Mathematics 8

DRAFT • DRAFT • DRAFT •
June 2015

CURRICULUM GUIDE

NOVA SCOTIA

Mathematics 8

Implementation Draft
June 2015

Website References

Website references contained within this document are provided solely as a convenience and do not constitute an endorsement by the Department of Education and Early Childhood Development of the content, policies, or products of the referenced website. The Department does not control the referenced websites and subsequent links, and is not responsible for the accuracy, legality, or content of those websites. Referenced website content may change without notice.

School boards and educators are required under the Department's *Public School Programs' Internet Access and Use Policy* to preview and evaluate sites before recommending them for student use. If an outdated or inappropriate site is found, please report it to links@EDnet.ns.ca.

Mathematics 8, Implementation Draft

Prepared by the Department of Education and Early Childhood Development

Acknowledgements

The Nova Scotia Department of Education and Early Childhood Education wishes to express its gratitude to the following organizations for granting permission to adapt their mathematics curriculum in the development of this guide.

- Manitoba Education
- New Brunswick Department of Education
- Newfoundland and Labrador Department of Education
- The Western and Northern Canadian Protocol (WNCP) for Collaboration in Education

We also gratefully acknowledge the contributions of the following individuals toward the development of the Nova Scotia Mathematics 8 curriculum.

Darryl Breen
Strait Regional School Board

Bob Crane
Mi'kmaw Kina'matnewey

Paul Dennis
Chignecto-Central Regional School Board

Trisha Demone
South Shore Regional School Board

Robin Harris
Halifax Regional School Board

Darlene MacKeen Hudson
Chignecto-Central Regional School Board

Mark MacLeod
South Shore Regional School Board

Sonya O'Sullivan
Halifax Regional School Board

Fred Sullivan
Strait Regional School Board

Marlene Urquhart
Cape Breton-Victoria Regional School Board
Brad Pemberton
Annapolis Valley Regional School Board

Contents

Introduction

Background and Rationale

Mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society. It is essential that the mathematics curriculum reflect current research in mathematics instruction. To achieve this goal, Western and Northern Canadian Protocol's (WNCP) *The Common Curriculum Framework for K–9 Mathematics* (2006) has been adopted as the basis for the new mathematics curriculum in Nova Scotia.

The Common Curriculum Framework was developed by the seven ministries of education in the Western provinces (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan, and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and performance indicators agreed upon by the seven jurisdictions. The outcomes and performance indicators have been adapted for Nova Scotia. This document is based on both national and international research by the WNCP and the National Council of Teachers of Mathematics (NCTM).

There is an emphasis in the Nova Scotia curriculum on particular key concepts at each grade that will result in greater depth of understanding and, ultimately, stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

Purpose

This document provides sets of outcomes and performance indicators to be used as a mandated common base for defining mathematics curriculum expectations. This common base should result in consistent student outcomes in mathematics within the Province of Nova Scotia. It should also enable easier transfer for students moving within the province or from any jurisdiction that has adopted the WNCP framework. This document is intended to clearly communicate to all education partners across the province the high expectations for students' mathematical learning.

Program Design and Components

Assessment

Ongoing assessment for learning is essential to effective teaching and learning. Research has shown that assessment *for* learning (formative assessment) practices produce significant and often substantial learning gains, close achievement gaps, and build students' ability to learn new skills (Black and Wiliam 1998; OECD 2006). Student involvement in assessment promotes learning. Timely and effective teacher feedback and student self-assessment allow students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes

- providing clear goals, targets, and learning outcomes
- using exemplars, rubrics, and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning

(Davies 2000)

Assessment for learning practices act as the scaffolding for learning, which only then can be measured through assessment of learning (summative assessment). Assessment of learning tracks student progress, informs instructional programming, and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning, and produce achievement gains.

Assessment *of* student learning should

- align with curriculum outcomes
- clearly define criteria for success
- make explicit the expectations for students' performance
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction

Balanced Assessment in Mathematics

Conversations/Conferences/Interviews
- individual
- group
- teacher-initiated
- child-initiated

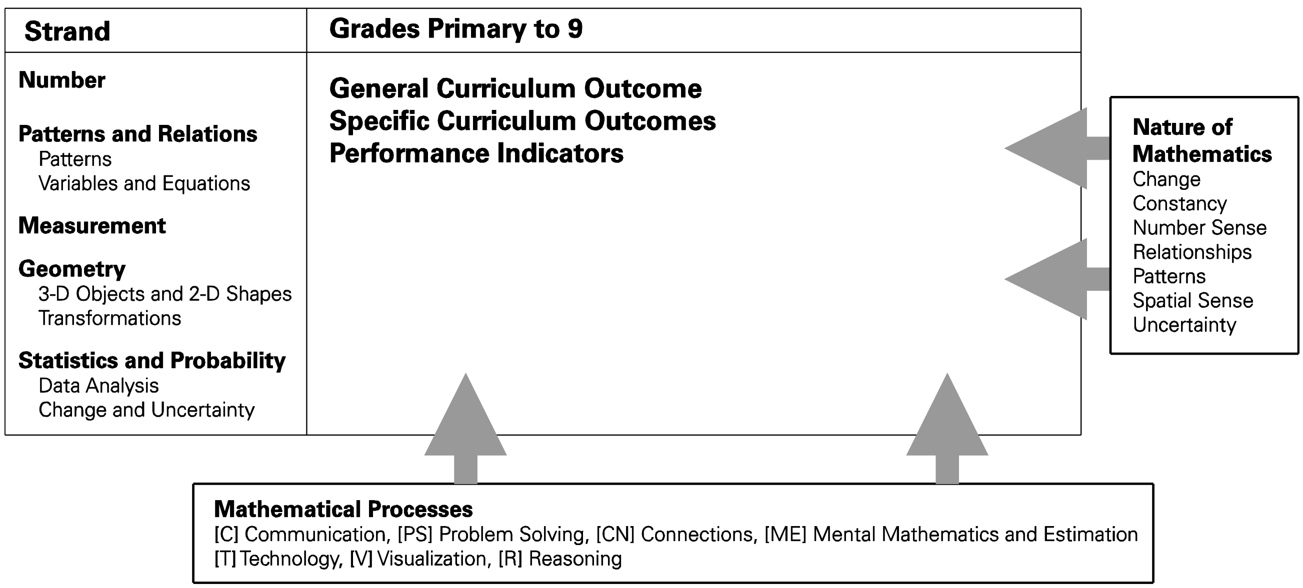
Products / Work Samples
- mathematics journals
- portfolios
- drawings, charts, tables, and graphs
- individual and classroom assessment
- pencil-and-paper tests
- surveys
- self-assessment

Observations
- planned (formal)
- unplanned (informal)
- read-aloud (literature with mathematics focus)
- shared and guided mathematics activities
- performance tasks
- individual conferences
- anecdotal records
- checklists
- interactive activities

Outcomes

Conceptual Framework for Mathematics Primary–9

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



Adapted from *The Common Curriculum Framework for K–9 Mathematics* (Western and Northern Canadian Protocol, 2005, 5).

Structure of the Mathematics Curriculum

Strands

The learning outcomes in the Nova Scotia Framework are organized into five strands across grades primary to 9.

- Number (N)
- Patterns and Relations (PR)
- Measurement (M)
- Geometry (G)
- Statistics and Probability (SP)

General Curriculum Outcomes (GCO)

Some strands are further subdivided into sub-strands. There is one general curriculum outcome (GCO) per sub-strand. GCOs are overarching statements about what students are expected to learn in each strand/sub-strand. The general curriculum outcome for each strand/sub-strand is the same throughout grades P–9.

NUMBER (N)

- Students will be expected to demonstrate number sense.

PATTERNS AND RELATIONS (PR)

Patterns

- Students will be expected to use patterns to describe the world and solve problems.

Variables and Equations

- Students will be expected to represent algebraic expressions in multiple ways.

MEASUREMENT (M)

- Students will be expected to use direct and indirect measure to solve problems.

GEOMETRY (G)

3-D Objects and 2-D Shapes

- Students will be expected to describe the characteristics of 3-D objects and 2-D shapes and analyze the relationships among them.

Transformations

- Students will be expected to describe and analyze position and motion of objects and shapes.

STATISTICS AND PROBABILITY (SP)

Data Analysis

- Students will be expected to collect, display, and analyze data to solve problems.

Chance and Uncertainty

- Students will be expected to use experimental or theoretical probabilities to represent and solve problems involving uncertainty.

Grade 8 Specific Curriculum Outcomes (SCOs) and Performance Indicators

Specific curriculum outcomes (SCOs) are statements that identify the specific conceptual understanding, related skills, and knowledge students are expected to attain by the end of a given grade.

Performance indicators are statements that identify specific expectations of the depth, breadth, and expectations for the outcome. Teachers use performance indicators to determine whether students have achieved the corresponding SCO.

Process Standards Key

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

NUMBER (N)

N01 Students will be expected to demonstrate an understanding of perfect squares and square roots, concretely, pictorially, and symbolically (limited to whole numbers). [C, CN, R, V]

Performance Indicators

N01.01 Represent a given perfect square as a square region, using materials such as grid paper or square shapes.
N01.02 Determine the factors of a given perfect square, and explain why one of the factors is the square root and the others are not.
N01.03 Determine whether or not a given number is a perfect square, using materials and strategies such as square shapes, grid paper or prime factorization, and explain the reasoning.
N01.04 Determine the square root of a given perfect square, and record it symbolically.
N01.05 Determine the square of a given number.

N02 Students will be expected to determine the approximate square root of numbers that are not perfect squares (limited to whole numbers). [C, CN, ME, R, T]

Performance Indicators

N02.01 Estimate the square root of a given number that is not a perfect square, using materials such as square shapes and graph paper and strategies such as using the roots of perfect squares as benchmarks.
N02.02 Approximate the square root of a given number that is not a perfect square using technology (e.g., a calculator or a computer).
N02.03 Explain why the square root of a number shown on a calculator may be an approximation.
N02.04 Identify a number with a square root that is between two given numbers.

N03 Students will be expected to demonstrate an understanding of and solve problems involving percents greater than or equal to 0%. [CN, ME, PS, R, V]

Performance Indicators

N03.01 Provide contexts where a percent may be between 0% and 1%, between 1% and 100%, and more than 100%.
N03.02 Represent a given fractional percent using concrete materials and pictorial representations.
N03.03 Represent a given percent greater than 100% using concrete materials and pictorial representations.
N03.04 Determine the percent represented by a given shaded region on a grid, and record it in decimal, fraction, and percent form.

N03.05 Express a given percent in decimal or fraction form.
N03.06 Express a given decimal in percent or fraction form.
N03.07 Express a given fraction in decimal or percent form.
N03.08 Solve a given problem involving percents mentally, with pencil and paper, or with technology, as appropriate.
N03.09 Solve a given problem that involves finding the percent of a percent.

N04 Students will be expected to demonstrate an understanding of ratio and rate. [C, CN, V]

Performance Indicators

N04.01 Explain the multiplicative relationship found within a ratio.
N04.02 Represent a two-term ratio from a given context concretely and pictorially and record using the forms 3:5 or 3 to 5.
N04.03 Express a three-term ratio from a given context in the forms 4:7:3 or 4 to 7 to 3.
N04.04 Express a part-to-part ratio as a part-to-whole fraction.
N04.05 Identify and describe ratios and rates (including unit rates) from real-life examples and record them symbolically.
N04.06 Express a given rate using words or symbols.
N04.07 Express a given ratio as a percent, and explain why a rate cannot be represented as a percent.

N05 Students will be expected to solve problems that involve rates, ratios, and proportional reasoning. [C, CN, ME, PS, R]

Performance Indicators

N05.01 Explain the meaning of $\frac{a}{b}$ within a given context.
N05.02 Provide a context in which $\frac{a}{b}$ represents a fraction, a rate, a ratio, a quotient, and a probability.
N05.03 Use pictures, models, or manipulatives to make sense of a proportional situation.
N05.04 Differentiate between proportional and non-proportional contexts.
N05.05 Use multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another.
N05.06 Use multiple methods to solve proportional tasks and understand that these methods are related to each other.
N05.07 Use estimation to determine the reasonableness of an answer.
N05.08 Solve a proportion using mental mathematics, pencil and paper, or technology, as appropriate.
N05.09 Solve a given problem involving rate, ratio, or percent using mental mathematics, pencil and paper, or technology, as appropriate.
N05.10 Create problems that are examples of proportional reasoning.

N06 Students will be expected to demonstrate an understanding of multiplying and dividing positive fractions and mixed numbers, concretely, pictorially, and symbolically. [C, CN, ME, PS]

Performance Indicators

N06.01 Identify the operation required to solve a given problem involving positive fractions.
N06.02 Provide a context that requires the multiplying of two given positive fractions.
N06.03 Provide a context that requires the dividing of two given positive fractions.

N06.04 Estimate the product of two given positive proper fractions to determine if the product will be closer to 0, $\frac{1}{2}$, or 1.
N06.05 Estimate the quotient of two given positive fractions, and compare the estimate to whole number benchmarks.
N06.06 Express a given positive mixed number as an improper fraction and a given positive improper fraction as a mixed number.
N06.07 Model multiplication of a positive fraction by a whole number concretely and/or pictorially and record the process.
N06.08 Model multiplication of a positive fraction by a positive fraction concretely and/or pictorially, using an area model, and record the process.
N06.09 Model division of a positive proper fraction by a whole number concretely and/or pictorially and record the process.
N06.10 Model division of a whole number by a positive proper fraction concretely and/or pictorially, using an area model, and record the process.
N06.11 Model division of a positive proper fraction by a positive proper fraction pictorially and record the process.
N06.12 Generalize and apply rules for multiplying and dividing positive fractions, including mixed numbers.
N06.13 Symbolically solve a given problem involving positive fractions, taking into consideration order of operations (limited to problems with positive solutions and that exclude exponents).

N07 Students will be expected to demonstrate an understanding of multiplication and division of integers, concretely, pictorially, and symbolically. [C, CN, PS, R, V]

Performance Indicators

N07.01 Identify the operation required to solve a given problem involving integers.
N07.02 Provide a context that requires multiplying two integers.
N07.03 Provide a context that requires dividing two integers.
N07.04 Model the process of multiplying two integers, using concrete materials or pictorial representations, and record the process.
N07.05 Model the process of dividing an integer by an integer, using concrete materials and/or pictorial representations, and record the process.
N07.06 Generalize and apply a rule for determining the sign of the product and quotient of integers.
N07.07 Solve a given problem involving the division of integers (two-digit by one-digit) without the use of technology.
N07.08 Solve a given problem involving the division of integers (two-digit by two-digit) mentally or with the use of technology, where appropriate.
N07.09 Symbolically solve a given problem involving integers, taking into consideration order of operations when necessary.

Patterns and Relations (PR)

PR01 Students will be expected to graph and analyze two-variable linear relations. [C, ME, PS, R, T, V]

Performance Indicators

PR01.01 Determine the missing value in an ordered pair for a given equation.
PR01.02 Create a table of values by substituting values for a variable in the equation of a given linear relation.
PR01.03 Construct a graph from the equation of a given linear relation (limited to discrete data).
PR01.04 Describe the relationship between the variables of a given graph.

PR02 Students will be expected to model and solve problems, concretely, pictorially, and symbolically, where *a*, *b,* and *c* are integers, using linear equations of the form

- $ax = b$
- $\frac{x}{a} = b,\ a \neq 0$
- $ax + b = c$
- $\frac{x}{a} + b = c,\ a \neq 0$
- $a(x + b) = c$

[C, CN, PS, V]

Performance Indicators

PR02.01 Model a given problem with a linear equation, and solve the equation using concrete models.
PR02.02 Verify the solution to a given linear equation, using a variety of methods, including concrete materials, diagrams, and substitution.
PR02.03 Draw a visual representation of the steps used to solve a given linear equation, and record each step symbolically.
PR02.04 Solve a given linear equation symbolically.
PR02.05 Identify and correct an error in a given incorrect solution of a linear equation.
PR02.06 Apply the distributive property to solve a given linear equation.
PR02.07 Solve a given problem, using a linear equation, and record the process.

Measurement (M)

M01 Students will be expected to develop and apply the Pythagorean theorem to solve problems. [CN, PS, R, T, V]

Performance Indicators

M01.01 Model and explain the Pythagorean theorem concretely, pictorially, or using technology.
M01.02 Explain, using examples, that the Pythagorean theorem applies only to right triangles.
M01.03 Determine whether or not a given triangle is a right triangle by applying the Pythagorean theorem.
M01.04 Determine the measure of the third side of a right triangle, given the measures of the other two sides, to solve a given problem.
M01.05 Solve a given problem that involves Pythagorean triples.

M02 Students will be expected to draw and construct nets for 3-D objects. [C, CN, PS, V]

Performance Indicators

M02.01 Match a given net to the 3-D object it represents.
M02.02 Construct a 3-D object from a given net.
M02.03 Draw nets for a given right cylinder, right rectangular prism, and right triangular prism, and verify by constructing the 3-D objects from the nets.
M02.04 Predict 3-D objects that can be created from a given net, and verify the prediction.

M03 Students will be expected to determine the surface area of right rectangular prisms, right triangular prisms, and right cylinders to solve problems. [C, CN, PS, R, V]

Performance Indicators

M03.01 Explain, using examples, the relationship between the area of 2-D shapes and the surface area of a given 3-D object.
M03.02 Identify all the faces of a given prism, including right rectangular and right triangular prisms.
M03.03 Identify all the faces of a given right cylinder.
M03.04 Describe and apply strategies for determining the surface area of a given right rectangular or right triangular prism.
M03.05 Describe and apply strategies for determining the surface area of a given right cylinder.
M03.06 Solve a given problem involving surface area.

M04 Students will be expected to develop and apply formulas for determining the volume of right rectangular prisms, right triangular prisms, and right cylinders. [C, CN, PS, R, V]

Performance Indicators

M04.01 Determine the volume of a given right prism, given the area of the base.
M04.02 Generalize and apply a rule for determining the volume of right cylinders.
M04.03 Explain the connection between the area of the base of a given right 3-D object and the formula for the volume of the object.
M04.04 Demonstrate that the orientation of a given 3-D object does not affect its volume.
M04.05 Apply a formula to solve a given problem involving the volume of a right cylinder or a right prism.

GEOMETRY (G)

G01 Students will be expected to draw and interpret top, front, and side views of 3-D objects composed of right rectangular prisms. [C, CN, R, T, V]

Performance Indicators

G01.01 Draw and label the top, front, and side views for a given 3-D object on isometric dot paper.
G01.02 Compare different views of a given 3-D object to the object.
G01.03 Predict the top, front, and side views that will result from a described rotation (limited to multiples of 90°), and verify predictions.
G01.04 Draw and label the top, front, and side views that result from a given rotation (limited to multiples of 90°).
G01.05 Build a 3-D block object given the top, front, and side views, with or without the use of technology.
G01.06 Sketch and label the top, front, and side views of a 3-D object in the environment, with or without the use of technology.

G02 Students will be expected to demonstrate an understanding of the congruence of polygons under a transformation. [CN, R, V]

Performance Indicators

G02.01 Determine the coordinates of the vertices of an image following a given combination of transformations of the original figure.

G02.02 Draw the original figure and determine the coordinates of its vertices, given the coordinates of the image's vertices and a description of the transformation (translation, rotation, reflection).

STATISTICS AND PROBABILITY (SP)

SP01 Students will be expected to critique ways in which data is presented. [C, R, T, V]

Performance Indicators

SP01.01 Compare information provided for the same data set by a given set of graphs, including circle graphs, line graphs, bar graphs, and pictographs, to determine the strengths and limitations of each graph.

SP01.02 Identify the advantages and disadvantages of different graphs, including circle graphs, line graphs, bar graphs, and pictographs, in representing a given set of data.

SP01.03 Justify the choice of a graphical representation for a given situation and its corresponding data set.

SP01.04 Explain how the format of a given graph, such as the size of the intervals, the width of the bars, and the visual representation, may lead to misinterpretation of the data.

SP01.05 Explain how a given formatting choice could misrepresent the data.

SP01.06 Identify conclusions that are inconsistent with a given data set or graph, and explain the misinterpretation.

SP02 Students will be expected to solve problems involving the probability of independent events. [C, CN, PS, T]

SP02.01 Determine the probability of two given independent events, and verify the probability using a different strategy.

SP02.02 Generalize and apply a rule for determining the probability of independent events.

SP02.03 Solve a given problem that involves determining the probability of independent events.

Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to

- communicate in order to learn and express their understanding of mathematics (Communication [C])
- develop and apply new mathematical knowledge through problem solving (Problem Solving [PS])
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines (Connections [CN])
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation [ME])

- select and use technologies as tools for learning and solving problems (Technology [T])
- develop visualization skills to assist in processing information, making connections, and solving problems (Visualization [V])
- develop mathematical reasoning (Reasoning [R])

The Nova Scotian curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning. The key to these process standards is presented in a box, as shown below, with each specific curriculum outcome within the strands.

Process Standards Key

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics.

Students also need to communicate their learning using mathematical terminology. Communication can help students make connections between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic—of mathematical ideas. Students must communicate *daily* about their mathematics learning. This enables them to reflect, to validate, and to clarify their thinking and provides teachers with insight into students' interpretations of mathematical meanings and ideas.

Problem Solving [PS]

Learning through problem solving should be the focus of mathematics at all grade levels. When students encounter new situations and respond to questions of the type, How would you ...? or How could you ...? the problem-solving approach is being modelled. Students develop their own problem-solving strategies by being open to listening, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement, perseverance, and collaboration.

Problem solving is also a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for and engage in finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive, mathematical risk takers.

When students are exposed to a wide variety of problems in all areas of mathematics, they explore various methods for solving and verifying problems. In addition, they are challenged to find multiple solutions for problems and to create their own problem.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to one another or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding. … *Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*." (Caine and Caine 1991, 5).

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates learning. Concepts and skills should also be connected to everyday situations and other curricular areas. For example, when developing literacy skills students learn to make text-to-world, text-to-text, and text-to-self connections. Students can also make connections to make mathematics come alive through math-to-world, math-to-math, and math-to-self connections.

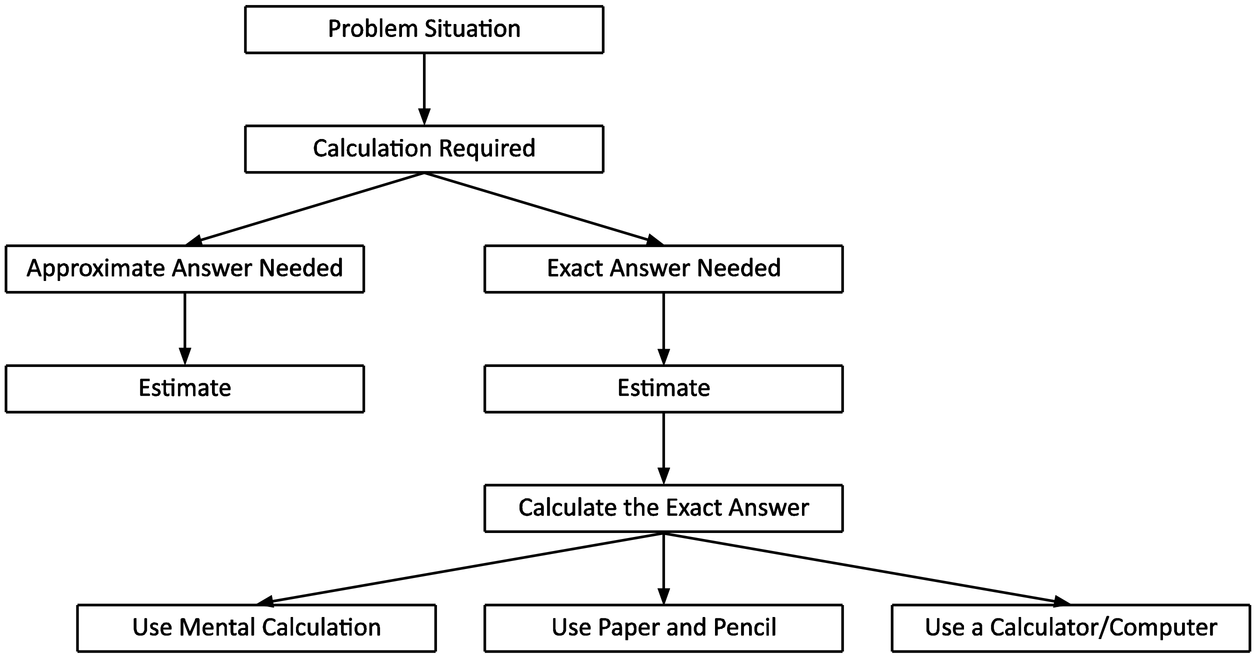
Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external aids. Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy, and flexibility. "Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math." (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers, and are more able to use multiple approaches to problem solving." (Rubenstein 2001) Mental mathematics "provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers." (Hope et al. 1988, v)

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when, and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process as illustrated below.



