
- Get students to choose their favourite vehicle and research the value and depreciation rate of the car.
- Using a graphing calculator, show students the graph of the decay curve that is representative of an automobile’s depreciation.

Suggested Models and Manipulatives

- auto buy and sell magazines
- computer with Internet access
- graphing calculator

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- depreciation
- extended warranty
- fixed costs
- insurance premiums
- lease
- lessee
- residual value
- variable costs
- warranty

Resources

Digital

- Financial institutions online loan calculators:
  - “Loan Calculator,” *BMO* (Bank of Montreal 2015): www4.bmo.com/popup/loans/Calculator.html
  - “Car Loan Calculator,” *CIBC* (Canadian Imperial Bank of Commerce 2015): www.cibc.com/ca/loans/calculators/car-loan-calculator.html
  - “Personal Loan Calculator,” *TD Canada Trust* (TD Canada Trust 2015): www.tdcanadatrust.com/loanpaymentcalc.form?lang=en

- Research the value of vehicles:
  - *Canadian Black Book* (Canadian Black Book 2015): www.canadianblackbook.com
  - "How to Look up a Car's Value in Canada," *eHow* (eHow 2015): www.ehow.com/how_5739452_look-up-car_s-value-canada.html
- "Take Charge of Your Future," *Hands on Banking* (Wells Fargo Bank 2015): www.handsonbanking.org/en (Free curriculum resources for a variety of financial mathematics topics, including obtaining a vehicle.)
- *GetSmarterAboutMoney.ca* (Investor Education Fund 2015): www.getsmarteraboutmoney.ca

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 4: Real-Life Decisions
    - > Sections 4.1 and 4.2
    - > Skill Check
    - > Test Yourself

SCO N03 Students will be expected to critique the viability of small business options by considering expenses, sales, and profit or loss. [C, CN, R]

[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

N03.01 Identify expenses in operating a small business.
N03.02 Identify feasible small-business options for a given community.
N03.03 Generate options that might improve the profitability of a small business.
N03.04 Determine the break-even point for a small business.
N03.05 Explain factors, such as seasonal variations and hours of operation, that might impact the profitability of a small business.

Scope and Sequence

Mathematics at Work 11	**Mathematics at Work 12**
N02 Students will be expected to solve problems that involve personal budgets. **N03** Students will be expected to demonstrate an understanding of compound interest.	**N03** Students will be expected to critique the viability of small business options by considering expenses, sales, and profit or loss.

Background

In today's society, it is important for students to develop an understanding of the business world and to develop critical-thinking skills that will assist them in possible future entrepreneurial opportunities. Before opening a small business, it is important to do a feasibility study on starting such a venture. This feasibility study includes researching the possible expenses and sources of revenue for the business. After this study, if a profit is predicted, then the business could be a viable one.

In addition to revenue and expenses, a variety of other factors should be considered when evaluating the feasibility of small business options in a given community. Possible factors to consider are

- number of customers
- set-up or equipment costs
- value of goods or services
- fixed and variable expenses
- weather
- number of employees required
- competition

Improved profitability can be achieved by increasing revenue or decreasing expenses. Increasing prices could theoretically generate more income, but this may exclude some customers. Extending operating hours and advertising may also generate more income, but this requires spending additional funds as well. To decrease expenses, a company looks at minimizing their operating costs.

Students will calculate sales revenue based on the number of items sold or services provided at a given price. They will also calculate the expenses associated with providing each good or service. Students calculate total expenses and revenue and use these values to determine the break-even point for a small business. Students should examine seasonal small businesses and evaluate the profitability of the business over the full year.

Since students are required to determine net income (profit or loss) and the break-even point based on revenue and expenses presented in a table or graph, it is important to incorporate outcomes A01 and N03. Students' ability to recognize patterns and trends, interpret graphs, and solve problems when provided with a table of values or graph (A01), will be built upon as students critique the viability of small business options.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

- If a family's total electrical bills for last year totalled $1749.80, what was their average monthly cost? Round your answer to the nearest dollar.
- Express $\frac{14}{25}$ as a percent.
- Estimate the sum of $215.49, $71.03, and $518.92. Explain your strategy.
- Mentally determine 25% of $600 and explain your strategy.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Sarah charges $65 for each print sold in her studio.
  (a) What is Sarah's sales revenue if she sells 50 prints?
  (b) If it costs Sarah $17.95 to produce each print, what are the expenses required to produce the 50 prints?
  (c) Assuming that Sarah makes 50 prints, how many does she need to sell in order to break even?
- Maya has a lawn-care business during the months of July and August. Business has been booming to the point that she has been turning away clients. What recommendations would you make for Maya to improve profitability? Discuss the advantages and disadvantages of each recommendation.

- Access the tourism website of a town of your choice.
  (a) After reading about the town, find business opportunities for residents who are new to the area.
  (b) Select two business options and list the resources needed to start each business.
  (c) Determine which of the two businesses requires the least amount of resources.
  (d) Explain how the number of resources needed could impact their decision in set up.
- Ralph is a painter. He is self-employed and runs a small painting business. He works for 10 months and goes on vacation in October and November. Last year's revenue and expenses for his business are shown in the table.

Month	Revenue	Expenses
January	$3000	$250
February	$5250	$1000
March	$1250	$1250
April	$3250	$850
May	$8000	$750
June	$7500	$850
July	$12,000	$3000
August	$12,000	$2000
September	$2000	$750
December	$10,000	$2250

  (a) During which month did his business show the greatest profit? How much was the profit?
  (b) Did his business show an overall loss or profit for the year? How much?
  (c) In June, Ralph spent $350 on supplies. What percent of his expenses in June were spent on supplies? Round off the answer to one decimal place.
- Generate a table of expenses (fixed and variable) for the following businesses:
  (a) snow removal service
  (b) summer daycare
  (c) electronics repair
- Patrick started an online business selling graphic T-shirts. He charges $19.99 per shirt. It costs Patrick $10.50 to produce each T-shirt. His operating costs are $451 per month. In one month, he made 75 T-shirts but he sold only 60 of them.
  (a) Has Patrick covered his expenses? What is his net income?
  (b) What must he have charged for his T-shirts in order to break even?
  (c) If he had produced only 60 T-shirts, what would be Patrick's net income?

- Analyze the revenue and expenses for Sven’s small business and answer the questions that follow:

Month	Revenue	Expenses
January	$400	$85
February	$350	$60
March	$275	$55
April	$475	$90
May	$540	$110
June	$710	$140
July	$710	$125
August	$680	$110
September	$500	$95
October	$425	$85
November	$345	$70
December	$400	$75

(a) Draw a graph showing both revenue and expenses for Sven’s business over the year. Explain why you chose this type of graph to represent the data.

(b) Which month was the most profitable? Which month was the least profitable?

(c) What type of good or service might Sven provide? How might this affect the profitability over the year?

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Discuss with students the difference between
  - start-up costs and on-going costs
  - fixed expenses and variable expenses
- As students explore small business options, a simulation game would be appropriate. Tycoon Games are a collection of games in which the player has to successfully run a business. Players must keep their workers happy, their clients coming back, and their resources properly functioning. They are available as online games and apps.
- Bring in guest speakers from local small businesses to talk about the different aspects of operating a small business and their experiences.
- Teachers could provide case studies for students to analyze and discuss. Teachers should complete an example of this prior to giving students their own study.

- As a class, discuss small businesses that already exist or may be considered for future development.
  - What small businesses currently operate locally?
  - For existing businesses, do you think they are operating effectively? Why or why not?
  - For businesses that are struggling, have students suggest options for improvement.
  - What new businesses do you think would be successful in this area? Provide some rationale for this.
- Have students work in groups to develop a small-business plan and discuss expenses, sales, and profits in a contextual situation.
- Discuss possible expenses with students for small businesses. The Canada Revenue Agency has sites for small business owners where possible business expenses are described in detail (www.cra-arc.gc.ca/tx/bsnss/tpcs/slprtnr/bsnssxpnss/menu-eng.html).
- When students are determining net income, caution them against assuming that the month with the greatest revenue is also the month with the greatest profit.
- If available in the school, take opportunities to work with other teachers in entrepreneurship or accounting classes.

Suggested Models and Manipulatives

- computer with Internet access
- graphing calculator

Mathematical Vocabulary

- break-even point
- expenses
- feasibility study
- loss
- net income
- profit
- revenue
- sales

Resources

Digital

- "Business Expenses," *Canada Revenue Agency* (Government of Canada): www.cra-arc.gc.ca/tx/bsnss/tpcs/slprtnr/bsnssxpnss/menu-ent.html
- "Take Charge of Your Future," *Hands on Banking* (Wells Fargo Bank 2015): www.handsonbanking.org/en (Free curriculum resources for a variety of financial mathematics topics, including starting a business.)
- *GetSmarterAboutMoney.ca* (Investor Education Fund 2015): www.getsmarteraboutmoney.ca

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 4: Real-Life Decisions
    - Section 4.3
    - Skill Check
    - Test Yourself
    - Chapter Project
    - Games and Puzzles

Algebra
18–20 hours

GCO: Students will be expected to develop algebraic reasoning.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

A01 Students will be expected to demonstrate an understanding of linear relations by
- recognizing patterns and trends
- graphing
- creating tables of values
- writing equations
- interpolating and extrapolating
- solving problems

[CN, PS, R, T, V]

[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

A01.01 Identify and describe the characteristics of a linear relation represented in a graph, table of values, number pattern, or equation.
A01.02 Sort a set of graphs, tables of values, number patterns, and/or equations into linear and non-linear relations.
A01.03 Write an equation for a given context, including direct or partial variation.
A01.04 Create a table of values for a given equation of a linear relation.
A01.05 Sketch the graph for a given table of values.
A01.06 Explain why the points should or should not be connected on the graph for a context.
A01.07 Create, with or without technology, a graph to represent a data set, including scatterplots.
A01.08 Describe the trends in the graph of a data set, including scatterplots.
A01.09 Sort a set of scatterplots according to the trends represented (linear, non-linear, or no trend).
A01.10 Solve a contextual problem that requires interpolation or extrapolation of information.
A01.11 Relate slope and rate of change to linear relations.
A01.12 Match given contexts with their corresponding graphs, and explain the reasoning.
A01.13 Solve a contextual problem that involves the application of a formula for a linear relation.