The skill of estimation requires a sound knowledge of mental mathematics. Both are necessary to many everyday experiences, and students should be provided with frequent opportunities to practise these skills.

Technology [T]

Technology can be effectively used to contribute to and support the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Technology can be used to

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- decrease the time spent on computations when other mathematical learning is the focus
- reinforce the learning of basic facts and test properties
- develop personal procedures for mathematical operations
- create geometric displays
- simulate situations
- develop number sense

The use of calculators is recommended to enhance problem solving, to encourage discovery of number patterns, and to reinforce conceptual development and numerical relationships. They do not, however, replace the development of number concepts and skills. Carefully chosen computer software can provide interesting problem-solving situations and applications.

Technology contributes to a learning environment in which the growing curiosity of students can lead to rich mathematical discoveries at all grade levels. While technology can be used in grades primary to 3 to enrich learning, it is expected that students will achieve all outcomes without the use of technology.

Visualization [V]

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world.” (Armstrong 1999). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers. These mental images are needed to develop concepts and understand procedures. Images and explanations help students clarify their understanding of mathematical ideas in all strands.

Being able to create, interpret, and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and to know several estimation strategies. (Shaw and Cliatt 1989)

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. High-order questions challenge students to think and develop a sense of wonder about mathematics. Mathematical experiences in and out of the classroom provide opportunities for inductive and deductive reasoning. Inductive reasoning occurs when students explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Deductive reasoning occurs when students reach new conclusions based upon what is already known or assumed to be true.

Mathematics reasoning involves informal thinking, conjecturing, and validating—these help students understand that mathematics makes sense. Students are encouraged to justify, in a variety of ways, their solutions, thinking processes, and hypotheses. In fact, good reasoning is as important as finding correct answers.

Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics, and these are woven throughout this document. These components include change, constancy, number sense, relationships, patterns, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen 1990, 184)

Constancy

Different aspects of constancy are described by the terms **stability**, **conservation**, **equilibrium**, **steady state**, and **symmetry** (American Association for the Advancement of Science 1993, 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy (British Columbia Ministry of Education 2000, 146). A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, and concepts. The search for possible relationships involves the collection and analysis of data, and describing relationships visually, symbolically, orally, or in written form.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all strands, and it is important that connections are made among strands. Working with patterns enables students to make connections within and beyond mathematics. These skills contribute to students' interaction with an understanding of their environment. Patterns may be represented in concrete, visual, or symbolic form. Students should develop fluency in moving from one representation to another. Students must learn to recognize, extend, create, and use mathematical patterns. Patterns allow students to make predictions and justify their reasoning when solving problems. Learning to work with patterns in the early grades helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics in higher grades.

Spatial Sense

Spatial sense involves visualization, mental imagery, and spatial reasoning. These skills are central to the understanding of mathematics. Spatial sense enables students to reason and interpret among and between 3-D and 2-D representations and identify relationships to mathematical strands. Spatial sense is developed through a variety of experiences and interactions within the environment. The development of spatial sense enables students to solve problems involving 3-D objects and 2-D shapes. Spatial sense offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations. Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions. For example,

- knowing the dimensions of an object enables students to communicate about the object and create representations
- the volume of a rectangular solid can be calculated from given dimensions
- doubling the length of the side of a square increases the area by a factor of four

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately.

Curriculum Document Format

This guide presents the mathematics curriculum so that a teacher may readily view the scope of the outcomes that students are expected to achieve during the year. Teachers are encouraged to examine teaching and learning that precedes and following this grade level to better understand how students' learning at a particular grade level is part of a bigger picture of concept and skill development.

The order of the outcomes does not prescribe a preferred order of presentation in the classroom, but provides the specific curriculum outcomes in relation to the overarching general curriculum outcomes (GCOs).

The table on the next page provided the structure of each specific curriculum outcomes section. When a specific curriculum outcome (SCO) is introduced, it is followed by the mathematical processes and performance indicators for that outcome. A scope and sequence is then provided, which relates the SCO to previous and next grade SCOs. Also for each SCO, there is background information, assessment strategies, suggested instructional strategies, suggested models and manipulatives, mathematical language, and a section for resources and notes. For each section, the guiding questions should be used to help with unit and lesson preparation.

Assessment Strategies

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcome and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO		
Mathematical Processes		
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections
[ME] Mental Mathematics and Estimation		
[T] Technology	**[V]** Visualization	**[R]** Reasoning

Performance Indicators

Describes observable indicators of whether students have achieved the specific outcome.

Scope and Sequence

Previous grade or course SCOs	Current grade SCO	Following grade or course SCOs

Background

Describes the “big ideas” to be learned and how they relate to work in previous grade and work in subsequent courses.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge
Sample tasks that can be used to determine students’ prior knowledge.

Whole-Class/Group/Individual Assessment Tasks
Some suggestions for specific activities and questions that can be used for both instruction and assessment

Follow-up on Assessment

Planning for Instruction

Choosing Instructional Strategies

Suggested strategies for planning daily lessons.

Suggested Learning Tasks

Suggestions for general approaches and strategies suggested for teaching this outcome.

Suggested Models and Manipulatives

Mathematical Language

Teacher and student mathematical language associated with the respective outcome.

Resources/Notes

Contexts for Learning and Teaching

Beliefs about Students and Mathematics Learning

“Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.” (National Council of Teachers of Mathematics 2000, 20).

The Nova Scotia mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning that have grown out of research and practice. These beliefs include the following:

- Mathematics learning is an active and constructive process.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.
- Learners are individuals who bring a wide range of prior knowledge and experiences and who learn via various styles and at different rates.
- Learning is most likely to occur when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and that nurtures positive attitudes and sustained effort.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Students develop a variety of mathematical ideas before they enter school. Children make sense of their environment through observations and interactions at home and in the community. Mathematics learning is embedded in everyday activities, such as playing, reading, storytelling, and helping around the home. Such activities can contribute to the development of number and spatial sense in children. Curiosity about mathematics is fostered when children are engaged in activities such as comparing quantities, searching for patterns, sorting objects, ordering objects, creating designs, building with blocks, and talking about these activities. Positive early experiences in mathematics are as critical to child development as are early literacy experiences.

Students learn by attaching meaning to what they do and need to construct their own meaning of mathematics. This meaning is best constructed when learners encounter mathematical experiences that proceed from the simple to the complex and from the concrete to the abstract. The use of models and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students and enhance the formation of sound, transferable, mathematical concepts. At all levels, students benefit from working with and translating through a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful discussions can provide essential links among concrete, pictorial, contextual, and symbolic representations of mathematics.

The learning environment should value and respect all students’ experiences and ways of thinking, so that learners are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore problem-solving situations in order to develop personal strategies and become mathematically literate. Learners must realize that it is acceptable to solve problems in different ways and that solutions may vary.

Goals for Mathematics Education

The main goals of mathematics education are to prepare students to

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- become mathematically literate adults, using mathematics to contribute to society

Students who have met these goals will

- gain understanding and appreciation of the contributions of mathematics as a science, a philosophy, and an art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for student success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations, and engage in reflective practices.

To experience success, students must be taught to set achievable goals or assess their progress as they work toward these goals. Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Engaging All Learners

“No matter how engagement is defined or which dimension is considered, research confirms this truism of education: *The more engaged you are, the more you will learn*.” (Hume 2011, 6)

Student engagement is at the core of learning. This is critical for teachers to take into account when planning and implementing instruction. Effective instruction engages, embraces, and supports all learners through a range of learning experiences. This curriculum is designed to provide learning opportunities that reflect culturally proficient instructional and assessment practices and are equitable, accessible, and inclusive of the multiple facets of diversity represented in today’s classrooms.

Engagement in learning occurs when students are provided with opportunities to become more invested in their learning. When teachers know their students as individual learners and as individual people, their students are more likely to be motivated to learn, participate in classroom activities, persist in challenging situations, and engage in reflective practices. Students often become more engaged when teachers demonstrate a genuine belief in each student’s potential to learn.

SUPPORTIVE LEARNING ENVIRONMENTS

A supportive and positive learning environment has a profound effect on students' learning. In classrooms where students feel a sense of belonging, are encouraged to actively participate, are challenged without being frustrated, and feel safe and supported to take risks with their learning, students are more likely to experience success. It is realized that not all students will progress at the same pace or be equally positioned in terms of their prior knowledge of and skill with particular concepts and outcomes. Teachers provide all students with equitable access to learning by integrating a variety of instructional approaches and assessment activities that consider all learners and align with the following key principles:

- Instruction must be flexible and offer multiple means of representation.
- Students must have opportunities to express their knowledge and understanding in multiple ways.
- Teachers must provide options for students to engage in learning through multiple ways.

Teachers who know their students well become aware of individual learning differences and infuse this understanding into planned instructional and assessment decisions. They organize learning experiences to accommodate the many ways in which students learn, create meaning, and demonstrate their knowledge and understanding. Teachers use a variety of effective teaching approaches that may include

- providing all students with equitable access to appropriate learning strategies, resources, and technology
- offering a range of ways students can access their prior knowledge to connect with new concepts
- scaffolding instruction and assignments so that individual or groups of students are supported as needed throughout the process of learning
- verbalizing their thinking to model comprehension strategies and new learning
- balancing individual, small-group, and whole-class approaches to learning activities
- involving students in the co-creation of criteria for assessment and evaluation
- providing students with choice in how they demonstrate their understanding according to learning styles and preferences, building on individual strengths, and including a range of difficulty and challenge
- providing frequent and meaningful feedback to students throughout their learning experiences

LEARNING STYLES AND PREFERENCES

The ways in which students make sense of, receive, and process information, demonstrate learning, and interact with peers and their environment both indicate and shape learning preferences, which may vary widely from student to student. Learning preferences are influenced also by the learning context and purpose and by the type and form of information presented or requested. Most students tend to favour one learning style and may have greater success if instruction is designed to provide for multiple learning styles, thus creating more opportunities for all students to access learning. The three most commonly referenced learning styles are:

- auditory (such as listening to teacher-presented lessons or discussing with peers)
- kinesthetic (such as using manipulatives or recording print or graphic/visual text)
- visual (such as interpreting information with text and graphics or viewing videos)

While students can be expected to work using all modalities, it is recognized that one or some of these modalities may be more natural to individual students than the others.

A Gender-Inclusive Curriculum

It is important that the curriculum respects the experiences and values of all students and that learning resources and instructional practices are not gender-biased. Teachers promote gender equity and inclusion in their classrooms when they

- articulate equally high expectations for all students
- provide equal opportunity for input and response from all students
- model gender-fair language and respectful listening in their interactions with students

Valuing Diversity: Teaching with Cultural Proficiency

Teachers understand that students represent diverse life and cultural experiences, with individual students bringing different prior knowledge to their learning. Therefore, teachers build upon their knowledge of their students as individuals and respond by using a variety of culturally-proficient instruction and assessment strategies. "Instruction that is embedded in socially meaningful contexts, and tasks that are meaningful and relevant to the lives of students, will engage students in high-level problem-solving and reasoning and enhance students' engagement (Frankenstein 1995; Gutstein 2003; Ladson-Billings 1997; Tate 1995)." (Herzig 2005)

Students with Language, Communication, and Learning Challenges

Today's classrooms include students who have diverse backgrounds, abilities, levels of development, and learning challenges. By observing and interacting with students as they work on assigned activities, teachers can identify areas where students may need additional support to achieve their learning goals. Teachers can then respond with a range of effective instructional strategies. Students who have English as an Additional Language (EAL) may require curriculum outcomes at different levels, or temporary individualized outcomes, particularly in language-based subject areas, while they become more proficient in their English language skills. For students who are experiencing difficulties, it is important that teachers distinguish between students for whom curriculum content is challenging and students for whom language-based issues are at the root of apparent academic difficulties.

Students who Demonstrate Gifted and Talented Behaviours

Some students are academically gifted and talented with specific skill sets or in specific subject areas. Most students who are gifted and talented thrive when challenged by problem-centred, inquiry-based learning and open-ended activities. Teachers may challenge students who are gifted and talented by adjusting the breadth, the depth, and/or the pace of instruction. Learning experiences may be enriched by providing greater choice among activities and offering a range of resources that require increased cognitive demand and higher-level thinking at different levels of complexity and abstraction. For additional information, refer to *Gifted Education and Talent Development* (Nova Scotia Department of Education 2010).

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in health education, literacy, music, physical education, science, social studies, and visual arts.

Number (N)

GCO: Students will be expected to demonstrate number sense.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Long-term Planning

- Yearly plan available at Mathematics Learning Commons: Grades 7–9: http://nsvs.ednet.ns.ca/nsps/nsps26/course/view.php?id=3875.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO N01 Students will be expected to demonstrate an understanding of perfect squares and square roots, concretely, pictorially, and symbolically (limited to whole numbers). [C, CN, R, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N01.01 Represent a given perfect square as a square region, using materials such as grid paper or square shapes.
N01.02 Determine the factors of a given perfect square, and explain why one of the factors is the square root and the others are not.
N01.03 Determine whether or not a given number is a perfect square, using materials and strategies such as square shapes, grid paper, or prime factorization, and explain the reasoning.
N01.04 Determine the square root of a given perfect square, and record it symbolically.
N01.05 Determine the square of a given number.

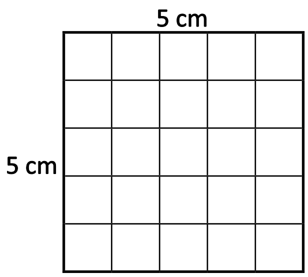
Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
N01 Students will be expected to determine and explain why a number is divisible by 2, 3, 4, 5, 6, 8, 9, or 10, and why a number cannot be divided by 0.	**N01** Students will be expected to demonstrate an understanding of perfect squares and square roots, concretely, pictorially, and symbolically (limited to whole numbers).	**N05** Students will be expected to determine the exact square root of positive rational numbers. **N06** Students will be expected to determine an approximate square root of positive rational numbers.

Background

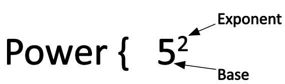
Students should recognize that a square is a quadrilateral with four congruent sides and four right angles. In Mathematics 7, students used formulas for finding the areas of certain quadrilaterals (squares, rectangles, and parallelograms) and used cm^2, m^2, mm^2, etc., for the units of area.

A perfect square is the product of two identical factors. Perfect squares, or square numbers, can be specifically connected to the areas of squares. In the figure below, students should be encouraged to view the area as the perfect square, and either dimension of the square as the square root.



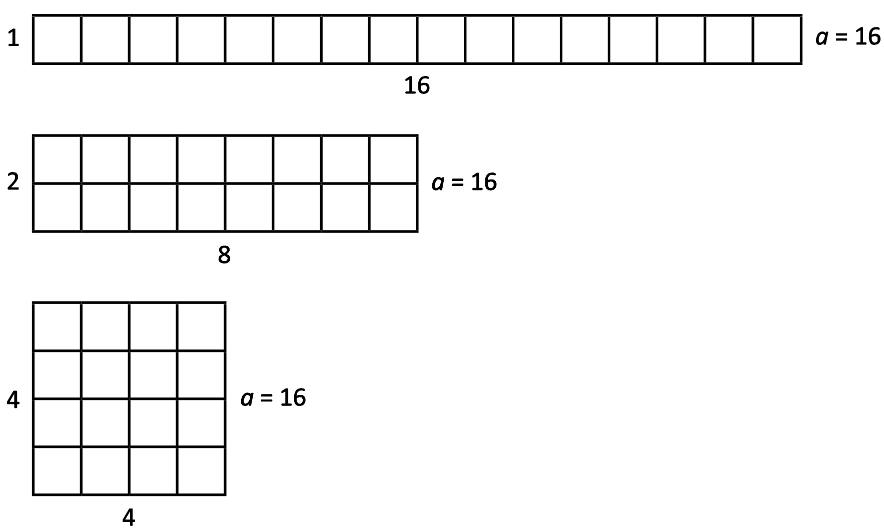
The area of this square is 5 cm x 5 cm = 25 cm^2. Therefore, 25 is a perfect square and 5 is its square root. A square root is a number that when multiplied by itself equals a given value, called the perfect square. Each perfect square has a positive and a negative square root. However, because this outcome is limited to whole numbers, the principal (positive) square root is the focus.

The use of exponents is introduced and limited to perfect squares in this outcome. For example 64 = 8^2.
$5^2 = 25$ is read as “5 to the exponent of 2 is 25” or “5-squared is 25.”
The number 5 is called the base, 2 is called the exponent, and 5^2 is called a power.



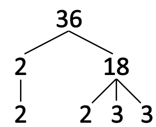
The expectation is that students are only introduced to square numbers. They are not exploring all powers. Exponential notation will be studied more thoroughly in Mathematics 9.

Students should be given opportunity to identify perfect squares using concrete and pictorial representations before moving into symbolic work with square roots. For example, give students a set area and have them construct as many rectangular regions as possible using grid paper, dot paper, or geoboards. Consider an area of 16 cm^2:



Have students decide if any of their rectangles are squares and use this square to make the connection between perfect squares and square roots by encouraging students to view the area as the perfect square and either dimension as the square root. Examples should not be limited to perfect squares. Have students complete this exercise for non-square numbers as well. This will help avoid the misconception that all numbers are square numbers.

Prime factorization can also be used to determine whether or not a number is a perfect square. This method builds on what students learned about prime factors and factor trees in Mathematics 6. A prime number is a whole number greater than 1 that has exactly two factors, 1 and itself. A composite number is a whole number greater than 1 that has more than two factors. Every composite number can be written as the product of prime numbers in exactly one way (if the order of the factors is ignored). This is called the prime factorization of the number. A factor tree can be used to list the prime factors.



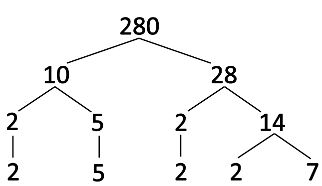
A perfect square has each distinct prime factor occurring an even number of times. Since $36 = 2 \cdot 2 \cdot 3 \cdot 3$, 36 is a perfect square. Since 36 also has factors 3 x 12, 4 x 9, 6 x 6 etc., there are other prime factorizations possible but all will result in $36 = 2 \cdot 2 \cdot 3 \cdot 3$.

Also prime factorization can be used to find the square root of perfect squares. Since each of the distinct prime factors occur an even number of times, they can be arranged in pairs.

$36 = 2 \cdot 2 \cdot 3 \cdot 3 = (2 \cdot 2) \times (3 \cdot 3)$
$\therefore \sqrt{36} = \sqrt{(2 \cdot 2)} \times \sqrt{(3 \cdot 3)}$
$\sqrt{36} = 2 \cdot 3$
$\sqrt{36} = 6$

It is in Mathematics 8 that students are introduced to using the dot ($\cdot$) as a way to indicate multiplication. Prior to this year, students used × when writing multiplication statements. Make students aware of this new notation for multiplication.

The prime factorization method can also be used to demonstrate that a number is not a perfect square. From the factor tree below, notice that none of the prime factors of 280 are present an even number of times. Therefore, 280 is not a perfect square.



$280 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7$

Students should be able to recognize each of the perfect squares from 1 through 144. This automatic recognition will be very useful when determining the reasonableness of results that involve square roots found using a calculator. It will also be valuable in later work with algebra and number theory.

It is also valuable to use patterns to determine square roots of larger numbers. Knowing that the square root of 25 is 5, can be used to determine that the square root of 2500 is 50.

Students have determined the factors of a number in previous grades through systematic trial and the application of the divisibility rules. To find a square root by using a list of factors, first arrange the factors in ascending order. Consider the factors of 36, a perfect square. Notice there is an odd number of factors.

middle number

1, 2, 3, 4, 6, 9, 12, 18, 36

1 × 36 = 36
2 × 18 = 36
3 × 12 = 36
4 × 9 = 36
6 × 6 = 36

1 2 3 4 6 9 12 18 36

The middle factor is the square root because the 6 will be multiplied by itself.
Since the 6 cannot be paired with a different number as a factor, it is the square root of 36. This is written as $\sqrt{36}$ = 6. This notation, $\sqrt{x}$, is called radical notation and is new to students in Mathematics 8. They must be introduced to the $\sqrt{\ }$ symbol to represent positive square roots.

This same method can also be used to determine when a number is not a perfect square. After examining the factors of a given number, students should conclude that if that number has an even number of factors, it is not a square number. For example, the factors of 35 are 1, 5, 7, 35. Since none of these factors pair with another identical factor to multiply to give 35, it (35) is not a perfect square.

The methods previously discussed are effective ways to identify perfect squares and to determine their square roots. It should be noted that none of these methods require the use of a calculator. Calculator usage can be discussed, but the focus should be on non-technological techniques referenced in the outcome.

Ultimately, using the various techniques, students should be able to make statements such as:

$\sqrt{49}$ = 7	or	7^2 = 49
$\sqrt{100}$ = 10	or	10^2 = 100
$\sqrt{144}$ = 12	or	12^2 = 144

A discussion of inverse operations is appropriate here. Students should be able to identify squaring a number and taking a square root as inverse operations. Ask them to consider other inverse operations (addition and subtraction, multiplication and division, etc.).

Assessment, Teaching, and Learning

Assessing Prior Knowledge

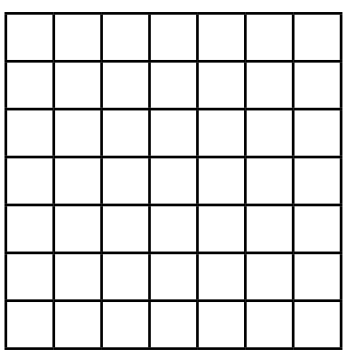
Tasks such as the following could be used to determine students' prior knowledge.

- Ask students to name the properties of a square and the properties of a rectangle. Ask them to describe the difference.
- Explore the sieve of Eratosthenes to identify the prime numbers to 100. Ask students to discuss any patterns they notice.
- Ask students to draw two different factor trees for 56. Ask them to explain why it is possible to draw two different factor trees for each number.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Tell students that a dance floor is square and has an area of 81 m^2. What are its dimensions?
- Explain how to determine each square root.
  - $\sqrt{15 \times 15}$
  - $\sqrt{2 \times 5 \times 2 \times 5}$
- Describe two strategies to calculate $\sqrt{196}$.
- 361 has only 3 factors: 1, 19, and 361. Explain how you can use this information to show that 361 is a perfect square.
- Tell students that Lydia listed all the factors of 7569 and wrote: 1, 3, 9, 87, 841, 2523, 7569. How can you determine a square root of 7569 using Lydia's list of factors?
- Have students explain why 97 is not a perfect square.
- Have students find a square root of 324 using prime factorization.
- Tell students that the prime factorization for a number is 2 × 2 × 3 × 3 × 7 × 7. Ask, what is the number, and what is its square root?
- Determine what number and its square root can be represented by this grid. Explain.