Scope and Sequence

Mathematics at Work 11	Mathematics at Work 12
S01 Students will be expected to solve problems that involve creating and interpreting graphs, including bar graphs, histograms, line graphs, and circle graphs.	**A01** Students will be expected to demonstrate an understanding of linear relations by ▪ recognizing patterns and trends ▪ graphing ▪ creating tables of values ▪ writing equations ▪ interpolating and extrapolating ▪ solving problems

Mathematics at Work 11 (continued)	Mathematics at Work 12 (continued)
A01 Students will be expected to solve problems that require the manipulation and application of formulas related to ▪ volume and capacity ▪ surface area ▪ slope and rate of change ▪ simple interest ▪ finance charges **A02** Students will be expected to demonstrate an understanding of slope ▪ as rise over run ▪ as rate of change ▪ by solving problems	

Background

Note: Basic graphing skills such as plotting points, labelling axes, and choosing an appropriate scale should be reviewed prior to beginning the outcome. Graphing technology and spreadsheets should be available for student use. The intent is not for students to become proficient at using various graphing software. Technology is meant to provide them with a quick and accurate visual to identify trends.

Students have worked with linear relations in previous grades. They used algebraic expressions to describe patterns, constructed graphs from a corresponding table of values, and examined the various ways a relation can be expressed, including ordered pairs, table of values, and graphs. They also used patterns to find missing values in a linear relation (7PR02, 7PR04, 8PR01). In Mathematics 9, students used patterns in the tables of values and graphs to learn that a linear relation occurs when there is a constant change in the independent and dependent variable (9PR01). Students also generalized patterns arising from a problem-solving context (9PR01) and graphed, analyzed, **interpolated** and **extrapolated** to solve problems (9PR02). They were introduced to rate of change in Mathematics 9, and related **slope** to **rate of change** in Mathematics at Work 11 (A02). In Mathematics 6 (SP01), students first learned about the concepts of **discrete** and **continuous** data, and in Mathematics at Work 11, they interpreted graphs by describing trends and interpolating and extrapolating values. These topics will be further explored and developed.

A **scatter plot** is a graph used to determine the type of relationship that exists (if any) between two variables. Scatter plots are especially useful with large quantities of data because they help in the visualization of trends. While the focus in this course is on linear trends, students will also look at non-linear trends. Students will see that as the **independent** variable increases the **dependent** variable will increase, decrease, or show no trend. Students have experience describing trends in the graph of a given data set from Mathematics at Work 11. The intent of this outcome is to examine and generalize trends in scatter plots.

Using given or collected data, students will create scatter plots. Although the terms **independent** and **dependent** variables are not new to students, it is important to reiterate that the independent variable is graphed along the horizontal axis while the dependent variable is represented on the vertical axis. By drawing a **line of best fit,** students will be able to determine linear or non-linear trends. A line of best fit is a straight line that represents a trend in a scatter plot that follows a linear pattern.

Students use the line of best fit to extrapolate and interpolate information in order to solve problems. **Interpolation** consists of estimating a value between two given values, while **extrapolation** consists of estimating a value beyond a given set of values. In order to extrapolate, students must extend the pattern beyond the given data. Interpolating is likely to be more accurate since it is between two known values. Extrapolating, on the other hand, is less reliable because a new trend could occur.

Caution students about making assumptions on the continuation of a pattern. By extending a graph, assumptions are being made that the pattern will continue. This is not always applicable in a given context. A graph may show a positive increase, but this trend may not continue. For example, a scatter plot comparing the time a weight lifter spends training to the amount she can bench press. The graph would show a positive trend up to a certain weight. Sports statistics and Olympic records can be used to illustrate that some linear trends will cease at some point. It is important, therefore, that students justify their interpolations and extrapolations as they make inferences from a graph.

When working with scatter plots and lines of best fit, students saw graphs with linear and non-linear trends. They should be exposed to a variety of graphs and asked to sort them according to these trends. Students should apply the trends they discover to contextual situations. For given graphs, discuss questions, such as

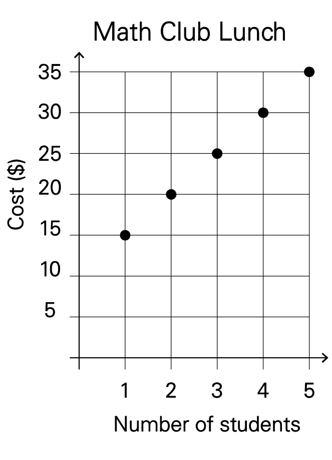
- Does the scatter plot show a trend?
- Is the trend linear or non-linear? Explain how you know.

Students should understand that a random scattering of data points indicates no relation between the variables, while a clustering with a general tendency indicates a relation. They should also note that how closely the data points are clustered along a line of best fit indicates how close the relation is or how strong the correlation.

Although students worked with discrete and continuous data in Grade 9 (9PR02), a review of these concepts is recommended. **Discrete** data are data values that are distinct and can be counted. When graphing data points that represent discrete data, points are not connected. If there are no valid values between the plotted points, then no line is drawn. Continuous data has an infinite number of values between data points. When graphing points that represent continuous data, points are connected with a solid line.

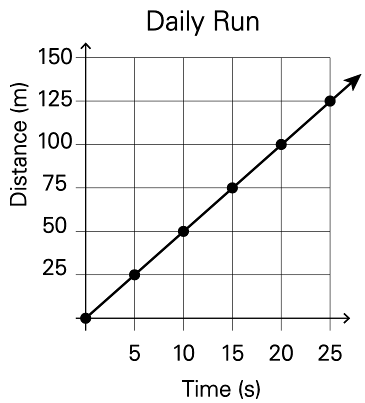
Contextual situations such as the following should make this idea more concrete for students.

- *Case 1:* No line drawn.



This graph has discrete data because it is not possible to have a fraction of a person. Since there are no valid data points between the plotted points, the points are not connected.

- *Case 2:* Solid line is drawn.



On this graph, the data are continuous because the distance and time are continuous. The points, therefore, are connected with a solid line.

In Mathematics at Work 11, students worked extensively with slope. They developed slope in terms of rise over run and explained slope as rate of change using examples and illustrations (A02). Students will now compare tables and graphs to make the connection between slope and rate of change. They should understand that the constant change in the independent variable in the table of values represents the horizontal change in the graph. Similarly, the constant change in the dependent variable represents the vertical change in the graph. The rate of change in the table, therefore, represents the slope of the line.

The concepts of **direct variation** and **partial variation** are introduced in this course. Students will identify and model direct variation and partial variation relationships with tables of values, graphs, and equations.

A **direct variation** is a linear relationship in which one variable is always a fixed multiple of the other variable. A direct variation is a linear relation of the form $y = ax$. When graphing, the line always passes through the origin. The **initial value** is the value of the dependent variable when the independent variable is zero. In a direct variation relationship, the y-intercept is always zero.

A **partial variation** is a linear relationship in which one variable is always a fixed multiple of the other variable plus a constant amount. It is an indication that a portion of the dependent variable is predetermined and does not change as the independent variable changes. A partial variation is a linear relation of the form $y = ax + b$. When graphing, the line does not pass through the origin. In a partial variation relationship, the y-intercept is not zero. In the equation $y = ax + b$, b represents the constant amount. b is the initial value and is the y-intercept value on the graph. **The rate of change** for direct variation and partial variation is the amount by which the dependent variable changes when the independent variable increases by one unit. The rate of change (the value of a in the equation) is constant and represents the **slope** of the graph.)

By comparing equations such as $y = 4x$ and $y = 4x + 5$, students should learn that the first equation, $y = 4x$, represents a relationship that has a direct variation with the y-value being a multiple (4) of the x-value. The second equation, $y = 4x + 5$ represents a relationship with a partial variation as the y-value is a multiple (4) of the x-value, plus five. As each x-value increases by one, the y-value increases by four.

Students should make the connection between the fixed multiple in the equation and the slope of the graph. They should also understand that the constant amount added is the y-intercept.

Students have used a table of values to construct a scatter plot and identified linear and non-linear trends in a graph. The relationship between linear and non-linear trends and the rate of change given a table of values will now be investigated. They will learn that a constant rate of change in the table indicates a linear relationship. Once students understand the characteristics of a linear relation, they should sort sets of graphs, tables of values, and/or number patterns into linear and non-linear relations and justify how they sorted the sets.

Students will use linear equations and formulas to set up tables of values to solve contextual problems. Initially the equation associated with a contextual situation is provided. Discuss with students how the given equation represents the situation. Have them consider the following:

- What the numerical coefficient represents in the situation (fixed multiple).
- What the numerical coefficient represents on the graph (slope).
- What the number being added represents in the situation (constant).
- What the number being added represents on the graph (y-intercept).

Students then apply these concepts to determine equations to represent given contexts. When solving contextual problems, they will substitute values into an equation to solve for the unknown.

When students are looking at a table of values for a linear relationship, such as the following table, they should look at the pattern and recognize a constant increase or decrease (here an increase of six) between the t-values.

Term Number (*n*)	1	2	3	4	5
Term (*t*)	2	8	14	20	26

Students should understand that multiplying the term number, n, by 6 always results in four more than the associated term, t. Therefore, they will need to subtract 4 from $6n$. As an equation, the pattern is represented by $t = 6n - 4$.

Students should verify their equation by substituting values from the table (for example, $n = 5$, $t = 26$) Students should use their equation to solve for any value of n or t.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- On graph paper draw a set of axes. Plot and label the following points:

  (2, 4) (0, 6) (5, 0)
- Describe the number pattern: 8, 5, 2, –1, …
- Give an example of a linear relation in real life. Explain why it is a linear relation.
- The values $A(4, 12)$ and $B(14, 28)$ were substituted into the slope formula by Thor, Liam, and Sarah as follows:

  Thor: $m=\frac{28-12}{14-4}$ Liam: $m=\frac{28-12}{4-14}$ Sarah: $m=\frac{12-28}{14-4}$

  (a) Which of these three versions of substitution are/is correct? Explain.

  (b) What is an important rule to remember about the order of the coordinates when substituting into the slope formula?
- Substitute $x = 4$ into each equation to solve for the value of y:

  (a) $y = 2x - 5$

  (b) $y = 12 + 0.5x$
- Substitute $y = 12$ into each equation to solve for the value of x:

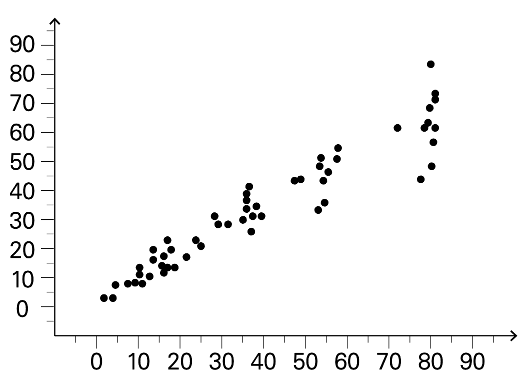
  (a) $y = 3x - 6$

  (b) $y = -2x + 8$

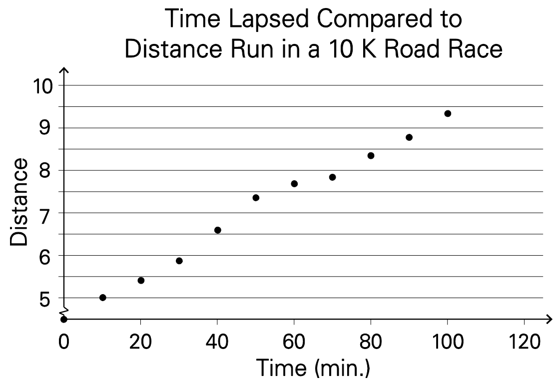
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine whether the relationship in the following scatter plot is a linear relationship or a non-linear relationship. Explain your reasoning.

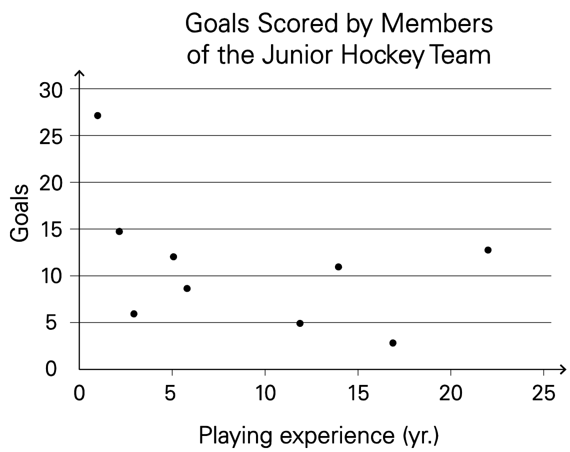


- The following scatter plot shows the distance covered by a runner in a 10 km road race.



  – Does the relationship between distance and time appear to be linear or non-linear? Explain.

- Describe the trend in the scatter plot:

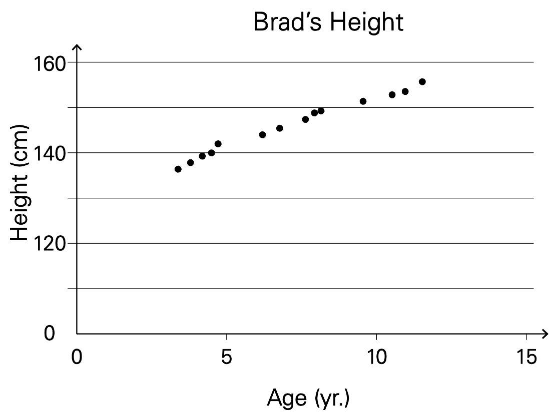


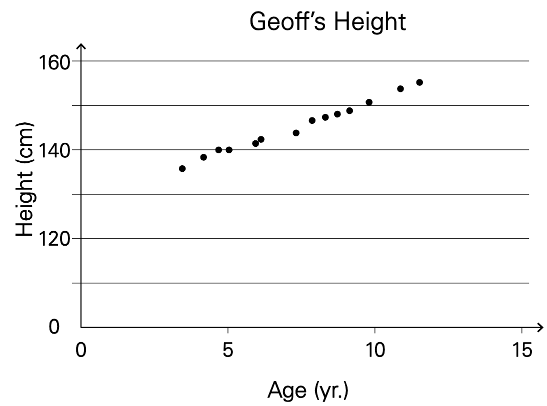
- Determine if the data point should or should not be connected in each of the following situations.

Day	1	2	3	4	5
# of Downloads	6	9	12	15	18

(a) The cost of 150 people renting a banquet hall at $6.50 per person.
(b) The cost of 10 students going to a movie if tickets cost $5 person.
(c) The amount of gas consumed after 500 km.
(d) The number of song downloads in iTunes.

- Brad and Geoff are brothers. Their parents track their growth on the following growth charts:





(a) Based on current growth trends, who will be taller at age 12, Brad or Geoff?

(b) Could the given chart be used to predict Brad's height at age 45? Explain your reasoning.

- Answer the questions below based on the data provided on a movie-streaming service for subscribers.

Number of Movies Streamed	Cost
0	$8
1	$10
2	$12
3	$14
4	$16

(a) Is this relation linear or non-linear?

(b) What is the rate of change for the relation?

(c) What is the slope of the relation and what does it represent?

(d) Does the relation have direct or partial variation?

- Liam is starting a summer lawn-care service. He is debating between whether to charge $10 per lawn plus $5 for each hour it takes to complete or $25 per lawn. Which setup is most beneficial to him? Use the terms **direct** or **partial variation** in your response.

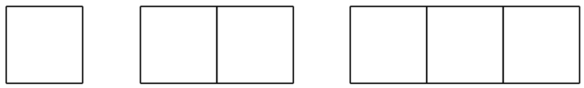
- On a spring day a balloon is released in Antigonish, recording the following temperatures as it increases in altitude:

Altitude (km)	0	0.3	1.5	3.0	4.5	6.0	9.0	10.8
Temperature (°C)	15	13	5	–5	–15	–26	–44	–56

(a) Draw a graph that describes how temperature varies with altitude.

(b) Explain whether or not you should join the points.

- *Partner Activity:* Using square tiles, construct rectangular shapes, as shown below. Continue the pattern. Complete the following table of values and answer the questions below.



Number of Tiles	Perimeter of the Rectangle
1	4
2	6
3	8
4	?
5	?
6	?
7	?

(a) Write a linear equation to represent this situation.
(b) Is the relationship a direct or partial variation? Explain.
(c) How many squares would be needed to make a rectangle with a perimeter of 24 units?
(d) Can you create a rectangle that has a perimeter of 41 units? Explain why or why not.
(e) Draw a graph to represent the equation. Did you connect the points? Explain why or why not.

Repeat this activity with hexagons and octagons. What do you notice about the perimeter of the shape as you increase the number of sides on the tile used by two?

- Use a Venn Diagram or a chart to compare and contrast direct variation and partial variation.
- Given the equation $y = 4x + 10$, create a table of values and draw the corresponding function. Using these representations, describe the patterns found between the table of values and the graphs.
- On the Amirault family road trip, the distance (km) over time (h) is given in the table. Graph the following data and draw a line of best fit.

Time (h)	Distance (km)
0	0
1	95
2	210
3	305
4	420
5	525

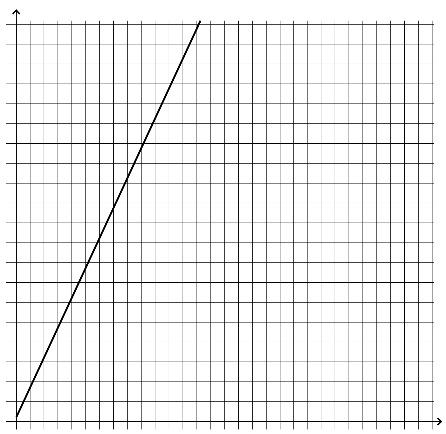
(a) How far were they after 250 minutes?
(b) How far will they be after 9 hours?

- A gas station raises its gas prices from \$1.28/L to \$1.31/L.

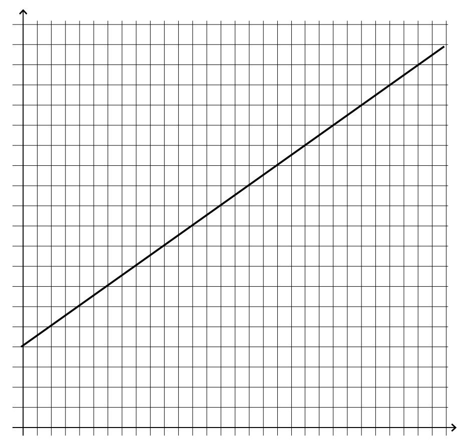
(a) Write an equation to show the relationship between the number of litres bought and the total price in each case.
(b) The gas tank in Beth's car holds 45 L. How much more money will it cost her to fill her tank?

- Jeremiah works at a grocery store and earns $12 per hour.
  (a) If he works for seven hours per day, create a table of values to show how much money he has earned after each hour on a given day.
  (b) Use this table of values to draw the graph of this relationship. Use the graph to determine whether this relationship is a direct variation.
  (c) What is the rate of change in his total earnings?
  (d) After how many hours will he earn $156?
- Determine whether each graph represents a direct variation or a partial variation. Explain your reasoning.

(a)



(b)



- Determine which of the following equations represent a direct variation and which represent a partial variation.

(a) $y = \frac{1}{2}x$

(b) $s = 45 - 3t$

(c) $y = 3x + 2$

(d) $d = \frac{r}{2}$

- Determine whether each table represents a partial variation. Explain your reasoning.

(a)

x	*y*
0	1
1	3
2	5
3	7

(b)

x	*y*
0	0
5	2.5
10	5
15	7.5

- John works part-time at a bicycle shop assembling bicycles. He is initially paid $10 per bicycle. After working there for one month, the bicycle shop changes the way he is paid. He is now paid a base salary of $50 per week and an additional $6 for each bicycle.
  (a) Write an equation that describes how much John would make in each situation if he assembled 20 bicycles per week.
  (b) Create a table of values to represent how much John makes per week for each pay scale, if he assembles 5, 10, 15, and 20 bicycles per week.

(c) On the same graph, graph the number of bicycles assembled in a week and John's salary for each pay scale.

(d) If John usually assembles 12 bicycles a day, which payment plan would he prefer?