- Is there a perfect square between 900 and 961? Explain. Would you use prime factors to determine whether 900 is a perfect square? Why or why not?
- Explain why the Prime Factorization Method cannot be used to find a whole number square root for numbers that are not perfect squares.
- Ask students: Ruth wants a large picture window put in the living room of her new house. The window is to be square with an area of 49 square feet. How long should each side of the window be?
- Ask students: The side length of a square is 11 cm. What is the area of the square?
- Ask students: A miniature portrait of your family is square and has an area of 196 square centimetres. How long is each side of the portrait?

CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

- Investigate the inverse relationship between a square (3^2) and a square root ($\sqrt{9}$).
- Have students model perfect squares using 2-D square tiles, and identify the perfect square and its factors.
- Provide students with many opportunities to explore a variety of concrete and pictorial models of perfect squares.
- Explore patterns related to perfect squares (e.g., the sum of the square roots of two perfect squares is equivalent to the difference between those two consecutive perfect squares), such as:

  $\sqrt{36} + \sqrt{25}$ = 6 + 5 = 11 *and* 36 – 25 = 11

  Use patterns to determine that the square root of 1600 is 40, since the square root of 16 is 4.

  Or, verify this: $\sqrt{1600} = \sqrt{16} \times \sqrt{100} = 4 \times 10 = 40$

Suggested Learning Tasks

- Provide students with 25 colour tiles. Have the students explore the number of rectangles they could make using 24 of the tiles and then using 25 tiles. Students should discover that they are only able to make a perfect square with the 25 tiles.
- Use the factors of a number, for example 36 or 45, to determine if the number is a perfect square or use the factors of several numbers to see if a pattern exists that will help predict whether or not the number is a perfect square. (Note: The pattern is that perfect squares have an odd number of factors and the middle factor is a square root.)
- Use grid paper or colour tiles to model all the perfect squares less than 150.
- Apply an effective strategy to determine the square root of a number. Determine when patterning is more effective than prime factorization.
- Find a square root of each of the following and justify your strategy: 900, 6400, 12 100, 676
- Investigate the pattern and reach a conclusion about the ones digit of perfect squares. (All perfect squares end in 1, 4, 9, 6, 5, or 0). Ask students why they think $\sqrt{\_\ \_8}$ cannot be a perfect square. Ask students if $\sqrt{\_\ \_6}$ will always be a perfect square.

  Create a factor list for the following and numbers decide if the original number is a perfect square, and if so, identify a square root: 2, 5, 9, 12, 16, 20, 81
- The factors of 81 are 1, 3, 9, 27, and 81. Use words and/or diagrams to explain how you know if 81 is a perfect square and, if so, which factor is the square root of 81.

Suggested Models and Manipulatives

- calculator
- colour tiles*
- geoboards or digital geoboard*
- grid paper
- number line*
- square dot paper or digital square dot paper

*also available in *Interactive Math Tools* (Pearson n.d.)

Mathematical Language

Teacher	Student
▪ base	▪ base
▪ exponent	▪ exponent
▪ factors	▪ factors
▪ perfect square	▪ perfect square
▪ power	▪ power
▪ prime factorization	▪ prime factorization
▪ prime numbers	▪ prime numbers
▪ principle square root	
▪ squared	▪ squared
▪ square root	▪ square root

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 1: Square Roots and the Pythagorean Theorem
  - Section 1.1: Square Numbers and Area Models
  - Section 1.2: Squares and Square Roots
  - Section 1.3: Measuring Line Segments
  - Section 1.5: The Pythagorean Theorem
  - Section 1.7: Applying the Pythagorean Theorem
  - Unit Problem: The Locker Problem
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 150.

SCO N02 Students will be expected to determine the approximate square root of numbers that are not perfect squares (limited to whole numbers). [C, CN, ME, R, T]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N02.01 Estimate the square root of a given number that is not a perfect square, using materials such as square shapes and graph paper and strategies such as using the roots of perfect squares as benchmarks.
N02.02 Approximate the square root of a given number that is not a perfect square using technology (e.g., a calculator or a computer).
N02.03 Explain why the square root of a number shown on a calculator may be an approximation.
N02.04 Identify a number with a square root that is between two given numbers.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
–	**N02** Students will be expected to determine the approximate square root of numbers that are not perfect squares (limited to whole numbers).	**N05** Students will be expected to determine the exact square root of positive rational numbers. **N06** Students will be expected to determine an approximate square root of positive rational numbers.

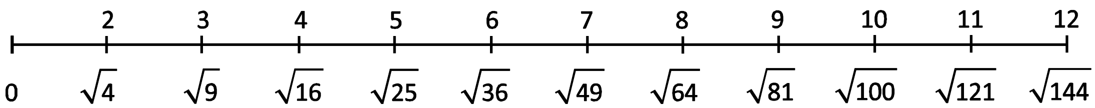
Background

As the discussion of perfect squares develops, students should notice that square numbers get farther apart on a number line as the numbers increase in value. That is, there are many whole numbers that are not perfect squares. It is very important to emphasize the difference between an exact square root and a decimal approximation of a square root.

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values.

To estimate a square root, identify the two consecutive whole numbers that the square root falls between. For example, when estimating the square root of 12, identify between which two consecutive perfect squares 12 falls (9 and 16). The square root of 12 is between $\sqrt{9}$ and $\sqrt{16}$, or between 3 and 4.

One effective model for estimating square roots is the number line. For numbers between 1 and 144, students should use benchmarks (roots of perfect square numbers) to identify between which two consecutive whole numbers the square root will fall, and to which whole number it is closer.

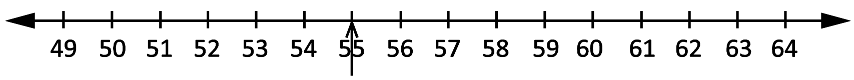


To estimate the square root of 55 using a number line, identify the two perfect squares closest to 55; the perfect square that is less than 55 is 49, and the perfect square that is greater than 55 is 64. Identify their square roots as 7 and 8 respectively. The square root of 55 must be between 7 and 8.

$\sqrt{49} < \sqrt{55} < \sqrt{64}$

$7 < \sqrt{55} < 8$

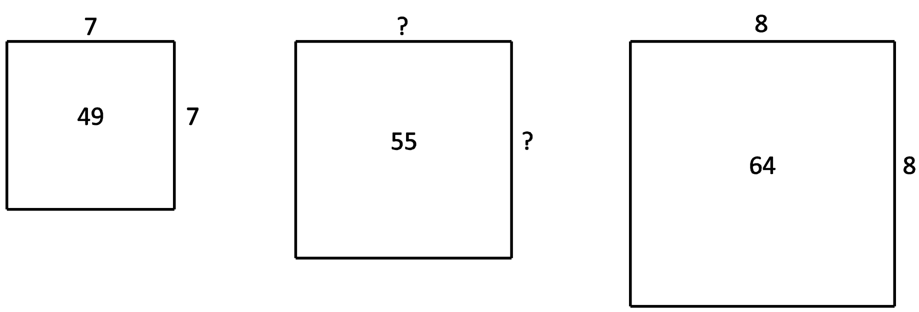
Develop the concept that numbers that are not perfect squares have approximate square roots that are decimal approximations. Draw another number line from 49 to 64 for students to be able to better approximate if 55 is closer to 49 or 64.



Since 55 is closer to 49, and is a little less than one-half of the way between 49 and 64, a good approximation of $\sqrt{55}$ is 7.4.

Alternatively, consider the following approach. To estimate $\sqrt{55}$ to one decimal place, students should recognize that 55 lies between the perfect squares 49 and 64. Therefore, the square root of 55 must be between 7 and 8. Since 55 is a little less than one-half of the way between 49 and 64, we can estimate that the square root of 55 is about one-half of the way between 7 and 8. Therefore, a good approximation of 55 is 7.4.

This method can be illustrated as follows (Van de Walle 2006, 150):



Any whole number between 49 and 64 has a square root between 7 and 8. There is more than a single correct answer. Using patterns and estimation, students should also recognize that the square root of 3200 is between 50 and 60, but closer to 60.

Calculators provide an efficient means of approximating square roots. They also provide a good opportunity to emphasize the difference between exact and approximate values.

Square roots can be approximated with calculators to any requested number of decimal places using rounding strategies.

$\sqrt{87} = 9.3$
$\sqrt{87} = 9.33$
$\sqrt{87} = 9.327$

Note: The symbol used in this document to represent "approximately equal to" is $=$. There are other symbols, such as $\approx$, that other resources may use.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Tasks such as the following could be used to determine students' prior knowledge.

- List the perfect squares from 1 to 144 and their square roots.
- Each of Eli's four friends has a code number. Keile's number is divisible by 3, 5, and 8. Max's number is divisible by 2 and 3. Jennifer's number is divisible by 4 and 5, but not 3. Ben's number is divisible by 3 and 5, but not 8. Eli receives a message with the code number 5384 from one of his four friends. Ask students to determine who sent the message.

Whole-Class/Group/Individual Assessment Tasks

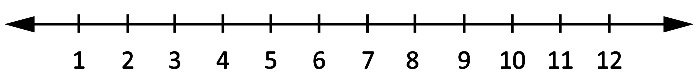
Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- If a whole number has an approximate square root of 5.66, is the whole number closer to 25 or 36? How do you know?
- In your own words, explain how you would estimate the square root of 75.
- Jim measures each side of his mother's vegetable garden to be 3.2 m. Explain how Jim could reasonably estimate the area of the vegetable garden.
- Estimate each square root to the nearest tenth.
  - $\sqrt{14}$
  - $\sqrt{35}$
  - $\sqrt{65}$

- $\sqrt{98}$

- Estimate to determine whether each answer is reasonable. Circle any that are unreasonable and modify the estimation. Justify your reasoning. Check your prediction using your calculator.
  - $\sqrt{11} = 3.3$
  - $\sqrt{27} = 9.5$
  - $\sqrt{46} = 6.8$
  - $\sqrt{82} = 9.6$
  - $\sqrt{99} = 10.1$

- Use a calculator to determine the square roots below and identify which of the numbers are perfect squares.
  - 2525
  - 1681
  - 999
- Ask students: If a whole number has an approximate square root of 7.75, is the whole number closer to 49 or 64? How do you know?
- Have students explain how they would estimate the square root of 40.
- Tell students that while Rebecca was shopping online she found a square rug with an area of 17 m^2. The dimensions of her bedroom are 4 m × 5 m. Will the rug fit in her room? Explain.
- Have students identify a whole number with a square root between 9 and 10.
- Ask students to locate the following numbers on a number line: $\sqrt{4}\sqrt{9}\sqrt{36}\sqrt{81}\sqrt{144}$



FOLLOW-UP ON ASSESSMENT

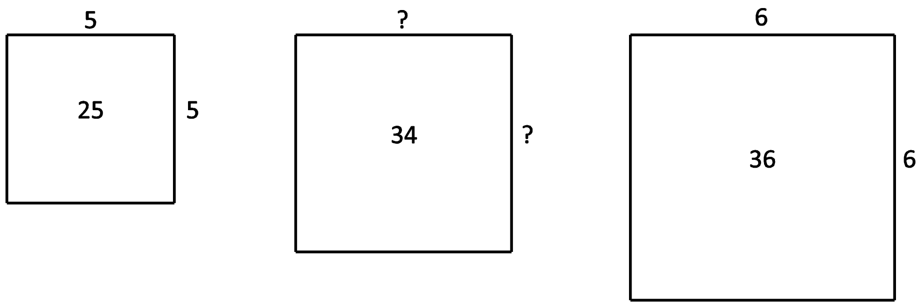
Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

- Ensure that students are comfortable with the perfect square benchmarks from 1 to 144 as these are used to establish an initial estimate when finding a square root. A number line is a very useful model for this.
- Have students draw squares to help them visualize an estimate of a square root between two perfect squares (Van de Walle and Lovin, 2006c, 150).

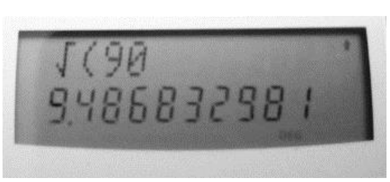


- Use a calculator to estimate the square root of a non-perfect square without using the $\sqrt{\ }$ key. If students are asked to estimate the square root of 110, they should know it is about halfway between 10 and 11, since 110 is almost halfway between 100 and 121. They might try 10.4 × 10.4 and then 10.5 × 10.5 on the calculator to determine which is closer to 110.

Suggested Learning Tasks

- Place pairs of numbers on the board. Ask students to use the strategy and identify a whole number whose square root lies between two given numbers. Have students write their answers on a card, or paper, and hold them up as a group. Discuss why all of the answers are not the same. (This could also be done with an interactive whiteboard and tablets.)
- Give students 22 colour tiles and have them attempt to build a square. What is the largest square they can build with the tiles and what does this tell them about the approximate square root of 22? To what whole number is the square root closer?
- Have students identify a whole number with a square root of approximately 4.9.
- Place pairs of numbers on the board using the numbers 2 to 9. Have students identify a whole number whose square root lies between two given numbers (e.g., if given 3 and 4, students could write any number from 10 to 15). Have students write their answers on a mini-whiteboard, card, paper, or tablet, and hold them up as a group. Discuss why the answers may not be all the same.
- Tell students that Nate was asked to estimate $\sqrt{62}$. He did not have his calculator. Demonstrate how Nate could estimate to the nearest tenth using his knowledge of perfect squares.
- Use a calculator to approximate these square roots, and identify which of the numbers under the radical signs are perfect squares.
  - $\sqrt{1600}$
  - $\sqrt{1681}$
  - $\sqrt{1212}$
  - $\sqrt{1000}$
  - $\sqrt{2468}$

- Kevin used his calculator to find the square roots of 90 and 169. Respectively his answers were:



Are these exact answers? Explain your reasoning.

- Emily wanted to find the area of a rectangle with a length of 9 cm. She knew that the width of the rectangle was the same as the lengths of the sides of an adjacent square. The area of the square was 38 cm^2. To find the side lengths of the square, she used her calculator as follows: $\sqrt{38} = 6.2$. Therefore the area of the rectangle is 9 cm $\times$ 6.2 cm = 55.8 cm^2. Andrew solved the same problem as follows: $\sqrt{38} \times 9 = 55.5$ cm^2. Account for the difference in results.

SUGGESTED MODELS AND MANIPULATIVES

- Calculator*
- colour tiles*
- geoboards or digital geoboards*
- 10 x 10 grid paper
- number line*

*also available in *Interactive Math Tools* (Pearson n.d.

MATHEMATICAL LANGUAGE

Teacher	Student
▪ approximate	▪ approximate
▪ base	▪ base
▪ estimate	▪ estimate
▪ exponent	▪ exponent
▪ factors	▪ factors
▪ perfect square	▪ perfect square
▪ power	▪ power
▪ squared	▪ squared
▪ square root	▪ square root

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: TBD)

- Unit 1: Square Roots and the Pythagorean Theorem
  - Section 1.4: Estimating Square Roots
  - Technology: Investigating Square Roots with a Calculator

  - Section 1.5: The Pythagorean Theorem
  - Section 1.7: Applying the Pythagorean Theorem
- *ProGuide* (CD [Word files]; NSSBB #: TBD)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD; NSSBB #: TBD)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006c), p. 150.

SCO N03 Students will be expected to demonstrate an understanding of and solve problems involving percents greater than or equal to 0%. [CN, ME, PS, R, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N03.01 Provide contexts where a percentage may be between 0% and 1%, between 1% and 100%, and more than 100%.
N03.02 Represent a given fractional percentage using concrete materials and pictorial representations.
N03.03 Represent a given percentage greater than 100% using concrete materials and pictorial representations.
N03.04 Determine the percentage represented by a given shaded region on a grid, and record it in decimal, fractional, and percent form.
N03.05 Express a given percent in decimal or fraction form.
N03.06 Express a given decimal in percent or fraction form.
N03.07 Express a given fraction in decimal or percent form.
N03.08 Solve a given problem involving percents mentally, with pencil and paper, or with technology, as appropriate.
N03.09 Solve a given problem that involves finding the percent of a percent.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
N03 Students will be expected to solve problems involving percents from 1% to 100% (limited to whole numbers).	**N03** Students will be expected to demonstrate an understanding of and solve problems involving percents greater than or equal to 0%.	—

Background

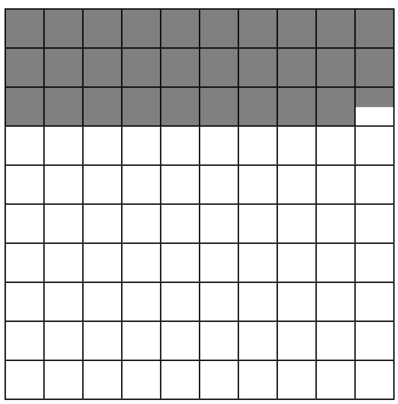
Percentages are ratios that compare a number to 100. Percentages can range from 0 to higher than 100. In Mathematics 7 (N03) students represent a quantity as a percentage, fraction, decimal, or ratio. Percentages have the same value as their fraction, decimal, and ratio equivalent, and this can be useful in solving problems with percentages.

In Mathematics 7 (N03), students worked with percentages from 1% to 100%. In Mathematics 8, students examine contexts where percentages can be greater than 100% or less than 1% (fractional percentages).

Students should be able to move flexibly between percentage, fraction, and decimal equivalents in problem solving situations. For example, when finding 25% of a number, it is often much easier to use

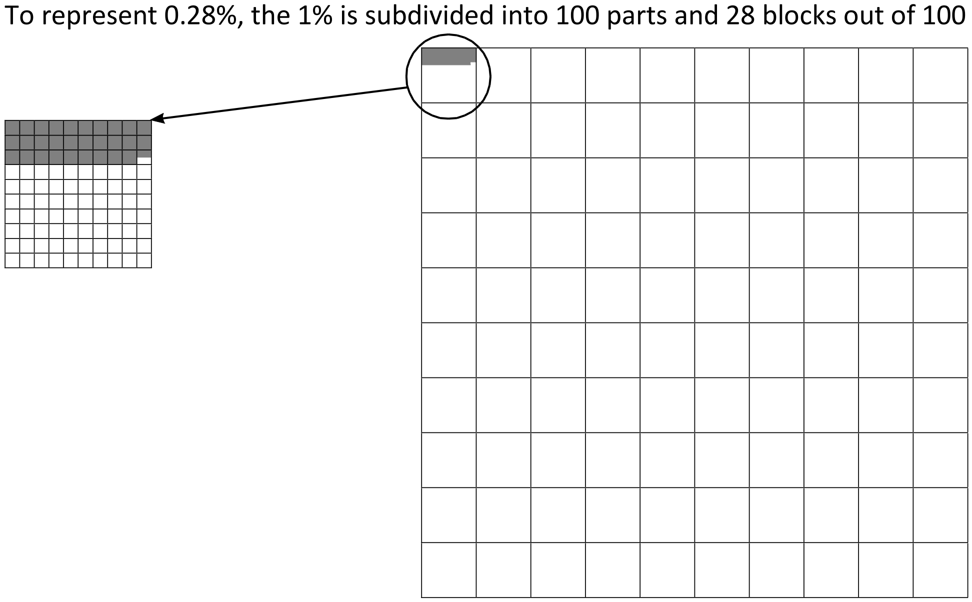
$\frac{1}{4}$ and then divide by 4 as a means of finding or estimating the percentage. If students can express fractions and decimals as hundredths, the term “percent” can be substituted for the term “hundredths.” The fraction $\frac{3}{2}$ can be expressed in hundredths, $\frac{150}{100}$ which has a decimal equivalent of 1.5, which is equivalent to 150%.

In previous grades, when working with whole number percentages from 1% to 100%, students represented them using 10 x 10 grid paper. In Mathematics 8, this is expanded to percentages between 0% and 1%, percentages greater than 100%, as well as other fractional percentages. Begin with a 10 x 10 grid to represent percentages. If the entire grid represents 100%, then each small square represents 1%. For fractional percentages that are easily recognizable, e.g., 0.5%, shade one-hald of one small square. To represent 29.5%, use grid paper and shade in 29 small squares and one-half of another small square.



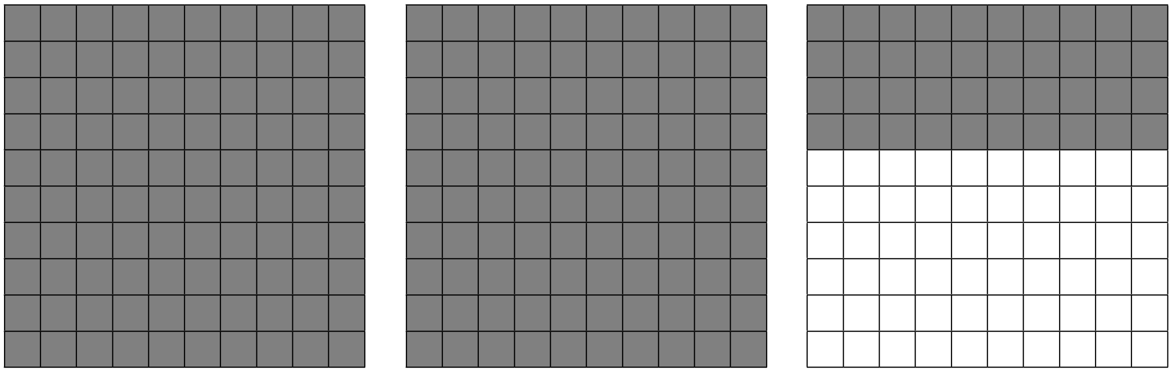
Fractional percentages less than 1% can be represented by zooming in on the 1% square, further subdividing it and shading in the appropriate area.

To represent 0.28%, the 1% is subdivided into 100 parts and 28 blocks out of 100 are shaded.



To represent $\frac{2}{3}$ %, the 1 % is subdivided into 3 parts and 2 are shaded.

Percentages greater than 100% are represented using more than one 10 x 10 grid chart. The diagram below represents 240%.



In this diagram, two full hundred grid charts and 40 blocks of another hundred grid chart are shaded.

The skills students learned in 7N03 taught students to convert between percentage, fraction, and decimal equivalents for whole number percentages between 1% and 100%. They will apply these skills to fractional percentages between 0% and 1%, percentages greater than 100%, as well as other fractional percentages.

Fractional percentages between 0% and 1% must be developed at a sensible pace. There is sometimes a tendency among students to see the percentage 0.1% as the decimal 0.1. It is important to distinguish the difference in these two forms. Similarly, students may confuse $\frac{3}{4}$ % with 75%. The hundreds and hundredths grid charts will help distinguish these differences. Given a shaded region on a grid, students will be expected to express the shaded region in fraction, decimal, or percentage form.

Another strategy that can be used when dealing with percentages greater than 100% and between 0% and 1% is patterning. For example:

Percent	Decimal	Fraction
0.3%	0.003	$\frac{3}{1000}$
3%	0.03	$\frac{3}{100}$
30%	0.3	$\frac{3}{10}$
300%	3	$\frac{3}{1}$

Percent	Decimal	Fraction
70%	0.7	$\frac{7}{10}$
7%	0.07	$\frac{7}{100}$
0.7%	0.007	$\frac{7}{1000}$
0.07%	0.0007	$\frac{7}{10000}$

Fractional and decimal percentages can be related to benchmark percentages. For example, 0.25% means one-fourth of 1%. If you know 1% of 400 is 4, then 0.25% of 400 would be a one-fourth of 4 or 1. It is also important to recognize that 1% can be a little or a lot depending on the size of the whole. For example, 1% of all of the population of a city is a lot of people compared to 1% of the students in a class.