- Create an advertisement for a product or an upcoming event for the community television channel. (Some suggestions: a used car for sale, a yard sale, car wash) The cost of the ad is $20.00 plus $0.25 per word. This includes one photo.
  (a) Create an advertisement.
  (b) Write an equation to represent the relationship between the number of words and the cost.
  (c) Is it a direct or partial variation relationship?
  (d) Calculate the cost of the advertisement.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- When constructing a scatter plot, students may not leave enough space to extrapolate. Encourage them to read an entire question before drawing a graph in case they need to extrapolate.
- Discuss a scatter plot comparing the height of individuals and their age. This graph would show a positive trend up to a certain age. Students should understand that this trend does not continue since growth does not continue throughout a person's life span.
- Have students match a scatter plot with a given trend. Ask them to provide examples of contextual situations that could be represented by the scatter plot.
- Present small groups of students with a scatter plot of mystery data. Students should hypothesize what is represented by each. Assistance may be offered by giving a list of the data sets represented and allowing groups to select which is depicted by their example.
- Discuss situations for each of the following scatterplots:
  - One that does not show any trend.
  - As one variable increases, so does the other.
  - A decreasing trend that cannot continue indefinitely.
- Provide examples of contextual situations that could be represented by a given graph. Ask students to match a given context with a graph. When matching graphs with contextual situations, they should identify the independent and dependent variable.
- Initiate a discussion by asking students to provide examples of a linear relation in real life. From this, a discussion about the specific characteristics of linear functions in the various forms could follow. This would also be an opportunity to look at non-linear functions and to compare the similarities and differences between linear and non-linear functions.
- Consider an activity where students are given linear and non-linear relations in the various formats. Ask students questions such as,
  - Why are the tables linear?
  - What makes the graph non-linear?

- A memory matching game could be played in small groups. Students have to find matching cards: one of the table of values and one of the graph. If they can correctly identify the relationship as linear or non-linear, they keep the matching pair.
- *Think-Pair-Share Activity:* Have students complete a chart listing the similarities and differences of a direct variation relationship and a partial variation relationship. Ask students to share their list with a partner. Ask them to discuss and have them collate their lists. Have groups share and discuss their information with the class by posting a master list. Check to make sure all the main points are discussed in the final chart.
- Have students research an Internet job search site (e.g., www.novascotiajobshop.ca). Have them find a job where they would be paid
  - a salary
  - an hourly wage
  - a straight commission
  - a salary plus commission
  - an hourly wage plus commission
  - by piece work

  From this, have students create a graphical representation and an equation to represent the earnings in each situation.
- Have students explore situations where they choose values for the independent variable. This would lead to a discussion about which values are appropriate when creating a table.

SUGGESTED MODELS AND MANIPULATIVES

- graph paper
- graphing calculator
- graphing technology, such as Autograph
- measuring tape
- metre sticks
- polydrons
- spreadsheets (Excel)
- square tiles

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- continuous data
- direct variation
- discrete data
- extrapolation
- initial value
- interpolation
- line of best fit
- linear relationship
- linear trend
- non-linear relationship
- partial variation
- rate of change
- scatter plot
- slope
- *y*-intercept

Resources/Notes

Digital

- Autograph (Autograph Canada 2015; NSSBB #: 51395–51397, 51500)
- Microsoft Excel (Microsoft 2015)
- *NovaScotiaJobShop.ca* (RegionalHelpWanted, Inc. 2015): www.novascotiajobshop.ca.

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 2: Working with Data
    - Sections 2.3
    - Skill Check
    - Test Yourself
    - Chapter Project
    - Games and Puzzles
  - Chapter 3: Linear Relationships
    - Sections 3.1, 3.2, and 3.3
    - Skill Check
    - Test Yourself
    - Chapter Project
    - Games and Puzzles
  - Chapter 4: Real-Life Decisions
    - Sections 4.1, 4.2, and 4.3
    - Skill Check
    - Test Yourself

Statistics
12–14 hours

GCO: Students will be expected to develop statistical reasoning.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO S01 Students will be expected to solve problems that involve measures of central tendency, including mean, median, mode, weighted mean, and trimmed mean. [C, CN, PS, R]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

S01.01 Explain, using examples, the advantages and disadvantages of each measure of central tendency.
S01.02 Determine the mean, median, and mode for a set of data.
S01.03 Identify and correct errors in a calculation of a measure of central tendency.
S01.04 Identify the outlier(s) in a set of data.
S01.05 Explain the effect of outliers on mean, median, and mode.
S01.06 Calculate the trimmed mean for a set of data, and justify the removal of the outliers.
S01.07 Explain, using examples such as course marks, why some data in a set would be given a greater weighting in determining the mean.
S01.08 Calculate the mean of a set of numbers after allowing the data to have different weightings (weighted mean).
S01.09 Explain, using examples from print and other media, how measures of central tendency and outliers are used to provide different interpretations of data.
S01.10 Solve a contextual problem that involves measures of central tendency.

Scope and Sequence

Mathematics at Work 11	Mathematics at Work 12
S01 Students will be expected to solve problems that involve creating and interpreting graphs, including bar graphs, histograms, line graphs, and circle graphs.	**S01** Students will be expected to solve problems that involve the measures of central tendency including mean, median, mode, weighted mean, and trimmed mean.

Background

Measures of central tendency allow a set of data to be described with a single meaningful number. In Mathematics 7, students calculated the measures of central tendency (mean, median, mode), discussed the effect of outliers on the measures of central tendency, and used them to solve contextual problems (SP01, SP02). In Mathematics at Work 11, students created appropriate data displays (S01). Students will now extend their understanding of measures of central tendency to include **weighted mean** and **trimmed mean**. The concepts of weighted mean and trimmed mean will be introduced, calculated, and analyzed in relevant situations. Students will evaluate when it is more appropriate to use a trimmed or weighted mean rather than just the mean.

A review of how to calculate each measure of central tendency will be needed.

- **Mode** is the observation that occurs most often (applies only to discrete data). Students should work with data sets that have one mode and those that have more than one mode. When determining mode, they should also explore data sets that are categorical in nature (e.g., favourite sport).
- **Median** is the middle data point of a data set, sorted from smallest to largest. If there is an even number of data points, the two middle observations are averaged to find the median.
- **Mean** is the sum of the data values, divided by the number of data values.

A set of data often contains **outliers**, or values that are significantly different from the others. The presence of outliers may affect which measure of central tendency best represents the data. Students have been previously exposed to this idea, but it will be necessary to revisit it. They should be able to identify the outlier(s) and explain the effect upon measures of central tendency. In some cases, the presence of outliers may not affect the measures of central tendency.

For example, as students explore the effect of 38 and 98 on the measures of central tendency for the data set {38, 64, 68, 68, 71, 72, 75, 98}, they should conclude that the values on opposite extremes of the data set will have virtually no effect on the average score. However, sometimes the median can be affected, as it would be in data sets such as {1, 2, 4, 6, 63} and {3, 5, 26, 33, 37, 42}.

A **stem-and-leaf plot** is a way to organize numerical data in order of place value. This representation allows for easy identification of the mode(s) and outlier(s) if they exist. It can also be useful in finding the median as well as for drawing a frequency table and a histogram. This will be new to students so an explanation of how to create and use a stem-and-leaf plot will be needed.

The following illustrates how to construct a stem-and-leaf plot for the data values 6, 8, 15, 12, 26, 20, 43.

- Construct a two-column chart.
- The tens' digit and greater is the stem. The ones' digit is the leaf.
- The stems and leaves must be placed in ascending order.

Stem (tens)	Leaf (ones)
0	6 8
1	2 5
2	0 6
3	
4	3

To plot decimal numbers, the ones' digit and greater is the stem. The tenths' digit and less is the leaf.

As students examine various contextual situations in which they are required to calculate the measures of central tendency, they should develop awareness that certain data sets are better represented by one or more measures of central tendency. As they evaluate the data, they must decide which measure or measures of central tendency allow a set of data to be best described with a single meaningful number. Although students may initially choose the mean to best describe the data, this discussion should demonstrate that it may be more advantageous to use either median or mode in certain situations. They should consider the appropriateness of each measure depending upon the situation presented.

Focus classroom discussion on realistic situations to explore the advantages and disadvantages of each of these measures of central tendency. This could include data sets such as test scores, shoe size, favourite music genre, or heights of trees. Sample advantages and disadvantages of the measures of central tendency are shown below. This list is not meant to be exhaustive.

Measure of Central Tendency	**Advantages**	**Disadvantages**
Mean	▪ Commonly used in familiar contexts. ▪ Easy to calculate. ▪ Useful when comparing sets of data.	▪ Skewed by extreme values (outliers).
Median	▪ Extreme values do not affect this measure as strongly as they do the mean.	▪ Tedious to arrange large sets of data in order without technology.
Mode	▪ Extreme values do not affect the mode. ▪ Useful when the data values are limited in scope (e.g., shoe size). ▪ Can be used with non-numerical data sets (e.g., favourite colour).	▪ A set of data may have no mode when no item appears more than once. ▪ There may be more than one mode that can be difficult to interpret.

In certain situations the straight forward calculation of mean does not adequately represent the given set of data. A weighted mean is a useful measure when all of the data are not all of equal significance. It is a calculation that takes into account the relative importance of the individual data values. **Weighted mean** is the mean determined by counting each category as a percentage of the total. Each data item is multiplied by a percentage that corresponds to its importance in the data set.

A discussion of how students' course marks are calculated will provide rationale for the use of weighted mean and why certain values in a data set are given greater weighting. If a student's overall grade, for example, is based on marks on two quizzes and one test, and the test is worth 60% of the grade while each quiz is worth 20%, it is the weighted mean that will give the student's final grade. After analyzing and calculating weighted mean for various sets of data, students should understand the difference between mean and weighted mean and be able to identify an appropriate context for each.

Trimmed mean is a calculation of the mean after removing the least and greatest value(s) in a data set. This is done to obtain a better representation of the central tendency of the data. In order to calculate trimmed mean, the same number of values are removed from each end of an ordered list and the mean is calculated from the remaining data values. These discarded values are not always outliers although they often can be. A trimmed mean will alleviate the misrepresentation of data caused by outliers. These outliers may represent a recording error, or an extreme event. The trimmed mean may give a better picture of what typically happens.

Trimmed mean is used to score Olympic events, such as gymnastics, to eliminate potential bias (outliers) from judges. The highest and lowest judges' scores are eliminated and the trimmed mean is calculated with the remaining scores.

Measures of central tendency are frequently cited in print and other forms of media. The choice of reporting mean, median, or mode is consciously made in order to emphasize a particular perspective. Even when a reported measure of central tendency is accurate, it may not always provide a true representation of the data being discussed. Students can use their knowledge of these measures of central tendency to interpret data presented in the media.

Students should analyze the reasons why particular measures of central tendency are reported and the change in perspective that would occur if a different measure was selected.

Reports of statistics in the media are often made with a particular audience in mind. Often, the median is reported for data that may be skewed one way or the other. An article on NHL player salaries, for example, may report the median salary. This information would give no indication of the salaries for some of the highest-paid players. On the other hand, the mean salary would indicate an overinflated answer to the question, How much does an NHL player make?