Students will continue to create and solve problems that they explored in Mathematics 7, which involve finding a, b, or c in a relationship of a% of b = c using estimation and calculation. However, the problem-solving situations will be more varied. As an application, students will be required to apply percentage increase and decrease in problem situations for self, family, and communities, in which percentages greater than 100 or fractional percentages are meaningful. They will apply their knowledge of percentages to find a number when a percent of it is known, and to find the percent of a percent.

A common example of combined percents is addition of percents, such as taxes. Students encounter combined percentages every day when they buy items at stores and pay sales tax. Although this tax appears to be just one percentage, it is a “harmonized sales tax” (HST), which includes both federal and provincial sales tax rates.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Tasks such as the following could be used to determine students’ prior knowledge.

- Ask students to change each of the following to a percentage mentally and to explain their thinking:
  $\frac{2}{5}$ $\frac{4}{25}$ $\frac{6}{50}$ $\frac{8}{20}$
- Ask students to estimate the percent for each of the following and to explain their thinking:
  $\frac{7}{48}$ $\frac{5}{19}$ $\frac{7}{20}$
- Give students 10 × 10 grid and ask them to shade percentages from 1 to 100 percent.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Estimate the percent for each fraction. Explain your reasoning.

- $\frac{125}{85}$
- $\frac{99}{95}$
- $\frac{2}{230}$

- Use mental mathematics to solve the following:
  - If 2% of a number is 4, what is the number?
  - What would 11.5% of that number be? (Hint: think 10% + 1% + 0.5%)
- Provide students with 10 × 10 grid paper and have them shade in amounts to represent given percentages greater than 100% (e.g., 124%, 101%, 150%).
- Ask students to explain the meaning of the following scenario statements and give reasons for their explanations.
  - When your coach tells you to “give 110%,” what does she mean?
  - What is the chance that the principal will give you the day off school because of your smile?
  - A newspaper article includes 200% in its headline. Give a situation to which the article may be referring.
  - The Cape Breton Screaming Eagles handed out T-shirts to the first 100 fans at a hockey game. This represented $\frac{1}{2}$ % of the fans who attended that game.
  - The school reached 150% of its goal in collecting food items for the local food hamper.
- Show students a hundredths grid that is shaded and have them record the percent, decimal, and fraction that is represented by the shading.
- Ask students to respond to the following: Jill predicted that the chance of Maple Academy winning the championship game against Evergreen Collegiate is 0.50%. Which school do you think Jill attends? Explain your choice.
- Have students express a variety of percentages, decimals, and fractions in all three forms. This could be done in a chart. They could also represent these by shading in a grid or using materials.

Percent	Decimal	Fraction
146%		
	0.003	
		$\frac{140}{100}$

- Ask students to solve the following problems and explain their reasoning:
  - Superstar basketball sneakers, which regularly sell for $185, were marked down by 25%. To further improve sales, the discount price was reduced by another 15%. What was the final selling price? Explain why this is not the same as a 40% discount. What would the difference be?
  - About 0.6% of Nova Scotia’s population lives in Wolfville. The population of Nova Scotia is about 750 000. What is the population of Wolfville? If the population in Wolfville increases by 1000 when students attend Acadia University, what percent increase would this be?
  - The price of a $250 video game console was increased by 25%. After two weeks the price was reduced by 25%. Explain why the final price is not $250.

- Ask students to respond to the following:
  - Two stores offer different discount rates as follows:
  - Store A: 50% off, one day only.
  - Store B: 25% off one day, followed by 25% off the reduced price the second day.
  - Which store has the better sale?

  - A jacket cost $100. The discount on the jacket is 15%. However you must also pay 15% sales tax. Would the jacket cost you $100, less than $100, or more than $100? Explain your reasoning.
  - Charlie works part-time at a local fast-food restaurant. On his next pay cheque, he will receive a 5% increase in pay on top of a 10% performance bonus. Charlie tells his friends he is receiving a 15% raise in pay. Is he correct? Explain.

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

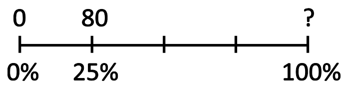
Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

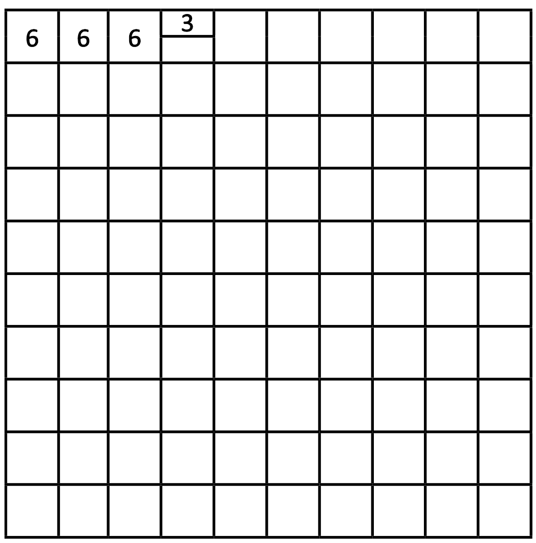
Consider the following strategies when planning daily lessons.

- Discuss with students the relevance of percentages in real-world applications (e.g., sales tax, discounts, sports statistics). Compile with students a list of situations where percentages are used. This list may include, but is not limited to:
  - test marks (78% on a science test)
  - sales tax (15% tax on all sales)
  - discount (25% off all purchases)
  - probability (10% chance of rain)
  - athletic statistics (scored 25% of shots on goal)
- Discuss with students situations that may result in percents greater than 100%. Ask students questions such as the following:
  - What percentage has the cost of soda pop increased when today’s cost is compared to the cost 50 years ago?
  - What percentage has the cost of a rare collectable item increased compared to its original value?
- Discuss with students situations that may result in percents between 0% and 1%. Ask students questions such as the following:
  - What is the percent chance that it will snow in August?

- Use a visual model for benchmark percentages like 10%, 25%, or 50% in questions, such as the following:
  - 25% of a number is 80. What is the number?



- Employ a variety of strategies to calculate the percentage of a number:
  - Represent with 10 × 10 grid:
    > $3\frac{1}{2}$ % of 600



  - Partition the percent:
    > 110% of 80:

      100% of 80 = 80
      10% of 80 = 8
      8 + 80 = 88

  - Represent the percent as a decimal equivalent and multiply:
    > finding 110% of 80:

      = 1.1 × 80 = 88

    > finding 0.5% of 800, find 1% and halve it:

      1% of 800 = 8

      $\frac{1}{2}$ of 8 = 4

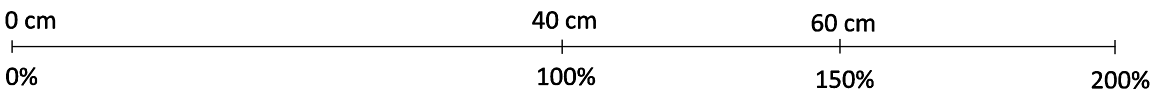
    > changing to a fraction and dividing:

      25% of 60 = $\frac{1}{4} \times 60 = 60 \div 4 = 15$

> using a proportion:

$$0.7\% \text{ of } 300 = \frac{0.1}{100} = \frac{x}{80}$$

- Use a 10 x 10 grid to represent 100%. Show examples of percentages greater than 100% such as, 260% and have students shade the grids to represent the amount or ask students to identify the percent shaded.
- Use a metre stick to represent 100%. Use more than one metre stick to show examples of percents greater than 100%.
- Use a double number line (vertical or horizontal) to solve percentage problems. For example, a length of 40 cm is increased by 50%. What is the new length?

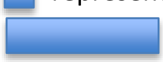


Suggested Learning Tasks

- As a decimal 140% = 1.40. Use patterning to write the following percents in decimal form.
  - 14%
  - 1.4%
  - 0.14%
- As a fraction 0.09% = $\frac{9}{10000}$, use patterning to write the following percents in fraction form.
  - 0.9%
  - 9%
  - 90%
  - 900%
- Substitute a variety of numbers in the following problems and solve.
- A school has a total enrolment of (527 or 200) students.
  - Yesterday (11.5% or 15%) were absent. How many came to school?
  - The current enrolment is 250% of the enrolment fifteen years ago. What was the enrolment then?
- Estimate the percent increase represented in the following problem. Derek's father said, "In my day, I could buy a chocolate bar and a soft drink for 25¢. Using your knowledge of the cost of these items today, what is the percent increase?
- Use a double number line to model: The student council president was elected with 215 votes. If she received 58% of the votes cast, about how many votes were cast?
- Construct a model to represent the following: Jimmy has an hour and 20 minutes to finish 5 tasks. What percentage of time can Jimmy spend on each task if each task takes an equal amount of time?
- Your friend was absent from school when your teacher explained fractional percentages. When he was studying for his test, he said that $\frac{1}{2}$ % was 0.5 as a decimal. How would you use a model or diagram to help him to understand the mistake he made?

- Trina received an 80% on a recent math test. If she answered 48 questions correctly, how many questions were on the test?
- Adam increased his song list by 40%. If he had 300 songs originally, how many songs does he now have?
- Shawn earned $85 and spent $15. What percent of his money did he spend?
- Last week the canteen sold 60 sandwiches. This week they sold 48 sandwiches. Calculate the percent decrease. How can you check that the percent decrease is correct?
- Copy and complete the following table.

Percent	Decimal	Fraction
148%		
$\frac{7}{20}$%		
26.4%		
	2.65	
	0.003	
	0.254	
		$\frac{8}{5}$
		$\frac{1}{250}$
		$\frac{3}{8}$

- Catherine said that her amount of homework increased 400% when it went from one half hour of work to two hours of work. Do you agree? Explain.
- Cyril collects hockey cards. He had 150 cards in his collection. His birthday was in June and his friends gave him hockey cards as presents which increased his collection by 20%. At Christmas his hockey card collection increased by another 15%. How many cards are in his collection after this 15% increase?
- If  represents 50%, what diagram represents 250%?
- If  represents 125%, what diagram represents 75%

SUGGESTED MODELS AND MANIPULATIVES

- Calculator*
- coins
- fraction circles*
- fraction pieces
- fraction strips
- hundredths grid
- thousandths grid
- metre stick
- number line / double number line*
- open number lines
- various objects for counting (e.g., beans, counters)

*also available in *Interactive Math Tools* (Pearson n.d.

MATHEMATICAL LANGUAGE

Teacher	Student
▪ combined percent ▪ decimal ▪ fraction ▪ fractional percent ▪ percent ▪ percentage ▪ ratio ▪ GST ▪ PST ▪ HST	▪ combined percent ▪ decimal ▪ fraction ▪ fractional percent ▪ percent ▪ percent increase ▪ percent decrease ▪ ratio ▪ GST ▪ PST ▪ HST

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 5: Percent, Ratio, and Rate
  - Section 5.1: Relating Fractions, Decimals, and Percents
  - Section 5.2: Calculating Percents
  - Section 5.3: Solving Percent Problems
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Math Matters: Understanding the Math You Teach, Second Edition (Chapin and Johnson 2006), 149–164.

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 175.

Proportional Reasoning (Erickson 2000; NSSBB #: 23187)

Digital

- "Interactive Hundredths Grid [unnamed]," *McGraw-Hill Education* (McGraw-Hill Education 2015): www.glencoe.com/sites/common_assets/mathematics/ebook_assets/vmf/VMF-Interface.html.

SCO N04 Students will be expected to demonstrate an understanding of ratio and rate. [C, CN, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N04.01 Explain the multiplicative relationship found within a ratio.
N04.02 Represent a two-term ratio from a given context concretely and pictorially and record using the forms 3:5 or 3 to 5.
N04.03 Express a three-term ratio from a given context in the forms 4:7:3 or 4 to 7 to 3.
N04.04 Express a part-to-part ratio as a part-to-whole fraction.
N04.05 Identify and describe ratios and rates (including unit rates) from real-life examples and record them symbolically.
N04.06 Express a given rate using words or symbols.
N04.07 Express a given ratio as a percentage, and explain why a rate cannot be represented as a percentage.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
N03 Students will be expected to solve problems involving percents from 1% to 100% (limited to whole numbers). **SP04** Students will be expected to express probabilities as ratios, fractions and percents.	**N04** Students will be expected to demonstrate an understanding of ratio and rate.	**N03** Students will be expected to demonstrate an understanding of rational numbers by comparing and ordering rational numbers and solving problems that involve arithmetic operations on rational numbers.

Background

Students have had previous experiences with **ratios**. In Mathematics 6 (N05), they defined, represented, and interpreted ratios of two numbers presented to them concretely. In Mathematics 7 (N03), they solved proportions within problem-solving situations involving percentage. In Mathematics 8, students will build on and extend their knowledge of ratios. They will also be introduced to **rate**. They will describe and record rates using real-life examples. Experiences with problem-solving situations using unit rates and unit prices should lead them to make connections between mathematics and everyday life.

Understanding that a ratio is a multiplicative comparison between two quantities is critical to the development of proportional reasoning. Encourage students to look for the multiplicative relationships within a ratio. For example, if two cans of blue paint are going to be mixed with six cans of yellow paint to produce green paint, the ratio of blue paint to yellow paint is 2:6. It can be said that the number of cans of blue paint is $\frac{1}{3}$ the number of cans of yellow paint, or that the number of cans of yellow paint is three times the number of cans of blue paint. These are examples of the multiplicative relationships found in this ratio. Students should be able to express these relationships in words. Students that understand the multiplicative relationship will be able to state other amounts of blue cans of paint and yellow cans of paint that will produce the same shade of green.

If a student describes the relationship between the number of blue cans of paint and yellow cans of paint this way; the number of blue cans of paint is four less than the number of yellow cans of paint, they see an additive comparison and not a multiplicative comparison. An additive comparison is not a ratio and students need to be able to think multiplicatively before they can be successful solving problems that will use proportional reasoning.

Ratios are classified by the type of comparison, either **part-to-part** (comparison of one part of the whole to another part of the same whole), **part-to-whole** (comparison of one part to the whole) for comparing measures or number or measures of the same type. Consider the following examples: 13 people rode the rollercoaster; 2 of them were girls. The ratio of girls to people was 2:13. This gives an example of a part-to-whole ratio: In an orchard with 80 trees, 43 of the trees were apple trees and the rest were pear trees. The ratio of apple trees to pear trees was 43:37, which represents a part-to-part ratio.

Part-to-whole relationships are fractions and can also be written in fractional form. Therefore 2:13 could also be written as $\frac{2}{13}$. Students need to understand that all fractions are also ratios but not all ratios are fractions.

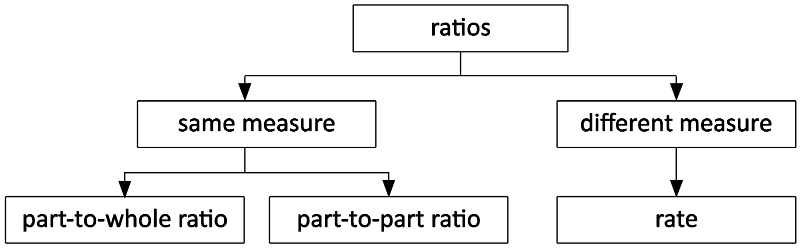
It should be emphasized that only part-to-whole ratios are expressed using fraction notation because a part-to-whole relationship is a fraction. For example, Daniel gets a hit 2 out of every 9 times he is at bat. The ratio of hits to bats can be written as 2 to 9, 2:9, or $\frac{2}{9}$.

Examples are also common using **three-term ratios**. If 13 people rode the rollercoaster and 2 of them were girls, the ratio Boys: Girls: Total People is 11:2:13, or 11 to 2 to 13. Similarly, in an orchard with 80 trees, if 43 of the trees were apple trees and the rest were pear trees, a resulting three-term ratio comparing Apple Trees to Pear Trees to Total Trees is 43 to 37 to 80 or 43:37:80.

Once students have an understanding that a part-to-part ratio compares one part of a set to another part of the same set, while a part-to-whole ratio compares one part of a set to the whole set, they should be able to write a part-to-part ratio as a part-to-whole ratio. For example, 1 can frozen juice concentrate to 4 cans of water can be represented as $\frac{1}{5}$, which is the ratio of concentrate to solution, or $\frac{4}{5}$, which is the ratio of water to solution.

Ratios are encountered frequently when describing real-world situations. Have students write ratios in words first. This will help students write the terms of the ratio in the correct order of comparison when expressing them symbolically.

Students have worked with ratios, which compare quantities with the same unit. Now the focus shifts to rates, which involve quantities with different units. However, the mathematics used to talk about rates is the same as the mathematics used to talk about ratios. Both represent comparisons. A graphic organizer, such as the following, may be helpful in making sense of the different types of ratios.



Remind students to include the units when writing rates.

Because a rate compares quantities measured in different units, a rate has no meaning without the units. Students should be familiar with numerous and various examples of rates already, even if they couldn't previously identify them as rates. Have the class brainstorm to identify as many real-life examples of rates as possible. Some examples that students should relate to include speed (km/h), text messaging rates ($/month), and school schedules (periods/day or days/cycle).

It is important to continue emphasizing that a rate compares two different quantities. In any context students may provide in which $\frac{a}{b}$ represents a rate, two quantities in different units are being compared.

The distinction between ratios and rates is subtle. As students work with rates, they should be encouraged to continue to examine the similarities and differences and make connections between ratios and rates. A fundamental difference between the two is the ability to represent a ratio, but not a rate, as a percent. Students should recall that part-to-whole ratios can be expressed as a percent, whereas part-to-part ratios cannot. For example, if Daniel gets a hit 2 out of every 9 times at bat, this is a ratio that can be written as a percent. If the hits are thought of as successful at bats, parts of the whole are being compared to the whole. A percent compares a number to 100. Because the units in a rate are different, there isn't a whole to make a comparison to. Therefore, a rate cannot be represented as a percent.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Tasks such as the following could be used to determine students' prior knowledge.

- Ask students to use pattern blocks to represent the following ratios and to write the ratio symbolically (e.g., trapezoids to triangles, triangles to all).
  - 3 trapezoids to 5 equilateral triangles
  - 4 rhombi to 6 equilateral triangles
- Ask students to express the following ratio comparisons from the diagram below:

☺☺☺☺♥

- The ratio of faces to hearts.
- The ratio of hearts to faces.
- The ratio of faces to all.
- The ratio of all to hearts.

Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Write a part-to-part and part-to-whole ratio for each situation below.
  - An aquarium has 3 guppies and 5 goldfish.
  - In the parking lot there are two types of vehicles: cars and trucks. There are 30 vehicles in total and seven of them are trucks.
  - You buy a mixed variety bag of apples containing 5 Honeycrisp apples and 7 Macintosh apples.
- Determine who would get the bigger portion of pizza if 9 girls share 4 pepperoni pizzas while 7 boys share 3 vegetarian pizzas. Explain your reasoning. What assumptions are you making?
- Ask the student to select three different colours of tiles, and model the following ratios:
  - 4 to 3
  - 2:1
  - $\frac{1}{3}$
  - 2:3:5
- Write a response to the following situation. If a tap is dripping at a rate of 50 mL an hour, can you describe that as a percentage? Explain why or why not.

Planning for Instruction

Choosing Instructional Strategies

Consider the following strategies when planning daily lessons.

- Present a collection of items that all have something in common (e.g., art supplies or types of sports balls). Ask students to describe the comparisons as ratios using part-to-part and part-to-whole ratio notation. (Note: part-to-part ratios cannot be written as fractions).
- Discuss with students that all fractions are ratios but not all ratios are fractions.
- Ensure students understand why rate cannot be represented as a percent. For example, if 200 out of 500 students in a school are going to the dance, we could also say that 40 percent will be attending. In this case the ratio is comparing the part (students attending) against the whole (the entire school population). This is different from a rate, which compares two quantities with different units such as speed in kilometres per hour. Since rates compare different quantities, they cannot be represented as a percent which compares part to whole of only one quantity (e.g. a school population).
- Discuss the best way to describe each of the following:
  - the speed you travel on the highway
  - the number of eggs a family uses in a day, a week, and a month

- hockey player achievements per minutes played

Suggested Learning Tasks

- In your school state these ratios:
  - students to teachers
  - teachers to students
  - classrooms to total number of rooms
  - soccer balls to basketballs to total number of balls
  - window to doors
  - desks to chairs
- Sort the following as ratios or rates:
  - a store clerk is paid $11.40 per hour
  - the average test score for the previous unit was 82%
  - a recipe for cleaner calls for 1 part baking soda, 2 parts vinegar and 4 parts water
  - for every hour I spend on homework, I spend 2 hours playing basketball
  - a car drives 240 km in 3 hours
- Shade in part of a 10×10 grid. Exchange your grid with a partner and then represent the portion shaded on your partner's grid in more than one way.
- Find examples of rates (e.g., in shops, in flyers, on the Internet) or develop a poster displaying rates. Describe in your math journals the meaning of the rates you chose using words and symbols.
- Write each ratio as a fraction in simplest form.
  - 14 to 6
  - 4:22
  - 18:12
  - 25 to 20
  - 18:21
  - 18:3
  - 7:21
  - 20 to 9
  - 4:10
  - 84 to 16
- Colour the following grid so that the ratio of shaded tiles to non-shaded tiles is 5:3.

SUGGESTED MODELS AND MANIPULATIVES

- colour tiles*
- hundredths grid
- pattern blocks*
- ratio tables

*also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ part-to-part ratio ▪ part-to-whole ratio ▪ rate ▪ three-term ratio ▪ two-term ratio ▪ ratio	▪ part-to-part ratio ▪ part-to-whole ratio ▪ rate ▪ three-term ratio ▪ two-term ratio ▪ ratio

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 5: Percent, Ratio, and Rate
  - Section 5.5: Exploring Ratios
  - Section 5.6: Equivalent Ratios
  - Game: *Triple Play*
  - Section 5.7: Comparing Ratios
  - Section 5.8 A: Proportional and Non-proportional Situations (Nova Scotia companion document only; found on Mathematics Learning Commons, Grades 7–9)
  - Section 5.8 B: Solving Problems That Involve Proportional Reasoning (Nova Scotia companion document only; found on Mathematics Learning Commons, Grades 7–9)
  - Section 5.9: Exploring Rates
  - Section 5.10: Comparing Rates
  - Unit Problem: What Is the Smartest, Fastest, Oldest?
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 154–175.

Math Matters: Understanding the Math You Teach, Second Edition (Chapin and Johnson 2006), 165–189.

N05 Students will be expected to solve problems that involve rates, ratios, and proportional reasoning. [C, CN, ME, PS, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N05.01 Explain the meaning of $\frac{a}{b}$ within a given context.

N05.02 Provide a context in which $\frac{a}{b}$ represents a fraction, a rate, a ratio, a quotient, and a probability.

N05.03 Use pictures, models, or manipulatives to make sense of a proportional situation.

N05.04 Differentiate between proportional and non-proportional contexts.

N05.05 Use multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another.

N05.06 Use multiple methods to solve proportional tasks and understand that these methods are related to each other.

N05.07 Use estimation to determine the reasonableness of an answer.

N05.08 Solve a proportion using mental mathematics, pencil and paper, or technology, as appropriate.

N05.09 Solve a given problem involving rate, ratio, or percent using mental mathematics, pencil and paper, or technology, as appropriate.

N05.10 Create problems that are examples of proportional reasoning.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
N03 Students will be expected to solve problems involving percents from 1% to 100% (limited to whole numbers). **SP04** Students will be expected to express probabilities as ratios, fractions, and percents.	**N05** Students will be expected to solve problems that involve rates, ratios, and proportional reasoning.	**N03** Students will be expected to demonstrate an understanding of rational numbers by comparing and ordering rational numbers and solving problems that involve arithmetic operations on rational numbers.

Background

Rate, ratio, and proportion are three related concepts. They are used in many problem-solving opportunities. For example, ratios, rates, and proportions are commonly used for scale models, altering a recipe, and comparison shopping.

A ratio is a comparison of two quantities or measures. In the previous outcome, students learned about the multiplicative relationships that are the basis of a ratio. This multiplicative thinking will be necessary as students now study proportions. A proportion is a statement of equality between two ratios. To be proportional thinkers, students

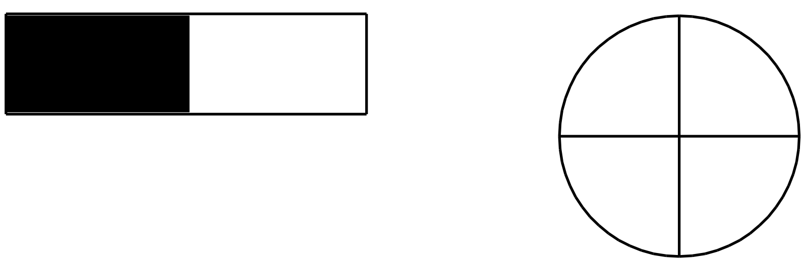
- need to develop an understanding of the multiplicative relationship within a ratio and between the equal ratios that make a proportion
- know the mathematical characteristics of proportional situations
- be able to differentiate mathematical characteristics of proportional thinking from nonproportional contexts
- understand realistic and mathematical examples of proportional situations
- realize that multiple methods can be used to solve proportional tasks and that these methods are related to each other
- know how to solve quantitative and qualitative proportional-reasoning tasks
- be unaffected by the context of the numbers in the task

While two ratios can represent the same relationship, different notations for proportions can be used. For example,

2:5 = 4:10 or 2 to 5 = 4 to 10 or $\frac{2}{5} = \frac{4}{10}$

These can all be read as "two is to five as four is to ten."

To effectively understand and solve ratio problems, comparing equivalent ratios or creating a proportion is necessary. Begin by using models or pictures to make sense of proportions. An example would be to use diagrams such as the following and ask students to complete a second diagram to show the same relationship as given in the first diagram.



To solve problems that can be represented as a proportion, students need to understand the multiplicative relationship found within a ratio or between two ratios. Understanding these relationships will help students solve proportions for missing information. Provide opportunities for students to gain a conceptual understanding of proportions using numbers that can be manipulated mentally with ease. Only after this understanding is gained can numbers that do not lend themselves to more intuitive methods be used.

Present students with a problem like the following.

- The ratio of indoor basketballs to outdoor basketballs at the recreation centre is 6:2. New basketballs have to be purchased. If the recreation centre is going to purchase 24 indoor basketballs, how many outdoor basketballs will they buy?

Ratios can be reviewed by asking students to describe in words the multiplicative relationship found in the ratio 6:2. We would want students to be able to say that the number of indoor basketballs is three times the number of outdoor basketballs or the number of outdoor basketballs is one-third the number of indoor basketballs. If a student says that the number of indoor basketballs is four more than the number of outdoor basketballs, they are thinking additively and not multiplicatively. Since additive comparisons are a common misconception and not part of proportional thinking, they will need more practice describing the multiplicative relationship in a ratio before they will be successful solving proportions.

A proportion that can be set up for this problem is

$$\frac{\text{6 indoor balls}}{\text{2 outdoor balls}} = \frac{\text{24 indoor balls}}{x\text{ outdoor balls}}$$

Students could get the answer by looking at the relationship within each ratio and thinking "6 is related to 2 in the same way that 24 is related to what number"?

$$\frac{\text{6 indoor balls}}{\text{2 outdoor balls}} = \frac{\text{24 indoor balls}}{x\text{ outdoor balls}}$$

They could also look at the relationship between each ratio and think "6 is related to 24 in the same way that 2 is related to what number." Either way, they will see the answer is 8.

$$\frac{\text{6 indoor balls}}{\text{2 outdoor balls}} = \frac{\text{24 indoor balls}}{x\text{ outdoor balls}}$$

Proportional reasoning is complex and involves the understanding that if one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship.

A ratio is usually a comparison of two quantities of like units. When the comparison is between two different units, such as litres of gas compared to kilometres travelled, that is defined as a **rate**. A **unit rate** is an equivalent rate where the second quantity is one. It allows us to compare varying sizes of quantities by examining the number of units of one quantity per 1 unit of the second quantity. To solve problems involving distance, time, and average speed, or to determine the better buy in consumer situations, it is often beneficial to use unit rates. An example would be, which is the better buy: 1.2 L of orange juice for $2.50, or 0.75 L of orange juice for $1.40? Explain why it is the better buy.

It is important for students to be aware that when they are comparing unit rates, the numbers must be in the same units. For example, if comparing one quantity measured in grams with another measured in kilograms, the options are to change both measurements to grams or kilograms. The unit of measurement used for such a unit rate is often the student's choice. A review of conversion from one unit of measurement to another may be necessary here.

Proportional reasoning can be developed through activities that compare and determine the equivalence of ratios and rates, and solving proportions in a wide variety of problem-based contexts. It is important that students see the usefulness of **proportions** and are able to efficiently set up and solve proportional reasoning problems.

Encourage estimation and thinking about a problem prior to calculating. Encourage students to think quantitatively (more than or less than a certain amount) when estimating.

Students should look for the most efficient method to set up and solve a problem, whether it be using the relationship found within a ratio, the relationship between the two ratios or the unit rate. Also if possible, determine if the problem can be solved mentally before using pencil and paper. Cross multiplication does not promote understanding and is therefore not recommend as an approach to solving problems.

"Students may need as much as three years' worth of opportunities to reason in multiplicative situations in order to adequately develop proportional reasoning skills. Premature use of rules encourages students to apply rules without thinking and, thus, the ability to reason proportionally does not develop." (Van de Walle and Lovin 2006c, 157)

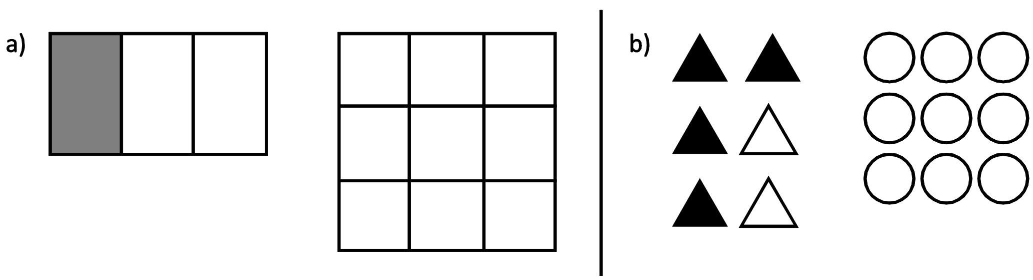
Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Tasks such as the following could be used to determine students' prior knowledge.

- Tell students that the length of one rectangle is double the length of another. Their widths are the same. What happens to the area and perimeter of the rectangle? Explain.
- Have students draw line segments of 3 cm and 4 cm respectively. Then have them draw two more line segments that are longer than, and equivalent ratios to, the segments they drew.
- For each of the following, have students complete the second diagram so that it shows the same relationship as the first and to explain their reasoning.



- Have students use concrete materials to make proportional pairs.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample tasks (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- During a heavy rain storm, 40 mm of rain fell in 30 minutes. How much rain would you expect to fall in one hour? In three hours? What assumptions are you making?
- Greg and Rose hammered a line of nails into different boards from one end to the other. Rose hammered more nails than Greg. Rose's board was shorter than Greg's. On which board are the nails closer together? Why?
- Jenson drives 100 km in 2 h and has 60 km to go. Will Jensen drive the other 60 km in more or less than 2 h? Justify your answer.
- Sue and Julie were running equally fast around a track. Sue started first. When she had run nine laps, Julie had run three laps. When Julie completed 15 laps, how many laps had Sue completed?
- A recipe uses 500 mL of flour for every 125 mL of sugar. How much flour would be needed when 500 mL of sugar is used?
- Provide a real-world context in which $\frac{a}{b}$ would represent: a fraction, a rate, a ratio, a quotient, and probability.

- If a package of 6 bottles of juice costs \$4.50 and an individual bottle costs \$1.50, how much would you save per bottle by purchasing a package?
- Dan can run 4 km in 15.2 minutes. If he keeps running at the same speed, how far can he run in 20 minutes?
- Use examples to explain how ratios and rates are the same. Then use examples to explain how ratios and rates are different.
- Jane found a good deal on soft drinks. She could buy 12 cans for \$2.99. She needs 72 cans for her party. Explain how she can calculate the cost.
- Explain why 1:20 000 000 is another way to describe the ratio of 1 cm representing 200 km on a map.
- Sam measures two flowers to find they are 8 cm and 12 cm tall. Two weeks later they are 12 cm and 16 cm tall respectively. Which flower grew more?
- Jane's father drove 417 km in 4.9 hours. Leah's father drove 318 km in 3.8 hours. Who was driving faster and by how much?

Follow-up on Assessment

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

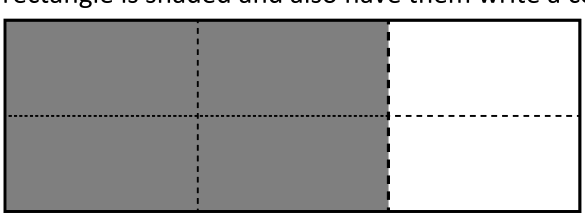
Planning for Instruction

Choosing Instructional Strategies

Consider the following strategies when planning daily lessons.

- Use problems that encourage the use of mental mathematics, such as quantities within a ratio or between ratios that are integral multiples of each other.

- Use diagrams, such as the following to have students write two different ratios that tell what part of the rectangle is shaded and also have them write a correct proportion.



- Make comparisons of two quantities that are measured by the same unit. For example, compare the ratio of students who wear wrist watches to the ratio of those who don't wear wrist watches in other classes and then to the entire grade.

	Wrist Watch Wearers	Non-Wrist Watch Wearers	Ratio of Wrist Watch Wearers to Non-Wrist Watch Wearers
Class 1	16	8	2:1
Class 2	24	12	2:1
Entire Grade 8	varies	varies	varies

- Encourage students to use their knowledge of equivalent fractions or unit rates to solve proportion problems. Students should select the method that is most efficient to them, and appropriate to, solve problems.
- Use tables with data that contain equivalent ratios to model proportions by looking for multiplicative relationships in the rows or column. Have students examine the tables to determine if relationships are proportional. For example:

Paper Mache

Number of cups of water	Number of cups of flour
2	3
4	6
6	9
8	12

Suggested Learning Tasks

- Use the Cuisenaire rods to find proportional pairs.
- Bring in a picture of yourself standing beside a person or object. Measure the height of the images in the picture (cm) and use that ratio to find the actual height of the other person or object in the photo. One of the actual heights must be determined.
- 3 British pounds can be exchanged for 2 Canadian dollars. How many British pounds would you get for 1 Canadian dollar?
- Fred received a gift card for his birthday. He used it to download some new music for his MP3 player. Fred downloaded 12 songs in 15 minutes. At this rate, how many songs could he download in 1 hour?
- Two carafes of juice are on the table. Carafe B contains weaker juice than carafe A. Add one teaspoon of instant juice crystals to carafe A and one cup of water to carafe B. Which carafe will contain the stronger juice? Why?
- Two carafes of juice are sitting on the table. Carafe B and carafe A contain juice that tastes the same. Add one teaspoon of instant juice crystals to both carafe A and carafe B. Which carafe will contain the stronger juice? Why?
- When making lemonade Sue uses 5 scoops of powder for 6 cups of water, and Sarah uses 4 scoops of powder for 5 cups of water.
  - Are the situations proportional to each other? Explain why or why not.
  - In which situation is it likely the lemonade will be more flavourful? What assumptions did you make?
- Discuss whether or not the following could be solved using a proportion:
  - David is 6-years-old and Ellen is 2-years-old. How old will Ellen be when David is 12-years-old?
- Determine the fuel economy for a vehicle. You can prepare and use a log such as the one that follows to track fuel purchases, kilometres driven, and fuel economy over several weeks.

Amount of Gas Purchased (L)	Beginning Odometer Reading (km)	Ending Odometer Reading (km)	Total Distance Travelled	Fuel Efficiency

- Read each of the following scenarios and determine whether the fraction $\frac{a}{b}$ is representing a fraction, rate, ratio, quotient, or probability. Explain your reasoning.
  - Jeremy has a $\frac{3}{6}$ chance of rolling an even number on a six-sided regular number cube.
  - Ling travelled $\frac{27km}{3h}$ on her bike.
  - Having three girls and five boys on the intramural volleyball team is shown as $\frac{3}{5}$.
  - Pedro scored $\frac{7}{10}$ on his math quiz.
  - Allan ate $\frac{3}{4}$ of a bag of chips.
- Using the rectangles in the Fraction Factory set, place an orange rectangle and a brown rectangle side by side. Beside this pair of rectangles, place two different rectangles that are proportional; to the first pair. Explain your reasoning.

SUGGESTED MODELS AND MANIPULATIVES

- Calculator*
- coins
- 10 × 10 grid
- number line*
- open number lines
- various objects for counting (e.g. beans, counters)

*also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ equivalent fraction ▪ part-to-part ratio ▪ part-to-whole ratio ▪ proportion ▪ rate ▪ three-term ratio ▪ two-term ratio	▪ equivalent fraction ▪ part-to-part ratio ▪ part-to-whole ratio ▪ proportion ▪ rate ▪ three-term ratio ▪ two-term ratio

▪ unit price ▪ unit rate	▪ unit price ▪ unit rate

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 5: Percent, Ratio, and Rate
  - Section 5.5: Exploring Ratios
  - Section 5.6: Equivalent Ratios
  - Section 5.7: Comparing Ratios
  - Section 5.8 A: Proportional and Non proportional Situations (Nova Scotia companion document only; found on Mathematics Learning Commons, Grades 7–9)
  - Section 5.8 B: Solving Problems That Involve Proportional Reasoning (Nova Scotia companion document only; found on Mathematics Learning Commons, Grades 7–9)
  - Section 5.9: Exploring Rates
  - Section 5.10: Comparing Rates
  - Unit Problem: What Is the Smartest, Fastest, Oldest?
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 154–175.

Math Matters: Understanding the Math You Teach, Second Edition (Chapin and Johnson 2006), 165–189.

Implementing the Common Core State Standards through Mathematical Problem Solving: Grades 6–8 (Gurl et al. 2013), 1–13.

Digital

- "Thinking Blocks: Model and Solve Word Problems, Ratio and Proportion Practice," *Math Playground* (MathPlayground.com 2014): www.mathplayground.com/tb_ratios/thinking_blocks_ratios.html
- "Thinking Blocks: Model Your Math Problems," Math Playground [iPad apps] (Math Playground LLC 2014): http://thinkingblocks.com/index.html

"Math Interactives: Rate/Ratio/Proportion," *LearnAlberta.ca* (Alberta Education 2003): www.learnalberta.ca/content/mejhm/index.html?ID1=AB.MATH.JR.NUMB&ID2=AB.MATH.JR.NUMB.RATE&lesson=html/video_interactives/rateRatioProportions/rateRatioProportionsInteractive.html.

SCO N06 Students will be expected to demonstrate an understanding of multiplying and dividing positive fractions and mixed numbers, concretely, pictorially, and symbolically. [C, CN, ME, PS]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N06.01 Identify the operation required to solve a given problem involving positive fractions.
N06.02 Provide a context that requires the multiplying of two given positive fractions.
N06.03 Provide a context that requires the dividing of two given positive fractions.
N06.04 Estimate the product of two given positive proper fractions to determine if the product will be closer to 0, $\frac{1}{2}$, or 1.
N06.05 Estimate the quotient of two given positive fractions, and compare the estimate to whole number benchmarks.
N06.06 Express a given positive mixed number as an improper fraction and a given positive improper fraction as a mixed number.
N06.07 Model multiplication of a positive fraction by a whole number concretely and/or pictorially and record the process.
N06.08 Model multiplication of a positive fraction by a positive fraction concretely and/or pictorially, using an area model, and record the process.
N06.09 Model division of a positive proper fraction by a whole number concretely and/or pictorially and record the process.
N06.10 Model division of a whole number by a positive proper fraction concretely and/or pictorially, using an area model, and record the process.
N06.11 Model division of a positive proper fraction by a positive proper fraction pictorially and record the process.
N06.12 Generalize and apply rules for multiplying and dividing positive fractions, including mixed numbers.
N06.13 Symbolically solve a given problem involving positive fractions, taking into consideration order of operations (limited to problems with positive solutions and that exclude exponents).

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
N05 Students will be expected to demonstrate an understanding of adding and subtracting positive fractions and mixed numbers, with like and unlike denominators, concretely, pictorially, and symbolically (limited to positive sums and differences).	**N06** Students will be expected to demonstrate an understanding of multiplying and dividing positive fractions and mixed numbers, concretely, pictorially, and symbolically.	**N03** Students will be expected to demonstrate an understanding of rational numbers by comparing and ordering rational numbers and solving problems that involve arithmetic operations on rational numbers.

Background

In Mathematics 7 (N05), students used models and an algorithm to add and subtract positive fractions. Benchmarks were used for estimation, and extensive work was done on equivalency, ordering, and expressing a fraction in simplest form.

The following guidelines should be kept in mind when developing computational strategies for fractions. It is important to not rush to computational rules.

- Begin with simple contextual tasks (include sets, area models, distance).
- Connect the meaning of fraction computation with whole-number computation.
- Let estimation and informal methods play a big role in the development of strategies.
- Explore each of the operations using models.

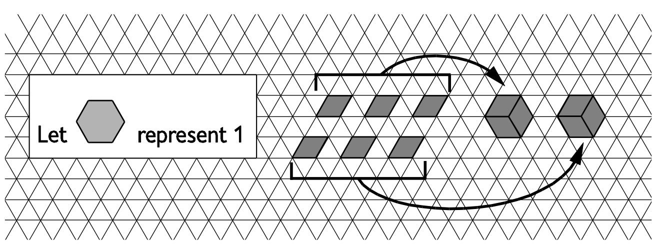
(Van de Walle and Lovin, 2006c, 88)

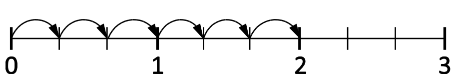
The teaching of fractions through memorizing rules poses significant problems; the rules by themselves do not help students think about the meanings of the operations or why they work. The mastery attained in the short term is often quickly lost.

Multiplication and division of fractions is similar to multiplication and division of whole numbers. It is important for students to realize that the meaning of the operation has not changed just because they are now working with fractions.

Investigating the operations on fractions through the use of models such as number lines, the area model, counters, fraction circles, and fraction strips helps solidify understanding of such concepts.

When multiplying a fraction by a whole number, a common misconception is that both the numerator and denominator must be multiplied by the whole number. The use of a concrete model should help address this. A model reinforces that a denominator indicates the number of equal parts that make up the whole and this does not change when multiplying by a whole number. The following pattern block model illustrates $6\times\frac{1}{3}$.

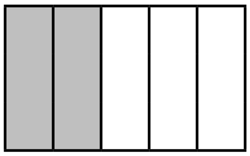




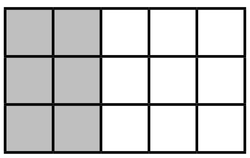
It is important that students start with a concrete model and then represent the concrete model pictorially, which helps develop an understanding of multiplying fractions symbolically. A variety of models can demonstrate the meaning of fraction multiplication. It is helpful to provide more than one type of model for the same activity; introducing an area model, a set model, and a linear (number line) model promotes deeper understanding and will reach a variety of learning styles. Models provide more meaning when the algorithm is introduced.

To use an area model to illustrate the solution $\frac{2}{3}\times\frac{2}{5}$, have students create a rectangle and shade in two-thirds of two-fifths of the rectangle. Putting $\frac{2}{3}\times\frac{2}{5}$ into words helps student see that the denominators determine the dimensions of the rectangle and the numerators indicate the required shading.

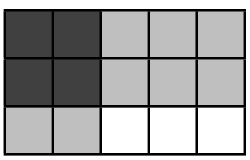
First, divide the rectangle vertically into fifths and shade two-fifths.



Next, to determine two-thirds of the shaded two-fifths, divide the rectangle into thirds horizontally.



Finally, shade two-thirds of the two-fifths horizontally. The product will be the area that is double shaded (four-fifteenths).



Therefore, $\frac{2}{3}\times\frac{2}{5}=\frac{4}{15}$.

Relating multiplication of fractions to real-life situations helps solidify student understanding. When asked to provide a context that requires the multiplication of two given positive fractions, some students may use original contexts for their problem and others may adopt the wording of earlier problems. Encourage students to share their problems so that they are exposed to some that show originality.

It should be shown that “of” means multiplication. This may be done by comparing results in examples such as $\frac{1}{2}$ of 6 and $\frac{1}{2}\times 6$.

Estimation keeps the focus on the meaning of the numbers and the operations, encourages reflective thinking, and helps build number sense with fractions.

To estimate products close to 0, $\frac{1}{2}$, or 1, consider the following properties:

$0\times n=0$, where *n* is any number

$1\times n=1$, where *n* is any number

$1\times 1=1$

Applying these properties and using benchmarks of 0, $\frac{1}{2}$, and 1 for given factors, students can estimate a product.

To estimate the product of $\frac{1}{9}$ and $\frac{8}{9}$, encourage students to think about $\frac{1}{9}$ being close to 0. Since $\frac{8}{9}\times 0=0$, $\frac{1}{9}\times\frac{8}{9}$ would be close to 0. Similarly, the products in the table below can be estimated using benchmarks.

Product	Determine Benchmarks	Multiply Mentally using benchmarks	Estimate Product
$\frac{8}{9}\times\frac{4}{9}$	$\frac{8}{9}<1$ and $\frac{4}{9}<\frac{1}{2}$	$1\times\frac{1}{2}=\frac{1}{2}$	$\frac{8}{9}\times\frac{4}{9}<\frac{1}{2}$
$\frac{8}{9}\times\frac{8}{9}$	$\frac{8}{9}<1$	$1\times 1=1$	$\frac{8}{9}\times\frac{8}{9}<1$

Estimation helps fraction computations make sense. It should play a significant role in the development of multiplication strategies.

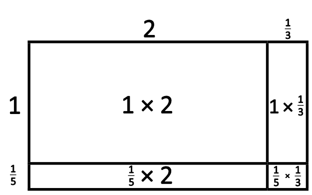
After working with models and patterns, students should conclude that when you multiply two fractions, the numerator in the product is found by multiplying the numerators in each factor, and also the denominator is the product of the denominators. For example, $\frac{2}{3}\times\frac{2}{5}=\frac{2\times 2}{3\times 5}=\frac{4}{15}$

Estimation is valuable once students have moved to the symbolic level.

To check the reasonableness of the solution, think about $\frac{2}{3}$ close to $\frac{1}{2}$ and $\frac{2}{5}$ as a little less than $\frac{1}{2}$, so it is a reasonable answer.

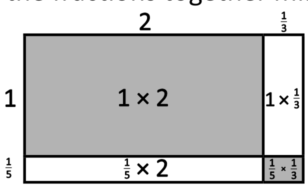
A common misconception among students is that multiplying always makes the product a larger number. When at least one of the factors is between zero and one, this is not the case. The use of models and estimation should help overcome this. Modelling multiplication of mixed numbers should be done prior to multiplying the equivalent improper fractions. No reference to improper fractions is necessary when using the models.