Another example is the reported average housing prices for a particular area during a given month. The decision to report the mean instead of the median price can create a completely different perspective. In the case where a number of expensive houses were sold in a given time frame, the mean price can indicate that the average house in that area is more expensive than it actually is.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- The following data set shows the number of birds at a feeder from Monday to Sunday. Find the range, mean, median, and mode.

M	T	W	T	F	S	S
3	4	6	9	2	11	13

- Arrange the data set in ascending order (least to greatest):

  53.3 55.3 53.17 50.9 53.05 50.88 55.21 53.13 55.09

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine whether mean, median, or mode would be most helpful in each situation. Justify your choice.
  (a) You want to know if you read more or fewer books per month than most people in your class.
  (b) You want to know the "average" amount spent per week on lunch in your class.
- For the following set of quiz marks out of 10, calculate the mean, median, and mode. Round off to one decimal place, where necessary. Compare the measures of central tendency.

8	7	9	10	8	6	5	6
8	9	8	10	7	7	7	8
6	10	10	4	7	9	7	9

- The following data set describes the heart rates of grade 12 students before they wrote a final exam, in beats per minute. Calculate the mean, median, and mode, and explain which measure is most useful and why.

63	59	71	64	68	76	88	72	85	83	93	73
90	57	68	75	67	61	72	88	63	74	72	84

- Suppose the mean of a set of scores is 89. One of the scores is missing from the report, and the other four scores are 90, 95, 85, and 100. Determine the value of the missing score.
- The mode of this set of data is 75. Determine the missing value, n.

  81 75 90 68 81 68 75 n
- Darryl, Heidi, and Juan are captains of the school mathematics teams. Their contest results are recorded in the table below.

	Darryl	**Heidi**	**Juan**
Contest 1	82	84	85
Contest 2	82	84	85
Contest 3	88	90	85
Contest 4	100	71	81
Contest 5	77	78	81
Contest 6	81	87	85
Contest 7	87	89	82
Contest 8	83	88	85
Contest 9	83	86	83

  – Which measure of central tendency would you choose to determine who has the best team? Why?

- *Journal Entry:* Describe how you would explain the difference between mean, median, and mode to a friend who missed the class when measures of central tendency were introduced. Use an example to support the explanation.
- Samantha is a hairdresser at a busy salon in Yarmouth, Nova Scotia. Her customers often leave tips for her service. One afternoon, Samantha gave four haircuts and earned tips of $4.50, $5.50, $5.00, and $7.25. She is hoping to earn a mean amount of $6.00 in tips per haircut. If she has one more appointment scheduled for the day, determine how much she must earn from the tip to reach her goal.
- Julia and Inge were both asked to determine the median of the following data set:

  5 8 3 14 21 16 9 18 4

  Inge stated that the median was 9 and Julia said it was 21.

  (a) Which student is correct? Explain your reasoning.

  (b) What error do you think the other student made?
- In your group, identify all errors that have been made in the solution below. Provide correct solutions.

  (a) Given the following set of data, determine the mean.

  3 4 0 9 8 7 6 7 7

  Student Solution:

  Mean is $\dfrac{3+4+0+9+8+7+6+7+7}{8}=\dfrac{51}{8}=6.375$

  (b) Given the following set of data, determine the median.

  6.2 7.4 5.9 6.0 6.3 6.9 7.5 7.2

  Student Solution:

  Order the data from least to greatest: 5.9 6.0 6.2 6.3 6.9 7.2 7.4 7.5

  Since there are eight numbers, the median is the fourth number, 6.3.

  (c) Given the following set of data, determine the mode.

  54 53 56 51 54 56 58 59

  Student Solution:

  Since both 54 and 56 occur most often, the mode is $\dfrac{54+56}{2}=55$.
- Identify the outlier(s) in the following data set.

  24 30 26 54 28 19

  Remove the lowest and highest scores, then calculate the trimmed mean.
- Mr. Trig recently marked a set of class assignments. The scores were as follows:

  45 58 78 69 0 25 14 74 85 96 96 85 100 12 46 78 65 70 41 55

  (a) Determine the class mean.

  (b) Determine a trimmed mean.

- Two candidates are applying for the same job. The following weightings were assigned.

Interview Questions: 25%	Presentation: 40%	Written Essay: 35%

  - Steven scored $\frac{8.0}{10}$ on his interview, $\frac{8.5}{10}$ on his presentation, and $\frac{6.0}{10}$ on his written essay.
  - Susan scored $\frac{9.0}{10}$ on her interview, $\frac{7.0}{10}$ on her presentation, and $\frac{6.5}{10}$ on her essay.

  (a) Based on these results, who will be hired?

  (b) If you were competing for this job, which of the three scoring categories do you think would be your strength?

- During the last six months, Steve, a real estate agent, sold nine houses at the following prices:

\$1,479,000	\$750,000	\$699,000
\$435,900	\$659,000	\$589,500
\$449,900	\$625,600	\$712,800

  (a) Calculate the mean and a trimmed mean.

  (b) Which would be a better indication of the average price of a house sold? Why?

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Use the following example to create a stem-and-leaf plot with students.
  - Using the following data values, create a stem-and-leaf plot.

    6.4 5.1 5.8 5.7 8.1 5.8 7.8 5.6 10.9 7.5 6.3 7.2 6.9

    Create a two-column stem-and-leaf by ordering the numbers least to greatest. The ones' digit and greater is the stem. The tenths' digit is the leaf.

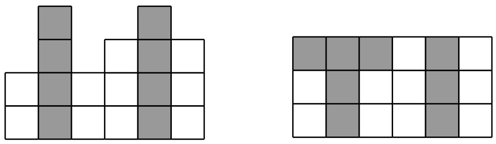
Stem (ones and greater)	**Leaf (tenths and less)**
5	1 6 7 8 8
6	3 4 9
7	2 5 8
8	1
9	
10	9

> Find the mode(s) values. Determine if any outliers exist.

> The mode value is 5.8 because it is the only number listed twice. 10.3 is an outlier.

> Discuss with students how the outlier affects the measures of central tendency.

- This activity uses blocks to determine the mean of a data. The blocks can be rearranged so each column has equal height. When using this strategy, ensure the data sets used have a small number of elements with low numbers, and that they balance to a whole number. The data set 2, 4, 2, 3, 4, 3, for example, has a mean of 3, as seen in the diagram below:



- Rolls of tickets can be used to review the concept of finding the median of a data set. On a strip of tickets, students write a data value on each ticket in ascending order. If there is an odd number of tickets, the strip is folded at the median.

- Present the following scenario to students.
  - A municipal government planning agency was interested in changing street parking zone regulations in an area of their city. They gathered information on the number of cars registered to households on the specific streets being affected by the proposed change. In each of the 17 homes on Elm Street, people were asked how many cars were registered to their households. The results were recorded as follows:

<table>
<tr><th>Number of Cars</th><th>Tally</th><th>Frequency</th></tr>
<tr><td>0</td><td>|</td><td>4</td></tr>
<tr><td>1</td><td>𝍸 |</td><td>6</td></tr>
<tr><td>2</td><td>𝍸</td><td>5</td></tr>
<tr><td>3</td><td>||</td><td>2</td></tr>
</table>

    > Ask students to use the data in the frequency table to calculate the measures of central tendency.

    > Ask students which measure of central tendency they think would best represent the typical number of cars registered to households on Elm Street and to explain their answer.

- The following set of data could be displayed in a stem-and-leaf plot as shown below.

  12 16 31 42 48 56 63 64 78 83 91

Stem	**Leaf**
1	2 6
2	
3	1
4	2 8
5	6
6	3 4
7	8
8	3
9	1

  From this, students should understand that there is no mode and that 56 is the median. This representation allows for easy identification of the number that occurs most frequently in a data set.

- Review with students the proper use of the calculator when determining the mean to eliminate possible calculation errors. Students may try to calculate the sum of the values in the data set and divide by the total number of data values in the same step. In this case, the answer given by the calculator may not be correct since order of operations may not have been followed. Students could be encouraged to write all steps in the solution to alleviate this potential error.
- Some common errors that occur when students are working with measures of central tendency are as follows:
  - They may not arrange the data in ascending or descending order prior to determining the median.
  - The median may be incorrectly identified in an even-numbered data set as both middle numbers or as one of the two.
  - Students report cases where no mode exists as having a mode of 0.
- When determining measures of central tendency, data sets should be limited to a manageable number of values and should contain both even and odd numbers of entries. Data sets can be sorted, but students should also be exposed to examples where they have to first order the data before calculating the median.
- Have students calculate the mean, trimmed mean, and median for a set of numeric data, with and without an outlier, to see the effect of outliers (change just the lowest or highest value to an outlier). They should understand that the median is unaffected but the mean gets either much higher or much lower.
- Have students find various articles and/or advertisements using measures of central tendency to report findings. Students should be able to determine the meaning of what was reported and if it was the best indicator of what was measured.

- Generate a class set of data by asking each student in the class to record the approximate time it takes him or her to get to school. This can be presented as a graffiti wall with sticky notes. The sticky notes can then be rearranged to help students visualize mode and median. Students can then use the data to answer the following questions:
  - Calculate the mean time students in the class spend travelling to school each day.
  - Calculate the median and mode for time spent travelling to school. How do they differ? Which is more useful? Explain your reasoning.
- Fill three identical containers with approximately 15 different-sized objects (e.g., paper clips, pencils). Each container should have one object significantly smaller and one significantly larger than the rest. Students are divided into three groups and each given a container. They should measure and record the length of each item, arrange their objects from shortest to longest, and identify objects that would be considered outliers. Ask students to calculate the mean length, eliminate the outliers, and calculate the trimmed mean. As a class, compare the arithmetic means and trimmed means and discuss any differences that are found.
- Students can collect data about the length of time it takes for each class member to run 100 m. They can then calculate the mean and trimmed mean and discuss whether it may be more appropriate to use a trimmed mean. Brainstorm examples where a trimmed mean may be more representative of the average time taken to run 100 m.
- Suggestions for research topics:
  - Students could research an article of interest that reports a measure of central tendency. They may consider such topics as car sales, salaries of actors, wages in a province, or sports statistics. Students can display their findings using a poster, PowerPoint presentation, etc., and present to the class.
  - Students might research the statistical career data for a professional sports figure, such as Sidney Crosby. If possible, find outliers that exist and give possible causes of these outliers. They could research the statistics on two players and justify which they think is the better player based on this data.
  - Students could choose a topic to research with the goal of creating an advertisement. For the advertisement, they should select a measure of central tendency that would skew the interpretation of the data.
- Students could collect a series of television advertisements / print advertisements relating to claims based on central tendencies. Examples could include the following:
  - Average increase in hair growth.
  - Average weight loss on a program.
  - Average savings on a cartload of groceries.
  - Average savings when buying a car outside of the urban area.
  - How much, on average, can be saved by using a mortgage broker.

  Ask students to rewrite one of the advertisements using a different measure of central tendency to make a different claim.

- To introduce the calculation of weighted mean, students could be provided with a sample course evaluation scheme and two fictional students' marks, such as in the table below.

Category	Weighting	Ashlyn's Average Marks	Sinead's Average Marks
Tests and quizzes	25%	75%	83%
Assignments	20%	90%	80%
Projects	20%	83%	95%
Games/puzzles	5%	95%	80%
Final examination	30%	82%	87%

Have students do a comparison of mean versus the weighted mean for the above data. The mean values for both students is 85%, but the weighted means are different. With students, calculate Ashlyn's weighted mean. A table, such as the following, might be useful when calculating weighted means.

Category	Weighting	Ashlyn's Average Marks	Product of weighting and mark
Tests and quizzes	25%	75%	$.25 \times 75 = 18.75$
Assignments	20%	90%	$.20 \times 90 = 18$
Projects	20%	83%	$.20 \times 83 = 16.6$
Games/puzzles	5%	95%	$.05 \times 95 = 4.75$
Final examination	30%	82%	$.30 \times 82 = 24.6$

- Find the sum of the last column: $18.75 + 18 + 16.6 + 4.75 + 24.6 = 82.7$
- Ashlyn's weighted mean is 82.7%
- Have students calculate Sinead's weighted mean.

This activity can lead to a discussion of how a student's average is obtained in each category compared to their overall average. It will help students distinguish between mean and weighted mean and to identify an appropriate context for each.

SUGGESTED MODELS AND MANIPULATIVES

- calculators
- newspapers or magazines

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- data set
- mean
- measures of central tendency
- median
- mean
- mode
- trimmed mean
- weighted

Resources/Notes

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 2: Working with Data
    - > Sections 2.1 and 2.2
    - > Skill Check
    - > Test Yourself
    - > Chapter Project
    - > Games and Puzzles

SCO S02 Students will be expected to analyze and describe percentiles. [C, CN, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

S02.01 Explain, using examples, percentile ranks in a context.
S02.02 Explain decisions based on a given percentile rank.
S02.03 Explain, using examples, the difference between percent and percentile rank.
S02.04 Explain the relationship between median and percentile.
S02.05 Solve a contextual problem that involves percentiles.

Scope and Sequence

Mathematics at Work 11	**Mathematics at Work 12**
–	**S02** Students will be expected to analyze and describe percentiles.

Background

Although students have previously worked with the concept of percent and its applications, the concept of a percentile and percentile rank will be new to them. Percentiles are often used to organize sets of data, such as test scores or birth weight. The link between percentile and median will be made. Percentile ranks compare individual values with the other values in a data set. This experience will be used to distinguish between appropriate uses of percentile ranks compared to percents.

A **percentile** is a value below which a certain percentage of the data set falls, and the rest of the data is at or above that value. Physicians often use percentiles, which are found in growth chart, to assess infant and child growth in comparison to national averages. For example, if a baby's weight is stated to be in the ninety-third percentile then 93% of babies at that age weigh less than this child. The percentile shows how an infant's weight compares to other infants.

Percentiles divide a distribution of data into two or more groups. Students will discuss the link between percentile and median. The median is the number (data point) at the fiftieth percentile. For example, if a teacher wishes to determine the exam score that divides the class in half, the median is determined. Since this score is the middle data point of the data set, the median is at the fiftieth percentile.

To determine a percentile for a set of data, the data is first arranged from smallest to largest. For example, if we wish to determine what data is in the seventy-fifth percentile for 7, 13, 9, 2, 5, 8, 6, 1, 5, 3, 11, 4, we would order the data as 1, 2, 3, 4, 5, 5, 6, 7, 8, 9, 11, 13. The median of the data values is 5.5. The numbers 5 or less would fall at or below the fiftieth percentile. The median of the top half of the data is 8.5. The numbers 9 or greater would fall at or above the seventy-fifth percentile because 75% of the data points are less than 9.

Outliers do not affect the calculation of percentiles. Percentiles are particularly useful statistics when working with a very large data set. They make it very easy to categorize such data based on rank. As a result, they can help make decisions based on rank.

A **percentile rank** is used to determine where a particular score or value fits within a broader distribution. It shows the percentage of values in a set that are at or below. Students should be encouraged to determine this rank with simple percentiles like the fiftieth or seventy-fifth percentile, before applying the following formula to determine percentile rank.

To determine the percentile rank of a specific value, *x*, the values in the data set should first be ordered from least to greatest. The percentage of values at or below that value gives the percentile rank. For example, to compare a score of 8 with the rest of the scores on a quiz marked out of 10, a percentile rank would be useful: 6, 6, 6, 7, 7, 7, 7, 8, 8, 10. Since 9 of the 10 scores are at or less than 8, the percentile rank for that score is ninetieth rank.

It is important that students understand that the percentile rank is not a percent score. An example such as the following highlights this difference.

Consider the following quiz scores:

75 77 78 78 80 81 81 82 83 84 84 85 87 87 88 88 88 89 90 92

A score of 80% initially sounds good. However, this does not sound as impressive when it is thought of as the twenty-fifth percentile rank, or only 25% of the class scored 80% or less.

Students can use the following formula to determine percentile ranking:

$$PR = \frac{\text{number of values at or below the value being considered}}{\text{total number of data values}} \times 100$$

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- What is 65% of 52?
- 48 is what percent of 60?
- 320 is what percent of 400?
- Express $\frac{5}{6}$ as a percent.
- Express 45% as a reduced fraction.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the fiftieth percentile for each data set.

  (a) 62 45 39 82 51 29 66

  (b) 3 8 12 15 4 22 13 20

- Twenty students participated in a basketball shooting contest. Each student made 30 attempts. The results were as follows:

15	17	19	23	19
23	9	21	20	19
12	22	26	17	21
19	16	13	8	7

  (a) Organize the data in a stem-and-leaf plot.

  (b) What result is at the fiftieth percentile? How does that relate to the median value?

  (c) Mary got 20 baskets.

    (i) At what percentile rank is her result?

    (ii) What percent of the students made more baskets than Mary?

  (d) What results are at the eighty-fifth percentile or above?

- The final exam scores for a class were as follows:

  62 66 71 75 75 78 81 83 84 85 85 87 89 89 91 92 93 94 95 99

  (a) What percent of the students scored over 85 on the exam?

  (b) Find the percentile rank for a score of 85 on this test.

  (c) Based on you answer in (b), would you consider 85 to be an exceptional score on the exam? Use the percentile ranking to justify your answer.

- If Jason graduated twenty-fifth out of a class of 150 students, what would his percentile rank be?

- Given the following set of data, find the number that lies at the twenty-eighth percentile and the seventy-second percentile.p

  20.1 18.5 21.5 20.3 19.0 18.4 18.2 18.0 17.6 18.5 16.8 20.1 19.0 18.0 18.5 20.0 22.0 18.4

- What conclusions can be made about a test where a mark of 50% is in the eightieth percentile?

- *Journal Entries:*

  (a) On a recent test, Laura received a score of 85%. This placed her in the ninety-sixth percentile of students who wrote the test. Explain what this means.

  (b) The following data shows the amount of time (in minutes) a student spends on Math homework during one week: 20, 0, 15, 30, 10. Determine the median, and explain what percentile this represents.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Have students line up at the front of the classroom based on how long it takes them to get to school each morning. Explore the concepts of percentiles by using the times by where they lie on the continuum. For example, use the student in the middle (or the two in the middle) to explain the connection between median and the fiftieth percentile.
- Growth charts for infants, children, and adolescents can be found online. They are based on percentiles and separated by gender.
- Have students research a situation of interest to them involving percentiles and report back to the class. Examples include aptitude test scores and salary ranks.
- Ensure students clearly understand the difference between percentages and percentiles by discussing and calculating examples of both for various sets of data.
- Begin work on percentile ranks with data values that are not percentages, since some students may struggle to understand the difference between percentage and percentile rank.

SUGGESTED MODELS AND MANIPULATIVES

- calculators

MATHEMATICAL VOCABULARY

- percentile
- percentile rank

Resources/Notes

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 2: Working with Data
    - Section 2.2
    - Skill Check
    - Test Yourself

Probability
10–12 hours

GCO: Students will be expected to develop critical-thinking skills related to uncertainty.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO P01 Students will be expected to analyze and interpret problems that involve probability. [C, CN, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

P01.01 Describe and explain the applications of probability (e.g., medication, warranties, insurance, lotteries, weather prediction, 100-year flood, failure of a design, failure of a product, vehicle recalls, approximation of area.)
P01.02 Calculate the probability of an event based on a data set.
P01.03 Express a given probability as a fraction, decimal, and percent and in a statement.
P01.04 Explain the difference between odds and probability.
P01.05 Determine the probability of an event, given the odds for or against.
P01.06 Explain, using examples, how decisions may be based on a combination of theoretical probability calculations, experimental results, and subjective judgements.
P01.07 Solve a contextual problem that involves a given probability.

Scope and Sequence

Mathematics at Work 11	Mathematics at Work 12
–	**P01** Students will be expected to analyze and interpret problems that involve probability.

Background

Students started calculating probabilities in Mathematics 6 (SP04). In Mathematics 7, they expressed probabilities as ratios, fractions, and percents (7SP04) and compared theoretical and experimental probabilities of independent events (7SP06). In Mathematics 8, they created and solved problems using probabilities (SP02), including the use of tree diagrams and simulations.

Although students have been introduced to probability in previous grades, this will be their first formal introduction to odds. Students will analyze problems that involve probability and distinguish between probability and odds. The focus of student work will be on independent events. They will calculate probability and odds in various situations, and use this information to make decisions. Students should understand that many day-to-day decisions are based on the combination of probability and subjective judgments.

When there is uncertainty about the occurrence of an event, we measure the chances of it happening with probability. **Probability** compares the number of times a favourable outcome will occur to the number of times all outcomes will occur (part:whole). Probability is determined by dividing the number of favourable outcomes by the total number of possible outcomes. This results in a value ranging from 0 to 1, where 0 refers to that event never happening and 1 refers to the event always happening.

Probability can be expressed as a fraction, decimal, and percent. Converting between these three representations should be reviewed.