An area model to multiply $1\frac{1}{5}$ by $2\frac{1}{3}$ is shown below:



$$
\begin{aligned}
&(1\times 2)+(1\times \frac{1}{3})+(\frac{1}{5}\times 2)+(\frac{1}{5}\times \frac{1}{3})\\
&=2+\frac{1}{3}+\frac{2}{5}+\frac{1}{15}\\
&=2+\frac{5}{15}+\frac{6}{15}+\frac{1}{15}\\
&=2\frac{12}{15}\\
&=2\frac{4}{5}
\end{aligned}
$$

A common error when finding the product of mixed numbers is to multiply the whole numbers together and multiply the fractions together. Use of the area model clearly demonstrates why this is incorrect. Since the product is the area of the entire rectangle, multiplying only the whole numbers together and the fractions together misses the two unshaded pieces.



After using the area model, students can move to rewriting the mixed numbers as improper fractions before finding the product. This conversion to the equivalent improper fraction was an outcome in Mathematics 6, and was revisited in Mathematics 7. As with multiplying proper fractions, it is essential that students check the reasonableness of an answer using estimation. Students worked with equivalent fractions in Mathematics 7. Fractions should be simplified in relation to the context used.

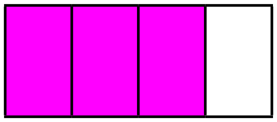
Work with concrete and pictorial models is necessary when students are first introduced to dividing fractions. It is not enough for students' knowledge of the division of fractions to be limited to the traditional invert-and-multiply algorithm. To develop students' conceptual understanding of division of fractions, teachers must carefully consider what students need to learn beyond this algorithmic procedure and well before any algorithm is introduced.

Students were introduced to division of whole numbers in two ways: sharing and grouping. This idea can be extended to division of fractions. Dividing a fraction by a whole number can be thought of as equal sharing. It is important to plan the progression of examples you present from those that are easier to understand to those that are conceptually more difficult.

Consider the following examples:

- You have $\frac{3}{4}$ of a rectangular pizza to divide evenly among 3 people. How much pizza would each person receive?
  - To model the solution to this problem using fraction factory, let the black fraction piece represent 1 whole.

– The magenta pieces each represent $\frac{1}{4}$, so model $\frac{3}{4}$ of the one whole.



– Then share the $\frac{3}{4}$ among the 3 people, each getting $\frac{1}{4}$.

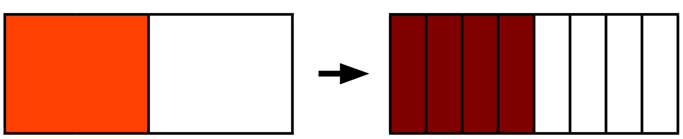
– Symbolically this is $\frac{3}{4} \div 3 = \frac{1}{4}$.

Progress to a more difficult problem.

▪ Model $\frac{1}{2}$ of a rectangular pizza shared among 4 people.

– Let the black fraction piece represent 1 whole.

– The orange piece represents $\frac{1}{2}$. One-half cannot easily be shared equally among 4 people. The orange piece is traded for an equivalent amount that can be shared equally among 4 people. That would be four brown pieces, each representing $\frac{1}{8}$, for a total of $\frac{4}{8}$.



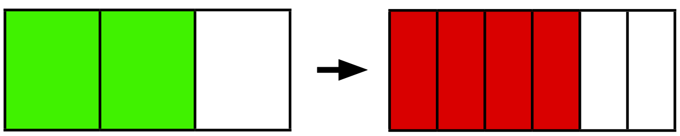
– Now the fraction pieces can easily be shared among the 4 students. Each person gets $\frac{1}{8}$.

Symbolically
$\frac{1}{2} \div 4$
$= \frac{4}{8} \div 4$
$= \frac{1}{8}$.

Again progress to a more difficult problem for students to understand.

- Model $\frac{2}{3} \div 4$.

  - Let the black fraction piece represent 1 whole.
  - Two green pieces represent $\frac{2}{3}$. Two-thirds cannot easily be share among four people. The two green pieces are traded for an equivalent amount that can be easily shared among four people. That would be four red pieces, each representing one-sixth for a total of four-sixths.

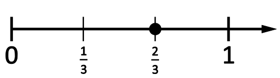


  - Now the fraction pieces can easily be shared among the 4 students. Each person gets one-sixth.

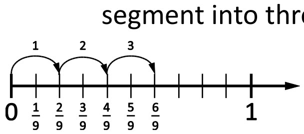
    Symbolically:
    $\frac{2}{3} \div 4$
    $= \frac{4}{6} \div 4$
    $= \frac{1}{6}$

A number line can also provide a useful model for division.

- To model, mark off thirds on the number line, between 0 and 1.



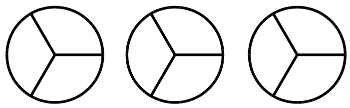
  - Dividing $\frac{2}{3}$ into 3 equal parts can be challenging. However, if we choose an equivalent fraction with a numerator evenly divisible by 3, such as $\frac{6}{9}$, then it is easier to see how to divide that segment into three equal lengths.



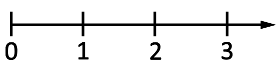
  - $\frac{2}{3}$ divided into 3 equal parts gives equal lengths of $\frac{2}{9}$. Therefore, $\frac{2}{3} \div 3 = \frac{2}{9}$.

When dividing a whole number by a fraction, ask "How many equal groups can be made?"

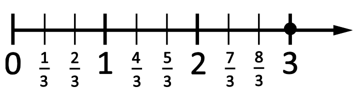
- You have 3 pizzas. Each person eats $\frac{1}{3}$ of a pizza and all pizzas are completely eaten. How many people eat the pizzas?
  - Start with 3 pizzas and divide them into thirds. How many groups of $\frac{1}{3}$ are there in the three pizzas?



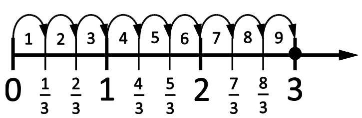
  - The pizzas can be divided into 9 equal groups of $\frac{1}{3}$. Therefore, nine people eat the pizza.
  - Use a number line to model.



  - Now divide each section of the number line into three equal parts and mark accordingly.

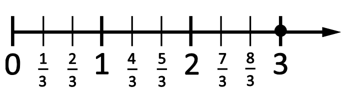


  - Starting at zero, mark off the number of groups (jumps) of $\frac{1}{3}$. There are 9 jumps.

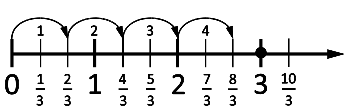


Students may experience more difficulty using number lines when the dividend is not evenly divisible by the divisor. Spend considerable time on examples such as $3 \div \frac{2}{3}$.

- Sarah bought 3 metres of fabric to make banners for the gym. If it takes $\frac{2}{3}$ of a metre to make one banner, how many banners can Sarah make with the fabric she bought?
  - Model $3 \div \frac{2}{3}$ on a number line.
  - Start with a number line, marking off 0, 1, 2, and 3 and then divide each section into thirds.



  - Starting at zero, mark off the number of groups of $\frac{2}{3}$ that can be found in 3.



  - There are 4 groups of $\frac{2}{3}$, leaving a remainder of $\frac{1}{3}$.
  - What part of the divisor ($\frac{2}{3}$) is the remainder ($\frac{1}{3}$)?

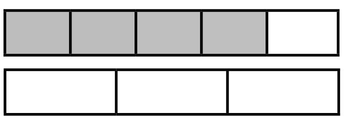
The remainder ($\frac{1}{3}$) is $\frac{1}{2}$ of the divisor.

Therefore, $3 \div \frac{2}{3} = 4\frac{1}{2}$.

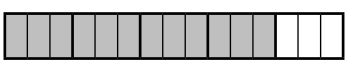
A common misconception about division is that it always leads to a quotient that is less than the dividend. Students should see here that this is not the case.

A good understanding of modelling division of a fraction and a whole number should provide a smooth transition to dividing positive proper fractions. An example of division of a fraction by a fraction using fraction strips follows.

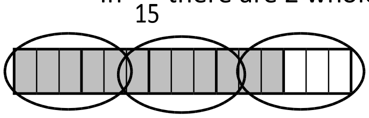
- For $\frac{4}{5} \div \frac{1}{3}$, students should use diagrams to determine how many groups of $\frac{1}{3}$ are in $\frac{4}{5}$. The diagram below shows that there are between 2 and 3 groups of $\frac{1}{3}$ in $\frac{4}{5}$ but it is difficult to determine exactly how many groups there are.



  – Subdividing the rectangle that represents $\frac{4}{5}$ into an amount that is divisible by both 3 and 5 will help students determine the number of groups. A common multiple for 5 and 3 is 15 so the rectangle can be divided into fifteenths.

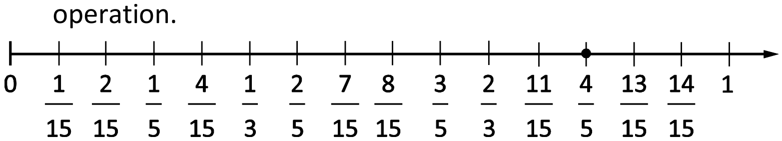


  In $\frac{12}{15}$ there are 2 whole groups of $\frac{5}{15}$ ($\frac{1}{3}$), plus $\frac{2}{5}$ of another group.

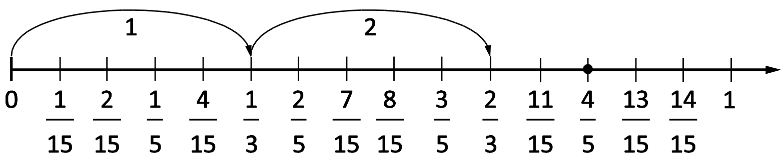


  Therefore, $\frac{4}{5} \div \frac{1}{3} = 2\frac{2}{5}$.

- Modelling division of a fraction by a fraction using a number line follows the same pattern as the fraction strip model. Using a common denominator to mark the intervals on the number line is beneficial. To model $\frac{4}{5} \div \frac{1}{3}$, use a number line with 15ths and mark $\frac{4}{5}$ ($\frac{4}{5} = \frac{12}{15}$), the first fraction in the operation.



  – Starting at zero, mark off groups of $\frac{5}{15}$($\frac{1}{3} = \frac{5}{15}$) until no more groups $\frac{5}{15}$ of can be made.



Two groups of $\frac{5}{15}$ are possible with two-fifteenths remaining. Two-fifteenths ($\frac{2}{15}$) is two parts of the divisor, five-fifteenths ($\frac{5}{15}$)?

Therefore, $\frac{4}{5} \div \frac{1}{3} = 2\frac{2}{5}$; the same result we saw with the fraction strips.

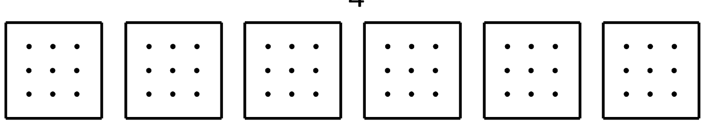
Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Tasks such as the following could be used to determine students' prior knowledge.

- Ask students a variety of "Always, Sometimes, Never True" questions such as the following and have them explain their reasoning:
  - The sum of two proper fractions is less than one.
  - The difference of two proper fractions is less than one-half.
  - If you double the numerator of a fraction, you double the value of the fraction.
  - If you double the denominator of a fraction, you double the value of the fraction.
- Have students create three addition and subtraction sentences that equal $\frac{2}{3}$.
- Using fraction strips or pattern blocks, ask students to model equivalent fractions and express them in simplest form.
- Ask students o shade in $\frac{1}{4}$ of the square in at least six ways.



Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

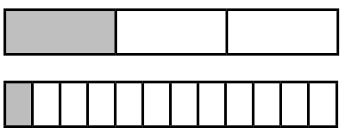
- Gabrielle filled 5 glasses with $\frac{7}{8}$ of a litre of juice in each glass. Use a model or draw a picture to determine how much juice Gabrielle used. Write the operation symbolically.
- Place the numbers 3, 4, 5, and 6 (or another set of numbers) in the boxes to get the least or greatest possible answer. $\frac{\square}{\square}\times\frac{\square}{\square}$
- Estimate each of the following and explain your thinking.
  - $6\frac{1}{4}\times 8$
  - $4\times 8\frac{3}{16}$
  - $\frac{1}{3}\times\frac{1}{12}$
- Model each of the following either concretely or pictorially and explain your thinking.
  - $4\times\frac{3}{5}$
  - $\frac{5}{8}\times 3$
  - $\frac{1}{4}\times\frac{2}{5}$
  - $\frac{2}{3}\times\frac{7}{8}$
- In a gymnasium, of the people present $\frac{1}{4}$ are men, $\frac{1}{3}$ are women, and the rest are children. If there are 840 people in the gymnasium, how many are children?
- Leah has $\frac{3}{4}$ of a large pizza. She gave $\frac{1}{3}$ of what she had to Jessie. What fraction of the whole does Jessie receive? What fraction of the whole pizza does Leah have left?
- Insert one set of brackets to make the following statements true and justify your answer.
  - $\frac{1}{2}+\frac{1}{4}\times\frac{2}{3}=\frac{1}{2}$
  - $\frac{3}{4}\times\frac{1}{5}+\frac{2}{3}\times\frac{5}{3}=1\frac{1}{2}$
- Represent your calculations pictorially to demonstrate understanding of the following:
  - Six containers of ice cream have been purchased for a birthday party. If each guest gets a serving of $\frac{3}{8}$ of a container of ice cream, how many guests can be served?
  - Hannah has $5\frac{1}{4}$ metres of ribbon to make bows for gift wrapping. If she needs $\frac{3}{4}$ of a metre for each gift, how many gifts will she be able to wrap? Show your work and explain your answer.

- Janet wants to plant 4 rows of garlic in her garden. If each bulb of garlic can be divided into cloves to seed $\frac{3}{8}$ of a row, how many bulbs of garlic does she need to buy?

- Use estimation to determine which expression has the greatest quotient:
  - $\frac{9}{5} \div \frac{3}{3}$
  - $2\frac{1}{5} \div 1\frac{7}{8}$
  - $\frac{1}{4} \div \frac{1}{2}$
  - $3\frac{1}{10} \div \frac{5}{6}$

- Create a problem that could be solved by dividing $\frac{7}{8}$ by 4.

- Explain how you could use the following model to determine the quotient for $\frac{1}{3} \div 4$.



- Represent the following equations using models or by drawing diagrams, and explain why each is true.
  - $3 \div \frac{1}{2} = 6$
  - $\frac{2}{5} \div \frac{3}{4} = \frac{8}{15}$

- Insert one set of brackets to make the following statement true. Justify your answer.

  $\frac{2}{3} \times \frac{1}{2} + \frac{1}{4} = \frac{1}{2}$

- Draw a number line to show why each of the following is true:
  - $\frac{1}{3} \times 3 = 1$
  - $3 \times \frac{1}{3} = 1$
  - $6 \times \frac{2}{3} = 4$
  - Use a different model to verify each equation above.
  - Create a word problem for one of the above equations.

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

Choosing Instructional Strategies

Consider the following strategies when planning daily lessons.

- Use problems as a context for multiplication of fractions, including those with and without subdivisions. Examples of these types of problems can be found on pages 94–96 in *Teaching Student-Centered Mathematics, Grades 5–8* (Van de Walle and Lovin 2006c).
- Begin with common benchmark fractions when introducing fraction multiplication.
- Use number lines to model how to combine groups of fractions or distances that occur when a fraction is multiplied by a whole number.
- Use an area model or multiply the equivalent improper fractions for modelling multiplying mixed fractions.

	3	$\frac{1}{3}$
2	2×3	$2 \times \frac{1}{3}$
$\frac{1}{2}$	$\frac{1}{2} \times 3$	$\frac{1}{2} \times \frac{1}{3}$

  – The distributive property applies here:

$$2\frac{1}{2}\times 3\frac{1}{3}=(2+\frac{1}{2})(3+\frac{1}{2})=(2\times 3)+(2\times\frac{1}{3})+(3\times\frac{1}{2})+(\frac{1}{2}+\frac{1}{3})=8\frac{1}{3}$$

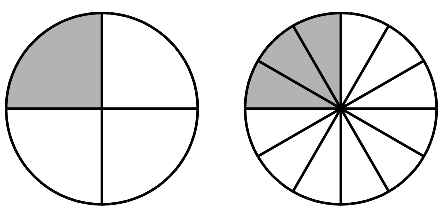
- Use real-world contexts and order or operations to further explore and practice multiplication of positive fractions.
- Use problems as a context for division of fractions, including those with partitioning and measuring. Examples of these types of problems can be found on pages 98–104 in *Teaching Student-Centered Mathematics, Grades 5–8,* Volume Three (Van de Walle and Lovin 2006c).
- Present division of a fraction by a whole number as a sharing situation.
- Present examples that can be modelled concretely and pictorially and then move to the symbolic representation once students understand the process. The common denominator method for division of fractions relates well to whole number division.
- Estimate the quotient of positive fractions by using whole number benchmarks (e.g., $7\frac{9}{10}\div 2\frac{1}{12}$ is approximately 8 ÷ 2, so the estimated quotient is 4).
- Ensure that students can compare the solutions of problems such as $8\div\frac{1}{2}$ and $8\times\frac{1}{2}$ as it is important for students to understand the concepts of multiplication and division of fractions.

- Review with students the order of operations rules for whole numbers. These rules apply to fractions, and problems should be created using "friendly" fractions so the process is not overwhelming (e.g., $\frac{51}{117}$ is not a friendly fraction).

Suggested Learning Tasks

- What division expression does this diagram represent?

- Estimate each of the following and explain your thinking.
  - $5\frac{1}{6}\times 8$
  - $4\times 8\frac{3}{8}$
- Compare the solutions to 2 × 4 and $\frac{1}{2}\times\frac{1}{4}$ and discuss your observations.
- Explain how the following diagram can be used to calculate $\frac{1}{4}\div 3$.



  Are there other manipulatives or diagrams you could use? Explain.

- If you cut $\frac{1}{4}$ of your lawn before lunch, and then $\frac{2}{3}$ of the remaining lawn after lunch, how much (if any) of the lawn remains to be cut? (This question could be changed to cutting one third before and three quarters of the remaining lawn after lunch, and compare the results.)
- Compare the two following situations: two thirds of John's fifteen cars are red and fifteen glasses that are two thirds full. Discuss the differences.
- Demonstrate the following by drawing diagrams, and explain why each is true:
  - $2\div\frac{1}{4}=8$
  - $\frac{1}{2}\div 2=\frac{1}{4}$
- Have students solve problems such as the following. Katlin decided to make muffins for the school picnic. Her recipe requires 2 $\frac{1}{4}$ cups of flour to make 12 muffins. Katlin found there were exactly 18 cups of flour in the canister and decided to use it all. How many muffins can she expect to make?

- Casey had 5 $\frac{1}{4}$ metres of material to make headbands for 7 friends. How much material should she use for each headband if she wants to use the same length of material for each?
- Use Fraction Factory to determine a common denominator when modelling division questions (e.g., $\frac{5}{3} \div \frac{1}{2}$).
- Explain the difference between "six divided by one-half" and "six divided in half." Write a division statement for each phrase and find each quotient.
- Play the Spinner Game: using a 4-section spinner. Label each section with fractions such as $\frac{1}{9}, \frac{9}{10}, \frac{11}{12}, \frac{5}{11}$.
  - Spin twice and estimate the product. Score 0 points if the closest benchmark is zero, 1 point if the closest benchmark is $\frac{1}{2}$, and 2 points if the closest benchmark is 1. The student who scores 20 points first wins the game.
- Jared calculated $\frac{3}{5} \times \frac{2}{5}$ as follows $\frac{3}{5} \times \frac{2}{5} = \frac{6}{5}$
  - What mistake did Jared make?
  - How could you use estimation to show Jared that he made a mistake?
  - What is the correct procedure? Show in two different ways.
- In your job as a gardener, you must decide how to use your garden. You mark $\frac{1}{2}$ of the garden for potatoes. You use $\frac{1}{4}$ of the remaining area for corn. Then you plant cucumbers in $\frac{1}{3}$ of what is left. The rest of your garden is used for carrots. What fraction of your garden is used for carrots?
- Joanne gave the following answer on her homework assignment:

  $2\frac{1}{3} \times 1\frac{3}{4} = 3\frac{1}{12}$
  - Use an area model to show why this answer is incorrect.
  - What mistake did Joanne make?
  - What is the correct answer?

Suggested Models and Manipulatives

- area model
- counters
- dot paper
- fraction pieces/circles/bars*
- Fraction Factory
- geoboards*
- grid paper
- number line*

- pattern blocks*

* also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ denominator ▪ fraction ▪ improper fraction ▪ inverse ▪ mixed number ▪ numerator ▪ order of operations ▪ proper fraction ▪ simplest form	▪ denominator ▪ fraction ▪ improper fraction ▪ inverse ▪ mixed number ▪ numerator ▪ order of operations ▪ proper fraction ▪ simplest form

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 3: Operations with Fractions
  - Section 3.1: Using Models to Multiply Fractions and Whole Numbers
  - Section 3.2: Using Models to Multiply Fractions
  - Section 3.3: Multiplying Fractions
  - Section 3.4: Multiplying Mixed Numbers
  - Game: Spinning Fractions
  - Section 3.5: Dividing Whole Numbers and Fractions
  - Section 3.6: Dividing Fractions
  - Section 3.7: Dividing Mixed Numbers
  - Section 3.8: Solving Problems with Fractions
  - Section 3.9: Order of Operations with Fractions
  - Unit Problem: Sierpinski Triangle
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)

- Projectable Student Book Pages
- Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 93–106.

Digital

- “Virtual Manipulatives: Fractions, Decimals, Percents,” *abcya.com* (ABCya.com LLC 2015): http://media.abcya.com/games/fraction_tiles/flash/fraction_tiles.swf
- “Equivalent Fractions,” *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/Activity.aspx?id=3510

SCO N07 Students will be expected to demonstrate an understanding of multiplication and division of integers, concretely, pictorially, and symbolically. [C, CN, PS, R, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

N07.01 Identify the operation required to solve a given problem involving integers.
N07.02 Provide a context that requires multiplying two integers.
N07.03 Provide a context that requires dividing two integers.
N07.04 Model the process of multiplying two integers, using concrete materials or pictorial representations, and record the process.
N07.05 Model the process of dividing an integer by an integer, using concrete materials and/or pictorial representations, and record the process.
N07.06 Generalize and apply a rule for determining the sign of the product and quotient of integers.
N07.07 Solve a given problem involving the division of integers (two-digit by one-digit) without the use of technology.
N07.08 Solve a given problem involving the division of integers (two-digit by two-digit) mentally or with the use of technology, where appropriate.
N07.09 Symbolically solve a given problem involving integers, taking into consideration order of operations when necessary.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
N06 Students will be expected to demonstrate an understanding of addition and subtraction of integers, concretely, pictorially, and symbolically.	**N07** Students will be expected to demonstrate an understanding of multiplication and division of integers, concretely, pictorially, and symbolically.	**N03** Students will be expected to demonstrate an understanding of rational numbers by comparing and ordering rational numbers and solving problems that involve arithmetic operations on rational numbers.

Background

Multiplication and division of integers are abstract concepts, therefore it is very important that concrete models be used to support learning before progressing to symbolic representations.

Addition of integers and the multiple representations of an integer using the zero principle that are part of Mathematics 7 establish a foundation for the understanding of multiplication of integers. Just as addition and subtraction are connected to whole-number concepts, the multiplication of integers should be a direct extension of the multiplication of whole numbers. Whole-number multiplication can be represented as repeated addition. The first factor tells how many sets there are or how many are added in all, beginning with 0. This translates to integer multiplication quite readily when the first factor is positive, regardless of the sign of the second factor.