While probability compares the number of favourable outcomes to the total number of outcomes, **odds** compare the number of times a favourable outcome will occur to the number of times an unfavorable outcome will occur (part:part). Since a fraction compares a part to a whole, odds can only be expressed as a **ratio**.

It is important to highlight the difference between probability and odds. For example, the odds that a randomly chosen day of the week is Sunday are 1 to 6 (1:6). However, the probability of it being Sunday is 1 in 7, or $\frac{1}{7}$, and the outcome of interest is indicated first. It is important to be consistent in vocabulary used, with odds using “to,” and probability using “in.”

Students will be required to express both the odds for an outcome and the odds against an outcome, and be clear on which they are describing.

Odds for an event are written as a ratio of
number of favourable outcomes:number of unfavourable outcomes

Odds against an event are written as a ratio of
number of unfavourable outcomes:number of favourable outcomes.

Students should understand that the ratio of the odds against an event is the reverse of the ratio for an event. If the odds for selecting a red marble from a jar is 5:3, for example, then the odds against selecting a red marble is 3:5.

Students will apply this knowledge to determine the probability of an event, given the odds for or against that event. For example, if an electrical circuit has 50:1 odds against failure, the probability of a defective electrical circuit is $\frac{1}{51}$.

Students will distinguish between theoretical and experimental probability. **Theoretical probabilities** are those that result from theory (what should happen mathematically), while **experimental probabilities** are those that result from experiments or repeated trials of performing the event. Students should think about how decision making is affected by a combination of probability and **subjective judgment** (personal thinking).

The theoretical probability of an event is the ratio of the number of **favourable outcomes** in an event to the total number of **possible outcomes**, when all possible outcomes are equally likely. (For example, a non-weighted die versus a weighted die.) It can only be used to predict what will happen in the long run, when events represented are equally likely to occur. Students should understand that the probability in many situations cannot be characterized as equally likely, such as tossing a thumb tack to see if it lands with the point up or down, and therefore, theoretical probability is more difficult to determine.

A discussion of games of chance could lead to an explanation of how theoretical probability, experimental probability, and subjective judgment would play a role in decisions made when playing such games. Calculations of probability are always based on assumptions. Students should be encouraged to identify and examine the assumptions to help them determine whether the calculated probability is meaningful when making a decision. Students should engage in evaluating situations that lend themselves to reasonably accurate predictions, those that are questionable, and those for which the unknowns are not quantifiable. Road accidents with/without seatbelts are good examples for survival prediction. Health professionals predicting that people of lower socio-economic status will have more health problems is a more questionable situation.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- What are the chances of
  (a) drawing a king from a deck of cards
  (b) getting tails in one toss of a coin
  (c) it raining today
- Determine the missing values in the table below.

Fraction	Decimal	Percent
$\frac{2}{5}$		
	0.15	
		72%

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Two number cubes are rolled. What is the probability of getting
  (a) a double
  (b) a sum of 7
  (c) a sum greater than 9
- Express the probability in each of these statements in two other forms.
  (a) There is a 10% chance of snow.
  (b) Four out of five dentists recommend this toothbrush.
  (c) Goalie Roberto Loungo's save percentage statistic was 0.927 at the 2010 Olympic Winter Games.

- The Rogers Centre, which is home to the Toronto Blue Jays, has the world's first fully retractable roof. The decision to host an event with an open or closed roof is based upon detailed weather information gathered by the roof technicians. If the technicians were given the following forecast, do you think they would leave the roof open? Use probability to justify your choice.

  Short-term Forecast
  Tuesday Afternoon
  Risk of thunderstorms
  Temperature: 36°C
  Feel Like: 43
  Wind: W 25 km/h
  Relative Humidity: 44%
  P.O.P.: 40%
  Rain: close to 1 mm

- Explain the difference between odds and probability.
- Determine the probability of each of the following events:
  (a) Odds in favour of the event are 1:3.
  (b) Odds against the event are 5:1.
  (c) Odds in favour of the event are 50:1.
  (d) Odds against the event are 1:1.
- The following data set lists the number of hours worked by twenty grade 12 students in a week.

12	25	0	19	2
15	4	8	11	19
30	4	40	3	6
7	1	10	40	21

  (a) What is the probability of selecting a student who works 30 or more hours per week? Express the answer as a fraction, decimal, and percent.
  (b) What are the odds of selecting a student who works fewer than 30 hours per week?
  (c) Write a probability statement in sentence form based on the answers from (a) and (b).
- The odds of winning a prize are 1:24. What is the probability of winning? What is the probability of losing?
- Every time a coin is flipped, there is a 50/50 chance of landing on heads or tails. So, if a coin is flipped 50 times, the result should be 25 heads and 25 tails. Do you agree with this statement? Why or why not? To test this, with a partner flip a coin 50 times and create a tally chart of the number of times you got heads and the number of times you got tails. Compare the experimental probability calculated from your data with the theoretical probability. Are they the same? What might you do to make your experimental results closer to the theoretical probability?

- Answer the questions below, based on the weather table shown.

	Tuesday Afternoon *Cloudy periods*	**Tuesday Evening** *Variable cloudiness*	**Tuesday Overnight** *Variable cloudiness*	**Wednesday Morning** *Cloudy with showers*	**Wednesday Afternoon** *Cloudy with showers*
Temperature	23°C	21°C	14°C	16°C	19°C
Wind	S 15 km/h	S 10 km/h	SE 5 km/h	SE 10 km/h	NE 10 km/h
Relative Humidity	46%	53%	72%	77%	77%
P.O.P.	20%	20%	30%	40%	60%
Rain	–	–	–	less than 1 mm	close to 1 mm

(a) What is the probability of precipitation on Wednesday morning?
(b) What are the odds of precipitation on Tuesday evening?
(c) What is the probability of no precipitation on Tuesday afternoon?

- In order to attract more customers to a store, they run a contest where the probability of winning a prize is $\frac{1}{5}$,

(a) Express the probability of winning a prize as a percent.
(b) Calculate the probability of not winning a prize. Express the answer as a decimal.
(c) What are the odds of winning a prize in this contest?
(d) If Meika enters the contest 10 times, is there any guarantee that she will win at least once?

- The odds in favour of drawing an ace in a standard deck of cards are 4:48 (or 1:12).

(a) What does this ratio represent?
(b) What does the sum of the parts of the ratio represent?
(c) What are the odds against selecting an ace?
(d) What is the probability of selecting an ace?
(e) Why is it important to know if the ratio represents odds against or odds in favour when using odds to determine the probability?

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Use ordering (e.g., least likely to most likely) as a way of familiarizing students with the various expressions of odds and probability.
- Incorporate discussion about interpretations. For example, ask, Does this mean that the most likely event will happen or that the least likely event will not happen?
- Incorporate discussion about the nature of subjective judgment. For example, ask, Why do people still decide to go to the beach when there is a reported 60% chance of rain?

- Statements of probability and odds are referenced frequently in various media and texts such as newspapers, magazines, websites, and television. Provide students with a newspaper, magazine editorial, or an opinion piece. These may relate, for example, to political polls, games of chance, sports, and social statistics. Ask them to identify examples of probability or odds present in the article. Ask them to describe how the author uses probability to support the argument being put forth. This activity promotes student discussion of the similarities and differences between these two concepts using interesting contexts found in the media.
- In discussions it is important to focus on how valid odds and probability statements are. Make it a common part of your teaching to ask, Is this valid? This should incorporate discussion about the source of the statement, and if that source is reliable and reputable.
- Lotto 6/49 is a popular lottery in Canada. To play the game, a person selects 6 of 49 numbers (1 through 49) on the playing card. Six numbers are drawn. To win the jackpot a person must select all six of the numbers that are drawn. Investigate the odds of winning this lottery or one similar to it such as Lotto Max. Ask students, Why do you think people play these lotteries every week if the chances of winning are so slim? In their summary, ask them to include some of the issues associated with regular gambling.
- To incorporate outcome N01 with outcome P01, have student play First Nations Stick Game.

First Nations Stick Game

Materials:

- 4 wooden stir sticks
- markers
- 50 toothpicks or beans to keep score

Decorate each stir stick on one side with markers. Use a different pattern or colour for each one. The decorated side is the “face” of the piece. Players take turns. Each player holds the sticks in one hand and lets them fall to the ground or the table.

Scoring:

- All four up = 5 points
- Three up and one down = 2 points
- Two up and two down = 1 point
- One up and three down = 2 points
- All four down = 5 points

Count the number of points and take that many toothpicks or beans from the pile. The player with the most toothpicks or beans at the end of the agreed-upon number of rounds is the winner.

Once the game is over, consider the following:

(a) Is this a fair way to score the game?

(b) Describe another way to score.

There are 16 different ways that four sticks can fall. Three ways are shown in the table.

#1	#2	#3	#4
up	up	up	up
down	up	up	up
up	down	up	up

Finish the table and determine the theoretical probability of two sticks landing up and two sticks landing down.

- Online or graphing calculator probability simulators can be used to generate data for large numbers of trials.
- Discuss how experimental probability and theoretical probability might approach each other as the number of trials increases. Some students may have difficulty with this concept and may need to see it played out many times.
- Probability questions involving playing cards are common. It might be helpful to post a reminder, such as the following, in the classroom and have decks of cards available for student use.

A regular deck of cards has

- 52 cards total
- 26 red (13 diamonds, 13 hearts) and 26 black (13 spades, 13 clubs)
- four suits, each with the cards 2–10, J, Q, K, and A

- Many games of chance use odds and probability. Examples can also be found or created based on school statistics (e.g., the number of biology students expected in grade 12) or sports (e.g., the chance of a team winning a championship). This would also be a good opportunity for students to conduct in-class surveys and calculate odds or probability based on results. This will allow them to focus on situations that are familiar and contextual to them.
- Students may need guidance to be able to distinguish between odds for and odds against. If a student is trying to calculate the odds of getting a heart in a deck of cards, for example, they will have to consider the probability that they will draw a heart from the deck. Since the probability of choosing a heart is $\frac{1}{4}$, the odds in favour would be 1:3. Teachers may wish to ask students questions, such as the following:
  - How would this ratio change to determine the odds against a student choosing a heart?
  - How are the odds in favour related to the odds against?
- Provide students with a variety of examples where they are asked to express odds as a probability and vice versa. Examples can be found or created based on school statistics (e.g., the number of biology students expected this year in grade 12) or sports (e.g., the chance of a team winning a championship). All odds and probability calculations begin with two of three values: total possibilities, favourable outcomes, and non-favourable outcomes.
- This is a good opportunity for students to conduct in-class surveys and calculate the odds and probability based on the results. A question on the survey could be, How many students use Twitter daily. Other student interests may include music, celebrities, or school teams.