The sign of the first factor of the expression tells us if we add or subtract the groups to zero. The sign of the second factor indicates the size and type of the group. When the rules for multiplying and dividing integers are just memorized, they are often soon forgotten. Understanding why the rules make sense is very important for future work with integers.

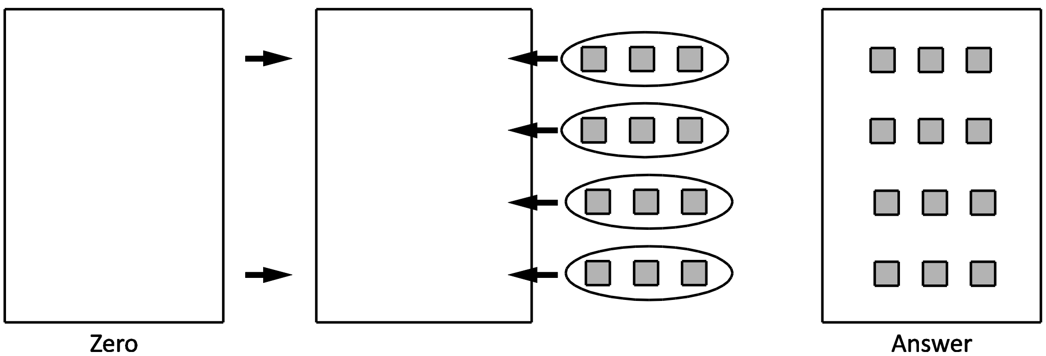
Common models that are used with integers are counters and number lines. To represent integer multiplication using counters consider the following.

- Positive multiplied by a positive:

represents positive 1

(+4) × (+3) reads "add four groups of positive three" represents negative1

represents positive 1
represents negative 1

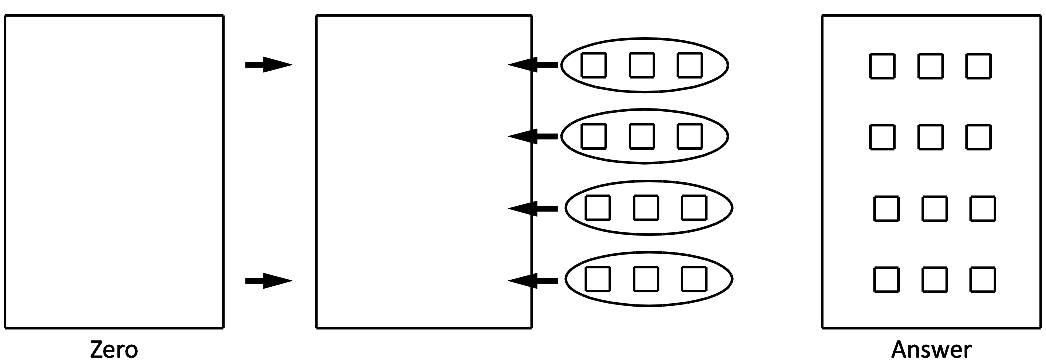


Therefore (+4) × (+3) = (+12)

- Positive multiplied by a negative:

(+4) × (–3) reads "add four groups of negative three."

represents positive 1
represents negative 1

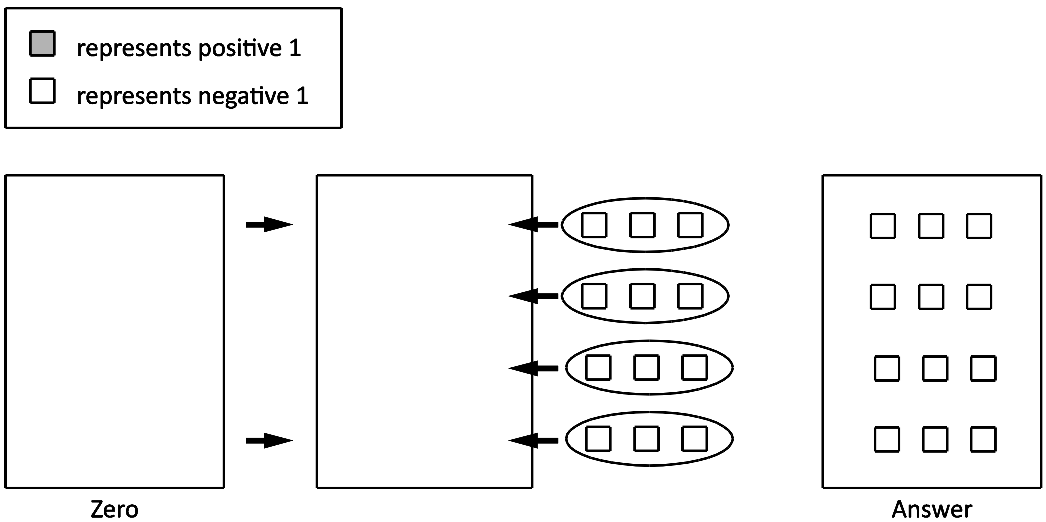


Therefore (+4) × (–3) = (–12).

- Negative multiplied by a positive:

  (–4) × (+3) reads “subtract four groups of positive three.”

  Use the Zero Principle to model what zero looks like. Four groups of +3 are needed to be removed from the zero. By starting the problem with a zero modelled as 12 positive and 12 negative tiles, there are the four groups of +3 available for removal.



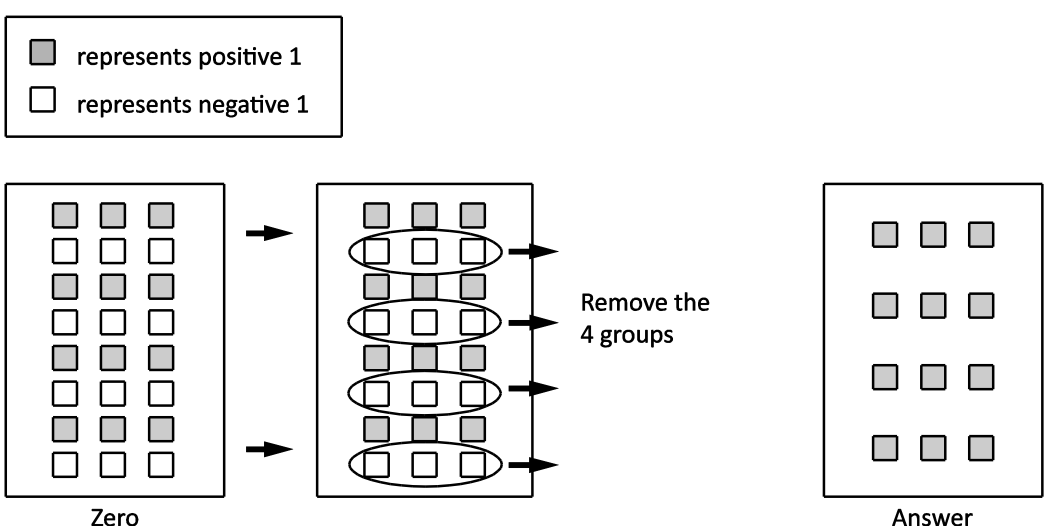
Therefore (-4) × (+3) = (–12)

Adding 4 groups of negative 3 gives the same answer as subtracting 4 groups of positive 3. This establishes (after many examples) that integers can be multiplied in any order without affecting the product (commutative property).

- Negative multiplied by a negative

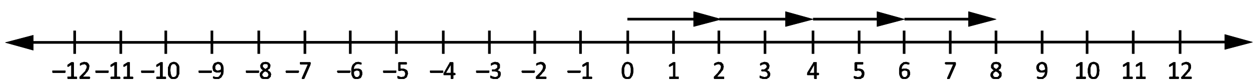
  (–4) × (–3) this reads “remove four groups of negative three”

  To perform this type of problem use the Zero Principle to model what zero looks like. Four groups of –3 are needed to be removed from the set, so add 12 positive and 12 negative tiles to the starting set. Remove the four groups of negative 3, leaving four groups of positive three.

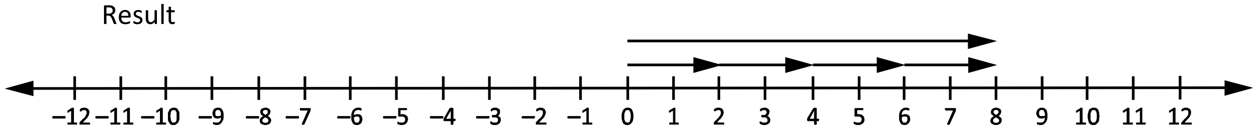


Therefore (–4) × (–3) = (+12).

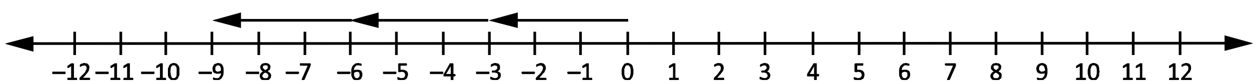
We can also show the multiplication on the number line.
For the expression (+4) × (+2), a positive 2 is being multiplied by a positive 4. This can be shown on the number line as four groups of +2 to the right of zero.



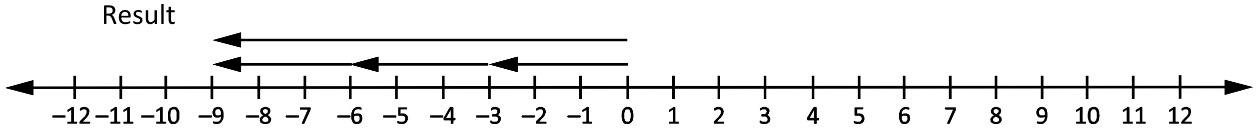
Result



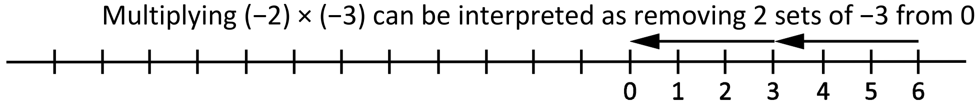
- For the expression (+3) ×(−3). a negative 3 is being multiplied by a positive 3. This is modelled on the number line as three groups of –3 to the left of zero.



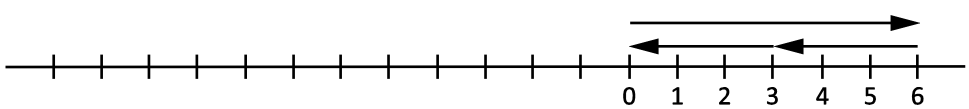
Result



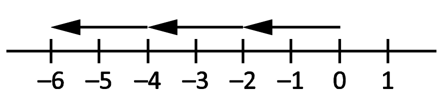
Multiplying (−2) × (−3) can be interpreted as removing 2 sets of −3 from 0



Back up from 0 with two –3 arrows



The result goes from 0 to the end



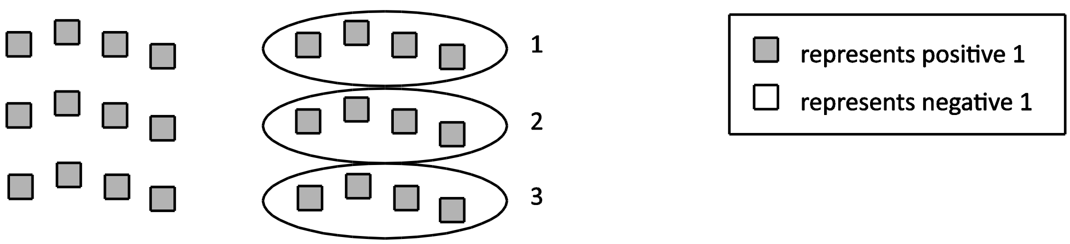
It is important to remember that integers can be multiplied in any order without affecting the product (commutative property). Using this property helps students with the multiplication (−4) × 5 because the answer will be the same as the answer to 5 groups of –4. When the rules for multiplying and dividing integers are just memorized, they are often soon forgotten. Understanding why the rules make sense is very important doe future work with integers.

With division of integers, again whole numbers are explored first. Recall that a whole number example, such as 8 ÷ 4, has two possible meanings corresponding to two missing factor expressions: 4 × ? = 8 asks, “Four sets of what make eight?” whereas ? × 4 = 8 asks, “How many fours make eight?” Generally, the measurement approach (? × 4) is the one used with integers, although both concepts can be exhibited with either model.

The models for division of a positive divided by a positive and a negative divided by a negative can be modelled if the dividend is considered to be the size of the group and not the number of groups.

- To model a positive divided by a positive:

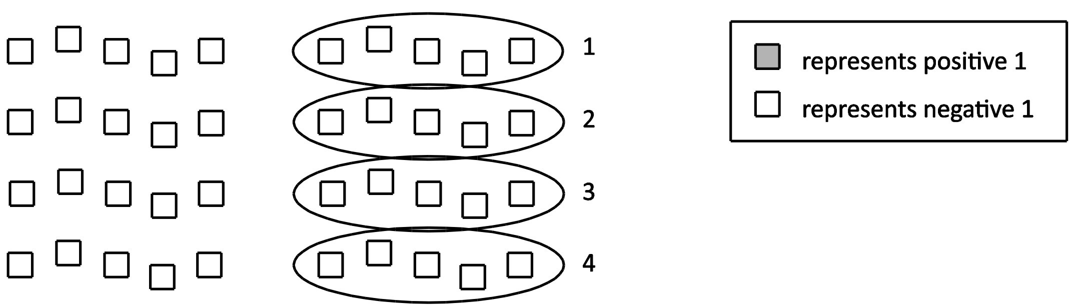
+12 ÷ +4 reads "how many groups of positive four are there in positive 12"



Therefore, (+12) ÷ (+4) = (+3). There are three groups of (+4) in (+12).

- To model a negative divided by a negative:

–20 ÷ –5 reads "how many groups of negative five are there in negative twenty"



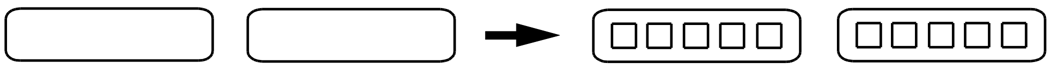
Therefore, (–20) ÷ (–5) = (+4). There are four groups of (–5) in (–20).

The model for division of a negative by a positive can be modelled if the dividend is considered to be the number of the groups and not the size of the groups.

- To model a negative divided by a positive:

−10 ÷ +2

−10 is divided into 2 equal groups, how many in each group?



Therefore, −10 ÷ +2 = −5.

Patterning can also be used in determining the product or quotient:

3 × (+2) = 6
2 × (+2) = 4
1 × (+2) = 2
positive products

0 × (+2) = 0
zero product

−1 × (+2) = −2
−2 × (+2) = −4
−3 × (+2) = −6
negative products

3 × (−2) = −6
2 × (−2) = −4
1 × (−2) = −2
negative products

0 × (−2) = 0
zero product

−1 × (−2) = 2
−2 × (−2) = 4
−3 × (−2) = 6
positive products

Comparison of multiplication and division situations can also be very useful in helping students understand division of integers. After multiplication has been fully developed, the fact that multiplication and division are inverse operations can be utilized. Students should understand the connection between multiplication and division of integers, as well as division and grouping/sharing. The use of the number line can be extended to model the division of integers. To make the connection, it may be beneficial to write a related multiplication statement.

For example, since −4 × 3= −12, it must be true that the product divided by either factor should equal the other factor; therefore, −12 ÷ (−4) = 3 and −12 ÷ 3= −4. Likewise, if −4 × (−3) = 12, then 12 ÷ (−4) = −3 and 12 ÷ (−3) = −4. Using a missing factor can also be useful.

Division Statement		**Related Multiplication Statement**
−16 ÷ (−4) =	→	? × (−4) = −16

As with multiplication, the models that have been used would lead to the general “sign rules” for division of integers. This provides another opportunity to explore inverse operations. Comparison of multiplication and division can be useful in helping students understand division. For example, since (−2) × 3 = (−6) , it must be true that the product divided by either factor should equal the other factor. Therefore, (−6)÷ (−2) = 3 and (−6)÷ 3 = −2. Students should conclude that when two signs are the same, the quotient is positive, and when two signs are different, the quotient is negative.

Useful relevant contexts for making work with integers meaningful for students, for example, temperature, deposits or withdrawals, golf scores that are below and above par, and floors that are above and below a main floor.

Discuss the meaning of the negative product or quotient. Once multiplication and division of integers have been explored concretely, pictorially, and symbolically, students should be encouraged to develop rules for determining the signs for products and quotients.

Students should be able to apply their knowledge of calculations with integers and the order of operations (excluding exponents) to solve problems. Using the order of operations maintains consistency in results.

To make meaningful connections between real-world contexts and integer multiplication, students must understand the use of positive and negative integers to represent the quantities that are multiplied. When solving problems, emphasize the importance of a summary statement to explain the meaning of the integer product, for example:

- Matthew has committed his support to a charity for 2 years. If he has $25/month deducted automatically from his bank account, what is the total of his deductions?

  First, students must decide what integers to multiply.
  −25 represents the monthly $25 deduction
  +24 represents the number of months in two years
  (+24) x (−25) = −600

A complete solution requires an explanation of the negative sign in the context of the problem. In this case, Matthew's total deductions will be $600.

Students will have already used order of operations, excluding exponents, but limited to whole numbers and decimals. This will now be extended to calculations with integers. It is important to note that work with exponents is not a specific outcome until Mathematics 9. For this unit of work, students will complete calculations based on the following order of operations:

- brackets
- division/multiplication (in the order they appear)
- addition/subtraction (in the order they appear)

The order of operations convention is necessary in order to maintain consistency of results. It is important to provide students with situations in which they can recognize the need for the order of operations.

Technology is useful for situations involving more than 1-digit divisors or 2-digit multipliers. Students should understand how to use the key on the calculator and how negatives are dealt with differently on certain calculators. Specific instruction should be given on calculator use with regard to the order of operations. Students should recognize the necessity of preparing problems for calculator entry. They should also be aware that different calculators process the order of operations in different ways. Some calculators are programmed to address the order of operations automatically, and others are not.

The appropriate use of brackets should be discussed:

- Brackets can be used to show integers as positive or negative such as (−3) or (+4). These brackets require no operation.
- There is a need for brackets around (−4) in the expression −5 – (−4), however brackets are not necessary for (−5).
- For positive integers the positive sign is often understood. For example, 4 is understood to be the same as (+4), so brackets are not necessary in this case.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

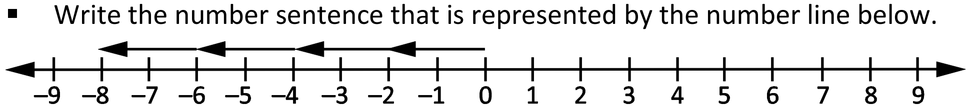
Tasks such as the following could be used to determine students' prior knowledge.

- Ask students: Can +4 be modelled with an odd number of counters? Explain your reasoning.
- Have students use a number line or two-colour counters to explain why the following calculations are correct.
  - (−3) + 8 = 5
  - (−5) − 3 = −8
  - (−4) − (−6) = 2
- Ask students: Can you model +2 with an odd number of counters? Explain why or why not.
- Ask students: Is the sum of a negative number and a positive number always negative? Explain your reasoning.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

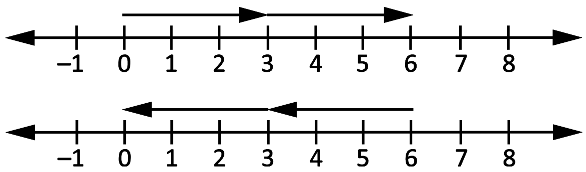
Consider the following sample tasks (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Without doing any calculations, Katie said that the quotients (−468) ÷ (−26) and (+468) ÷ (+26) must be the same. How did she know?
- Write a number sentence for each of the following problems and use a diagram to model them.
  - Michelle withdrew $25 from her bank account each week for 16 weeks. How much did she withdraw in total?
  - Fran lost 3 points in each round (hand) of cards that was played. If she played 4 rounds, what was her score at the end of the game?
  - The temperature in Springhill was falling 2°C each hour. How many hours did it take for the temperature to fall 10°C?
  - Mike and his three friends together owe $12. They agree to share the debt equally. What is each person's share of the debt?
- Write the number sentence that is represented by the number line below.



- The sum of two integers is −2. The product of the same two integers is −24. What are the two integers? Explain your reasoning.
- To win a free trip, the following skill-testing question must be answered correctly:

  −3 x (−4) + (−18) ÷ 6 – (−5). The contest organizers say that the answer is +4.
  - Write a note to the organizers explaining why there is a problem with their solution.
- Find the mean temperature for your town for the past twelve months.
- What multiplication statement does each diagram represent?



Follow-up on Assessment

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

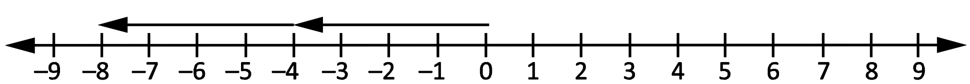
Planning for Instruction

Choosing Instructional Strategies

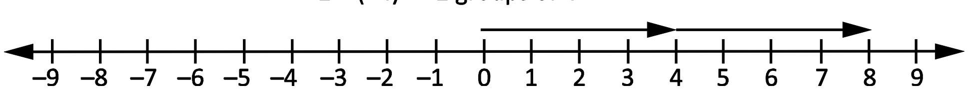
Consider the following strategies when planning daily lessons.

- Use a variety of models to represent multiplication and division of integers: use of counters, number lines, the idea of net worth, and patterning. Of these models, patterning is perhaps the most effective method for presenting the multiplication or division of two negative numbers in a way that is easiest for the students to understand. The number line can also be used to model problems such as:

2 × (–4) → 2 groups of –4

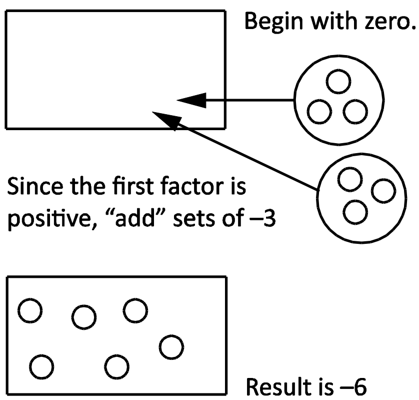


2 × (+4) → 2 groups of 4

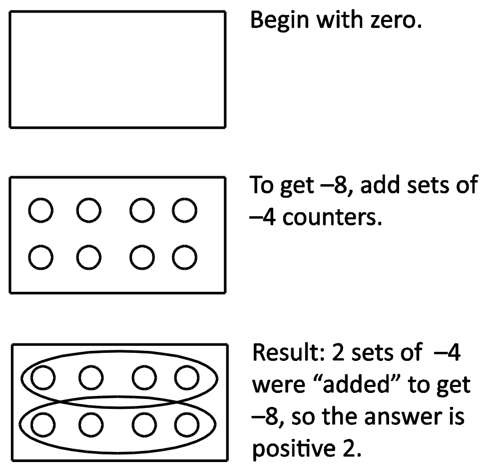


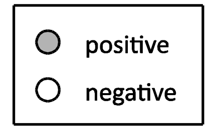
- Use relevant contexts for multiplication and division of integers (e.g., the impact on net worth if a person owes $6 to each of 3 friends, or if a debt of $6 to each of 3 friends is forgiven).
- Use counters to model multiplication and division as described on pages 145–146 of *Teaching Student-Centered Mathematics, Grades 5–8,* Volume Three (Van de Walle and Lovin 2006c). Have students write the number sentence.