- Have students investigate the contributions to probability theory made by the Chevelier de Méré, a 17th-century French gentleman, and Blaise Pascal, an eminent mathematician of the time. Chevelier de Méré's questions to Pascal concerned two games using dice. In the first game, he bet on getting at least one six on four rolls of a fair die and won more often than he lost. In the second game, he bet on getting at least one double six on 24 rolls of a pair of dice and lost more often than he won.

SUGGESTED MODELS AND MANIPULATIVES

- coins
- decks of cards
- graphing calculators
- number cubes

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- experimental probability
- fair game
- favourable outcomes
- odds
- odds against an event
- odds for an event
- possible outcomes
- probability
- random
- ratio
- simulation
- subjective judgment
- tally
- theoretical probability
- tree diagram

Resources/Notes

Print

- *Math at Work 12* (Etienne et al. 2012)
  - Chapter 1: Measurement and Probability
    - Sections 1.2, 1.3, and 1.4
    - Skill Check
    - Test Yourself
    - Chapter Project
    - Games and Puzzle

References

Abbott, Robert. 2015. "Logic Mazes." *Robert Abbott*. www.logicmazes.com.

American Association for the Advancement of Science [AAAS-Benchmarks]. 1993. *Benchmark for Science Literacy*. New York, NY: Oxford University Press.

Armstrong, T. 1993. *Seven Kinds of Smart: Identifying and Developing Your Many Intelligences*. New York, NY: Plume.

Autodesk Inc. 2015. AutoCAD. *Autodesk*. Computer Software. www.autodesk.com/education/free-software/autocad.

Autograph. 2015. Autograph. Waterford, ON: Autograph Canada. Computer Software. (NSSBB #: 51395–51397, 51500)

Bank of Montreal. 2015. "Loan Calculator." *BMO*. www4.bmo.com/popup/loans/Calculator.html.

Black, Paul, and Dylan Wiliam. 1998. "Inside the Black Box: Raising Standards through Classroom Assessment." *Phi Delta Kappan 80*, No. 2 (October 1998), pp. 139–144, 146–148.

British Columbia Ministry of Education. 2000. *The Primary Program: A Framework for Teaching*. Victoria, BC: Province of British Columbia.

Britton, Jill. 2013. Nim. Jill Britton. http://britton.disted.camosun.bc.ca/nim.htm.

Caine, Renate Numella, and Geoffrey Caine. 1991. *Making Connections: Teaching and the Human B*rain. Reston, VA: Association for Supervision and Curriculum Development.

Canada Revenue Agency. 2015. "Business Expenses." *Government of Canada.* www.cra-arc.gc.ca/tx/bsnss/tpcs/slprtnr/bsnssxpnss/menu-eng.html.

———. 2013. "Payroll Deductions On-line Calculator." *Government of Canada.* www.cra-arc.gc.ca/esrvc-srvce/tx/bsnss/pdoc-eng.html.

———. 2013. *Government of Canada*. www.cra-arc.gc.ca/menu-eng.html.

Canadian Black Book. 2015. *Canadian Black Book*. www.canadianblackbook.com.

Canadian Imperial Bank of Commerce. 2015. "Car Loan Calculator." *CIBC*. www.cibc.com/ca/loans/calculators/car-loan-calculator.html.

CoolMath.com LLC. 2015. " Geometry and Trig Reference Area." *CoolMath.com* www.coolmath.com/reference/geometry-trigonometry-reference.html.

———. 2014. "Logic Games." *Coolmath.com*. www.coolmath-games.com/0-b-cubed/index.html.

Davies, Anne. 2000. *Making Classroom Assessment Work*. Courtenay, BC: Classroom Connections International, Inc.

eHow. 2015. "How to Look up a Car's Value in Canada." *eHow*. www.ehow.com/how_5739452_look-up-car_s-value-canada.html.

Escher, M.C. 2015. "Picture Gallery 'Symmetry.'" *The M.C. Escher Company BV.* www.mcescher.com/Gallery/gallery-symmetry.htm.

Etienne, Steve, Emily Kalwarowsky, Carol Jaap Klass, Melissa Marchesi, and Deborah Turner. 2012. *Math at Work 12*. Whitby, ON: McGraw-Hill Ryerson.

Frankenstein, Marilyn. 1995. "Equity in Mathematics Education: Class in the World outside the Class." *New Directions for Equity in Mathematics Education*. Cambridge, MA: Cambridge University Press.

Gardner, Howard E. 2007. *Frames of Mind: The Theory of Multiple Intelligences*. New York, NY: Basic Books.

GeoGebra. 2015. *GeoGebra*. International GeoGebra Institute. www.geogebra.org/cms/en.

Gutstein, Eric. 2003. "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School." *Journal for Research in Mathematics Education 34*, No. 1. Reston, VA: National Council of Teachers of Mathematics.

Hall, Tracey E., Anne Meyer, and David H. Rose (Ed). 2012. *Universal Design for Learning in the Classroom: Practical Applications*. New York, NY: The Guilford Press.

Hazard, Shelly. 2005. "Feeding Time ~ a logic problem," *PuzzlersParadise.* www.puzzlersparadise.com/puzzles/feedingtime.html.

Herzig, Abbe. 2005. "Connecting Research to Teaching: Goals for Achieving Diversity in Mathematics Classrooms." *Mathematics Teacher*, Volume 99, No. 4. Reston, VA: National Council of Teachers of Mathematics.

Hope, Jack A., Larry Leutzinger, Barbara Reys, and Robert Reys. 1988. *Mental Math in the Primary Grades*. Palo Alto, CA: Dale Seymour Publications.

Hume, Karen. 2011. *Tuned Out: Engaging the 21st Century Learner*. Don Mills, ON: Pearson Education Canada.

Investor Education Fund. 2015. *GetSmarterAboutMoney.ca*. www.getsmarteraboutmoney.ca.

Jefferson Lab. 2015. "The Nim Number Game." *Jefferson Lab*. http://education.jlab.org/nim/index.html.

Key Curriculum Press. 2013. The Geometer's Sketchpad. Columbus, OH: McGraw-Hill Education. Computer Software. (NSSBB #: 50474, 50475, 51453)

Ladson-Billings, Gloria. 1997. "It Doesn't Add Up: African American Students' Mathematics Achievement." *Journal for Research in Mathematics Education 28*, No. 6. Reston, VA: National Council of Teachers of Mathematics.

MathIsFun.com. 2014. *Math Is Fun, Enjoy Learning!,* "Four In A Line." MathIsFun.com. www.mathsisfun.com/games/connect4.html.

———. 2014. *Math Is Fun, Enjoy Learning!,* "Tic-Tac-Toe." MathIsFun.com. www.mathsisfun.com/games/tic-tac-toe.html.

———. 2015. "Interactive Quadrilaterals." *MathIsFun.com.* www.mathsisfun.com/geometry/quadrilaterals-interactive.html

———. 2014. *Math Is Fun, Enjoy Learning!,* "Dots and Boxes Game." MathIsFun.com. www.mathsisfun.com/games/dots-and-boxes.html.

Math Open Reference. 2009. "Dilation of a Polygon." *Math Open Reference*. http://mathopenref.com/dilate.html.

MathPlayground.com. 2014. "Mancala." MathPlayground.com. www.mathplayground.com/mancala.html.

Math Warehouse. 2015. "Triangles Side and Angles: Interior and Exterior Angles and Sides." *Math Warehouse*. www.mathwarehouse.com/geometry/triangles.

National Council of Teachers of Mathematics. 2005. "Computation, Calculators, and Common Sense: A Position of the National Council of Teachers of Mathematics" (position paper, May 2005). Reston, VA: National Council of Teachers of Mathematics.

———. 2000. *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

New Brunswick Department of Education and Early Childhood Development. 2014. *Financial and Workplace Mathematics 12*. Fredericton, NB: Province of New Brunswick.

Newfoundland and Labrador Department of Education. 2011. *Mathematics: Applied Mathematics 3202*, Interim Edition. St. John's, NL: Government of Newfoundland and Labrador.

Nova Scotia Department of Education. 2011. *Racial Equity / Cultural Proficiency Framework*. Halifax, NS: Province of Nova Scotia.

———. 2002. *Racial Equity Policy*. Halifax, NS: Province of Nova Scotia.

———. 2010. *Gifted Education and Talent Development*. Halifax, NS: Province of Nova Scotia.

RegionalHelpWanted, Inc. 2015. *NovaScotiaJobShop.ca.* www.novascotiajobshop.ca.

OECD Centre for Educational Research and Innovation. 2006. *Formative Assessment: Improving Learning in Secondary Classrooms*. Paris, France: Organization for Economic Co-operation and Development (OECD) Publishing.

Prince Edward Island Department of Education and Early Childhood Development. 2013. *Mathematics: MAT631A*. Summerside, PE: Government of Prince Edward Island.

Rubenstein, Rheta N. 2001. "Mental Mathematics beyond the Middle School: Why? What? How?" *Mathematics Teacher*, September 2001, Vol. 94, No. 6. Reston, VA: National Council of Teachers of Mathematics.

Shaw, J. M., and Cliatt, M. F. P. 1989. "Developing Measurement Sense." In P.R. Trafton (Ed.), *New Directions for Elementary School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

SketchUp. 2013. *SketchUp 2013*. Google.com. www.sketchup.com.

Steen, L. A. (ed.). 1990. *On the Shoulders of Giants: New Approaches to Numeracy*. Washington, DC: National Research Council.

System Improvement Group, Alberta Education. 2006. *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings*, January 25, 2006. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.

Tate, William F. 1995. "Returning to the Root: A Culturally Relevant Approach to Mathematics Pedagogy." *Theory into Practice 34*, Issue 3. Florence, KY: Taylor & Francis.

TD Canada Trust. 2015. "Personal Loan Calculator." *TD Canada Trust*. www.tdcanadatrust.com/loanpaymentcalc.form?lang=en.

Tomlinson, Carol Ann. 2001. *How to Differentiate Instruction in Mixed-Ability Classrooms*. Alexandria, VA.: Association for Supervision and Curriculum Development.

Wells Fargo Bank. 2015. "Take Charge of Your Future." *Hands on Banking*. www.handsonbanking.org/en.

Western and Northern Canadian Protocol (WNCP) for Collaboration in Education. 2008. *The Common Curriculum Framework for Grades 10–12 Mathematics*. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.

———. 2006. *The Common Curriculum Framework for K–9 Mathematics*. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.