Multiplication
$2 \times (-3) = (-6)$



Division
$-8 \div (-4) = 2$





Suggested Learning Tasks

- Solve the following problems using a model or diagram, and write the number sentence for each.
  - Greg borrowed $5 from each of his three friends. What is Greg's total debt?
  - Jan decided to donate $10 a month to her favourite charity for the next two years. The money deducted automatically from her bank account. What is the total of her deduction?
  - Nick has $20 and spends $4 per day for 7 days. What is his net worth at the end of the week?
- Kyle borrowed $6 from each of his two friends, Abdullah and Jayden. Because it was Kyle's birthday his friends each forgave Kyle's debt. Explain using pictures and words how this affected Kyle's net worth.
- Write each repeated addition as a multiplication.
  - $(-6) + (-6) + (-6) + (-6) + (-6)$
  - $(+4) + (+4) + (+4) + (+4)$
- Write each multiplication as repeated addition.
  - $(+7) \times (+2)$
  - $(+7) \times (-2)$
- The sum of two integers is −2. The product of the same two integers is −24. What are the two integers? Explain your reasoning.
- Make up a pattern to show a friend how to calculate $(+5) \times (-3)$.
- Play the "Operation Integers" Game:

  Players: 2 to 4
  Materials: A deck of cards (no face cards)
  Description: Deal all the cards face down on the table. Black suits are positive and red suits are negative. Each player turns over two cards and decides whether to add, subtract, multiply, or divide the two numbers on the cards. The player who has the greatest result wins all the cards that are face up.
  Goal: The play continues until one person (the winner) has all the cards.
- "Operation Integers" Game Variations:
  - Use fewer cards or cards with only certain numbers.

- Use fewer operations (limit to multiplication and division).
- Turn over three or four cards instead of two cards for each player.
- The player who has the least sum, difference, product, or quotient wins all the cards that are face up.
- Each player rolls two (or more) dice with integers on each face rather than using playing cards. The player with the greatest (or least) number resulting from the operations scores one point.
- The winner is the player with the most points.

- Evaluate each expression:
  - $(-4)-(+8)\times(-2)-(+15)$
  - $(+3)\times\left[(+14)\right]+(-18)-(+8)\div(-4)$
  - $\dfrac{\left[6+(-38)\right]\div 4(-2)}{(-2+4)(5-6)}$

SUGGESTED MODELS AND MANIPULATIVES

- algebra tiles*
- horizontal number lines*
- two-colour counters
- vertical number lines

* also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ commutative property ▪ integers ▪ negative ▪ opposite integer ▪ positive ▪ order of operations ▪ zero pair ▪ zero principle	▪ commutative property ▪ integers ▪ negative ▪ opposite integer ▪ positive ▪ order of operations ▪ zero pair ▪ zero principle

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 2: Integers
  - Section 2.1: Using Models to Multiply Integers
  - Section 2.2: Developing Rules to Multiply Integers
  - Game: What Is My Product
  - Section 2.3: Using Models to Divide Integers

  - Section 2.4: Developing Rules to Divide integers
  - Section 2.5: Order of Operations with Integers
  - Unit Problem: Charity Golf Tournament
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 139–141, 144–146.

Digital

- "Two-Colour Counters [unnamed]," *McGraw-Hill Education* (McGraw-Hill Education 2015): www.glencoe.com/sites/common_assets/mathematics/ebook_assets/vmf/VMF-Interface.html.

Patterns and Relations (PR)

GCO: Students will be expected to use patterns to describe the world and to solve problems.

GCO: Students will be expected to represent algebraic expressions in multiple ways.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Long-term Planning

- Yearly plan available at Mathematics Learning Commons: Grades 7–9: http://nsvs.ednet.ns.ca/nsps/nsps26/course/view.php?id=3875.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO PR01 Students will be expected to graph and analyze two-variable linear relations. [C, ME, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

PR01.01 Determine the missing value in an ordered pair for a given equation.
PR01.02 Create a table of values by substituting values for a variable in the equation of a given linear relation.
PR01.03 Construct a graph from the equation of a given linear relation (limited to discrete data).
PR01.04 Describe the relationship between the variables of a given graph.

Scope and Sequence

Mathematics 7	**Mathematics 8**	**Mathematics 9**
PR01 Students will be expected to demonstrate an understanding of oral and written patterns and their equivalent linear relations. **PR02** Students will be expected to create a table of values from a linear relation, graph the table of values, and analyze the graph to draw conclusions and solve problems.	**PR01** Students will be expected to graph and analyze two-variable linear relations.	**PR01** Students will be expected to generalize a pattern arising from a problem-solving context using a linear equation and verify by substitution. **PR02** Students will be expected to graph a linear relation, analyze the graph, and interpolate or extrapolate to solve problems.

Background

In Mathematics 7 students used algebraic expressions to describe patterns and constructed graphs from the corresponding **table of values**. They substituted values for unknowns and evaluated algebraic expressions. The distinction between expressions and equations was made and students worked with both.

Students in Mathematics 7 used input and output tables. It will be necessary to explain that the related pair of values in a table of values is called an ordered pair of the form (x, y) and that the input values correspond to x and the output values correspond to y. Students should also recognize that if the change in the x values is constant and the change in the corresponding y values is constant, then the relation is linear. Students will not formally encounter the slope and the y-intercept in Mathematics 8. It is important for students to be able to analyze linear relationships expressed graphically. If students make these generalizations, provide them with the correct terminology. In Mathematics 8 students need many experiences focusing on contextual problems and the relationship between the variables. Although there are different types of patterns, the focus is on increasing/decreasing linear patterns. The elements that make up these patterns are called **terms**. Consequently, these patterns are often referred to as growing patterns. For example, 2, 4, 6, 8, 10 . . . and 1, 2, 4, 8, 16 . . . are two common increasing patterns. Using a table to model an increasing/decreasing pattern can help students organize their thinking. It can also help them generalize the patterns symbolically.

Students will be expected to find both missing independent variables and dependent variables for **linear relations**. Finding a dependent variable, or y-coordinate in an ordered pair, is part of the process of creating a table of values. When students have to determine the value of an independent variable, or x-coordinate in an ordered pair, they will have to apply previous work with solving linear equations.

Constructing graphs from equations allows students to visualize linear relationships. When the ordered pairs resulting from a linear relation are graphed on a coordinate plane they fall along a straight line. Although many graphs are tied to everyday situations and are mainly located in Quadrant I, students need experiences graphing an equation on a four-quadrant grid. All work with graphing in this unit is limited to **discrete** data. Discrete data can only have a finite or limited number of possible values. Generally discrete data are counts: number of students in class, number of tickets sold, number of items purchased. **Continuous** data can have an infinite number of possible values within a selected range, such as quantities of temperature and time. A graph of discrete data has plotted points, but they are not joined together.

The following example demonstrates how one problem can be used to explore several performance indicators simultaneously.

- Zachary is planning a swimming party. Pool rental will cost him $30.00 for one hour. After the swim, everyone will have a snack. The snack costs $3.00 per person.

  Write an algebraic equation to represent this problem.

Students should be able to think about this relation as the cost being equal to three times the number of guests plus $30.00 for the pool rental.

Students are accustomed to choosing variables that represent a situation. For example, c represents the cost and g represents the number of guests (e.g., $c = 3g + 30$). Now, they must recognize that if the linear equation is written as $y = 3x + 30$, it means the same thing. The development of typical algebraic language and terminology is important for the introduction of the term ordered pairs. When variables such as x and y are chosen to be used in the equation, it is important for students to state what each variable represents. In this example, students would say that x represents the number of guests and y represents the total cost.

The equation above was used to complete the following table of values.

Number of guests x	**Total cost** y
0	30
1	33
2	36
3	39
4	42
5	45

Consider the situation where 8 guests attend Zachary's party. In this case, have students determine the cost of the party by substituting $x = 8$ into the equation $y = 3x + 30$.

$$
\begin{aligned}
&y = 3x + 30\\
&y = 3(8) + 30\\
&y = 24 + 30\\
&y = 54
\end{aligned}
$$

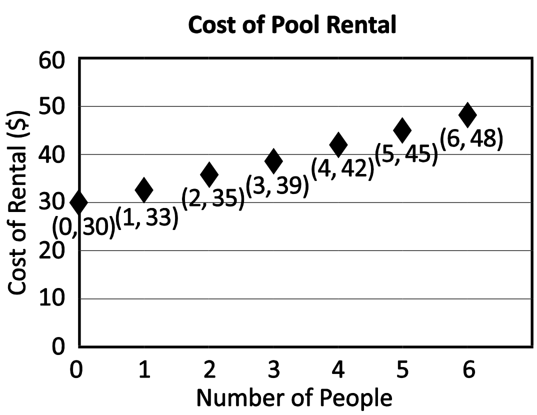
The cost of the party with 8 people is $54. Written as an ordered pair this would be (8, 54). Emphasize that all related pairs in a table of values may be written as ordered pairs. Using substitution, students may use the given equation to find missing values in any ordered pair.

Zachary has a budget of $60. Determine the maximum number of people he can invite to his party. Write this information as an ordered pair. In this case, have students determine the maximum number of guests by substituting *y* = 60 into the equation *y* = 3*x* + 30.

$$
\begin{aligned}
&y = 3x + 30\\
&60 = 3x + 30\\
&60 - 30 = 3x + 30 - 30\\
&30 = 3x\\
&\frac{30}{3} = \frac{3x}{3}\\
&10 = x
\end{aligned}
$$

For $60, Zachary can invite 10 guests. Written as an ordered pair, this would be (10, 60). Encourage students to verify the value of *x* in this case by using substitution.

Construct a graph from the equation of this linear relation.



Describe the relationship between the variables in the graph.

Students should be able to make statements such as the following:

- The variables represent the number of guests and the cost of the party.
- As the number of guests increases by one, the cost of the party increases by $3.
- The points lie on a line that goes up toward the right. The points are not connected as the data is discrete.
- The graph starts at the point (0, 30) and not at the origin (0, 0) because the pool rental costs $30 even if no guests are invited.

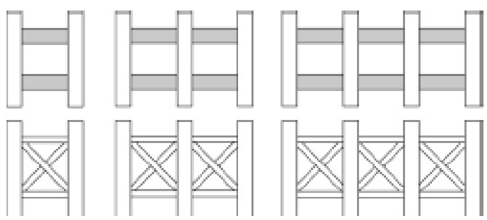
Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Tasks such as the following could be used to determine students' prior knowledge.

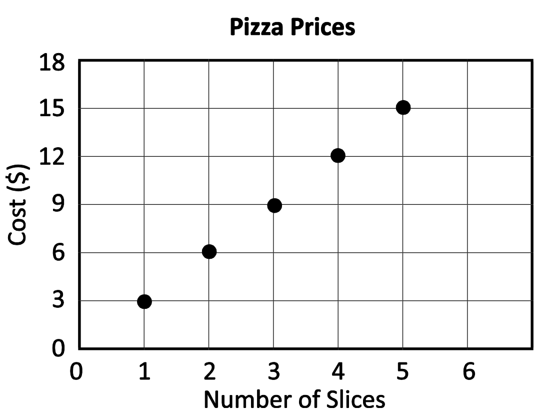
- For the following diagrams, ask students to
  - draw a representation of the pattern and extend the pattern
  - describe the pattern in their own words
  - develop a table
  - generate a graph



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample tasks (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Show students the Pizza Prices graph which shows a linear relation.
  - Describe the patterns on the graph.
  - What is the cost of one slice of pizza?
  - What is the relationship between the number of slices and the cost?
  - Make a table of values from the graph.
  - If 7 slices of pizza are purchased, what is the cost?



- Provide the following linear relation: $y = -2x + 3$.
  - Create a table for values of x beginning with 0.

- Draw the corresponding graph.
- Determine the value of y for the ordered pair (7, y).
- Determine the value of x for the ordered pair (x, 11).

- Tell students that Eric is organizing a skating party. He has to pay $50 to rent the rink and $4 for lunch for each person. He made a table of values, but he made an error in one of the costs. Identify the error and provide the correct value. Provide an explanation for the correction.

Number of people p	1	2	3	4	5	6	7	8
Cost in dollars c	54	58	62	68	70	74	78	82

- Mary has started a new exercise program. The first day she does 9 sit-ups, the second day she does 13, the third day 17, and the fourth day 21. This can be represented by $s = 4d + 5$.
  - Construct a graph of this linear relationship.
  - If she continues this way, how many sit-ups does she do, on the 5th day? 10th day? 20th day? 50th day?
- Provide students with a table of values that represents a linear relation.

x	−2	−1	0	1	2	3	4
y	−5	−3	−1	1	3	5	7

  - Have students graph the ordered pairs in the table of values.
  - Describe, in words, the relationship between the x–values and the y-values.
  - Write the linear relation using x and y.
- Explain the meaning of a linear relation using an example. What is the relationship between the variables?
- Use the equation $y = -3x + 4$ to complete the following:
  - Determine the missing values in the table.

x	−1	0	1	2	3	4
y						

  - Determine the value of y for the ordered pair (11, y).
  - Determine the value of x for the ordered pair (x, 13).
- James determined the mass of five pieces of a type of metal, representing a linear relationship. The table shows his results. James made one error in finding the masses.

Volume (cm^3)	8	9	10	11	12
Mass (g)	88	99	110	121	144

  - Using the relationship between the variables, identify the incorrect mass. Explain how you know this is the error. What is the correct mass?
  - Graph the ordered pairs from James' table of values.
  - How could you use the graph to show which value is incorrect?

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

- Encourage the use of proper algebraic vocabulary and terminology.
- Emphasize that all related pairs in a table of values may be written as ordered pairs.
- Have students arrange ordered pairs in ascending order based on the *x*-coordinate to establish a pattern to determine a missing value in a set.

SUGGESTED LEARNING TASKS

- Create a table of values for the equation, $k = 6(n + 2)$, by substituting values for 1 to 5 for *n*.

n	1	2	3	4	5
k					

- Using the table of values below, which represents a linear relation, complete the following:

x	1	2	3	4	5	6	7
y	4	8	12	16	20	24	28

  - Graph the ordered pairs in the table.
  - Determine the difference in value in consecutive *x*-values? *y*-values?
  - Describe the relationship between the *x*- and *y*-values.
  - Write an expression for *y* in terms of *x*.

x	−3	−2	−1	0	1	2
y				7	9	11

- Using the table of values below, which represents a linear relation with missing coordinates, complete the following:
  - How could you use a pattern to find the missing *y*-coordinates?
  - What are the missing coordinates?
- A community centre has a new banquet facility. It costs $8 per person to rent the centre.
  - Make a table of values showing the rental cost for 30, 60, 90, 120, and 150 people.
  - Graph the ordered pairs.
  - What is an expression for the rental cost in terms of the number of people?
- Determine the missing values in the following set of ordered pairs.

(0, 0), (1, 12), (2, 24), (3, __)
(–4, __),(–2, –6), (0, 2), (2, 10), (__, 18)

- An Internet company charges a basic monthly rate of $40 plus a per hour rate of $2. This can be described by the equation $c = 2h + 40$.
  - Determine the cost of using the Internet by completing the table below.

Hour (h)	Cost (c)
0	
1	
2	
3	
4	
5	
6	

  - Create a graph using the data from the table of values. How did the pattern in the table of values show up in the graph?
  - Matthew’s Internet bill for the first month was $100. How could you use the graph to find the number of hours he used the Internet? Use the equation to determine how many hours Matthew used the Internet for the first month.
- The table of value represents a linear relation.

x	1	2	3	4	5	6
y	5	10	15	20	25	30

  - Graph the ordered pairs in the table of values.
  - What is the difference in consecutive *y*–values? What is the difference in consecutive *x*–values?
  - Describe, in words, the relationship between the *x*–values and the *y*–values.
  - What is an expression for *y* in terms of *x*?

SUGGESTED MODELS AND MANIPULATIVES

- algebra tiles*
- grid paper
- pattern blocks*

* also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ continuous ▪ discrete ▪ equation ▪ expression ▪ formula ▪ linear relation	▪ continuous ▪ discrete ▪ equation ▪ expression ▪ formula ▪ linear relation

▪ pattern ▪ relations ▪ table of values ▪ term ▪ variable ▪ *x*-value ▪ *y*-value ▪ independent variable ▪ dependent variable	▪ pattern ▪ relations ▪ table of values ▪ term ▪ variable ▪ *x*-value ▪ *y*-value ▪ independent variable ▪ dependent variable

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Student Book Unit 6: Linear Equations and Graphing
  - Section 6.6: Creating a Table of Values
  - Section 6.7: Graphing Linear Relations
  - Technology: Using Spreadsheets to Graph Linear Relations
  - Unit Problem: Planning a Ski Trip
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), pp. 287–294

Digital

- *National Library of Virtual Manipulatives* (Utah State University 2015): http://nlvm.usu.edu (Algebra manipulatives for Grades 6–8.)

SCO PR02 Students will be expected to model and solve problems, concretely, pictorially, and symbolically, where *a*, *b,* and *c* are integers, using linear equations of the form

- $ax = b$
- $\frac{x}{a} = b,\ a \neq 0$
- $ax + b = c$
- $\frac{x}{a} + b = c,\ a \neq 0$
- $a(x + b) = c$

[C, CN, PS, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

PR02.01 Model a given problem with a linear equation, and solve the equation using concrete models.
PR02.02 Verify the solution to a given linear equation, using a variety of methods, including concrete materials, diagrams, and substitution.
PR02.03 Draw a visual representation of the steps used to solve a given linear equation, and record each step symbolically.
PR02.04 Solve a given linear equation symbolically.
PR02.05 Identify and correct an error in a given incorrect solution of a linear equation.
PR02.06 Apply the distributive property to solve a given linear equation.
PR02.07 Solve a given problem, using a linear equation, and record the process.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
PR06 Students will be expected to model and solve, concretely, pictorially, and symbolically, problems that can be represented by one-step linear equations of the form $x + a = b$, where a and b are integers.	**PR02** Students will be expected to model and solve problems, concretely, pictorially, and symbolically, where a, b, and c are integers, using linear equations of the form ▪ $ax = b$ ▪ $\frac{x}{a} = b,\ a \neq 0$ ▪ $ax + b = c$ ▪ $\frac{x}{a} + b = c,\ a \neq 0$ ▪ $a(x + b) = c$	**PR03** Students will be expected to model and solve problems, where a, b, c, d, e, and f are rational numbers, using linear equations of the form ▪ $ax = b$ ▪ $\frac{x}{a} = c,\ a \neq 0$ ▪ $ax + b = c$ ▪ $\frac{x}{a} + b = c,\ a \neq 0$ ▪ $ax = b + cx$ ▪ $a(x + b) = c$ ▪ $ax + b = cx + d$ ▪ $a(bx + c) = d(ex + f)$ ▪ $\frac{a}{x} = b,\ x \neq 0$
PR07 Students will be expected to model and solve, concretely, pictorially, and symbolically, where a, b, and c are whole numbers, problems that can be represented by linear equations of the form ▪ $ax + b = c$ ▪ $ax = b$ ▪ $\frac{x}{a} = b,\ a \neq 0$		

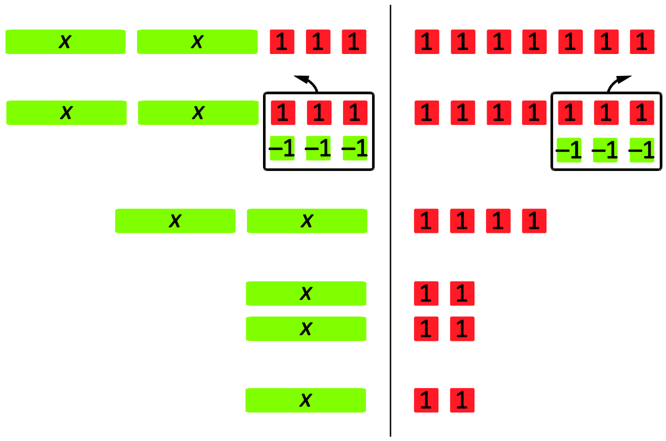
Background

Students have previously solved one-step equations of the form $x + a = b$, where a and b are integers. They have also solved two-step equations with two whole numbers. This unit builds on this previous experience and extends the numbers used to include integers. Students will also solve equations of the form $a\,(x + b) = c$, using the distributive property.

Significant work was done with concrete materials and diagrams in solving linear equations in Mathematics 7. Here, instruction should start with concrete materials and pictorial models, and then move to the symbolic, with the ultimate goal that students can solve one- and two-step equations with or without concrete or pictorial support.

When using concrete models to solve equations, the steps should also be recorded symbolically. In Mathematics 7, students applied the zero principle to add and subtract integers. Here they will use the same principle to isolate a variable in order to solve an equation. Equations must be kept balanced to maintain preservation of equality.

Algebra tiles can be used to solve $2x + 3 = 7$.



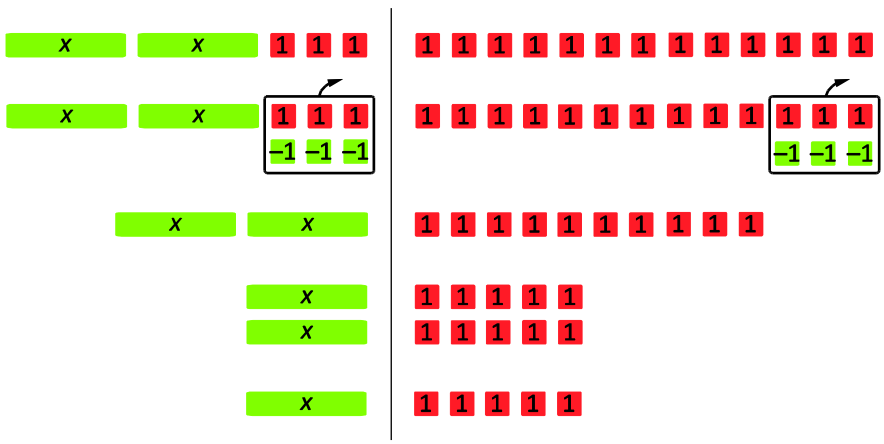
To maintain balance, record the operation that is being done to both sides of the equation on the same line as the equation. Recording the operation as a superscript (or subscript), which could be interpreted as exponents, is not an acceptable format. For example,

$$\begin{aligned}2x+3&=13\\2x+3-3&=13-3\end{aligned}$$

is an acceptable format,

$$\begin{aligned}2x+3&=13\\2x+3^{-3}&=13^{-3}\end{aligned}$$

is not acceptable.



$2x + 3 = 13$

$2x + 3 - 3 = 13 - 3$

$2x = 10$

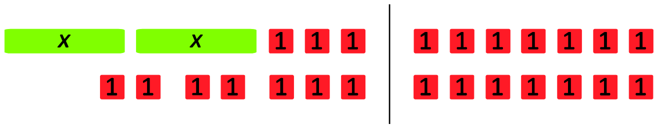
$\frac{2x}{2} = \frac{10}{5}$

$x = 5$

It is important to work with students to develop a procedure to solve equations so that they understand the mathematics involved. The important principles include the following:

- **Making zeros:** The reason you subtract 3 from both sides in the example above is to help isolate the variable. 3 – 3 creates an addition of zero, leaving 2*x* on the left-hand side. This is formally referred to as the *zero property of addition* or the *identity property of addition*.
- **Making ones:** The reason you would then divide by 2 is to isolate the variable, *x*. Dividing by 2 creates a multiplication by one, leaving just the variable on the left-hand side. This is formally referred to as the *inverse relationship* or the *identity property of multiplication*.
- **Preservation of equality:** To maintain balance and equality, you must do the same thing to both sides of the equation at the same time (on the same line of the equation).
- **Distributive property:** a property of real numbers that states that the product of the sum or the difference of two numbers is the same as the sum or difference of their products. Students should be familiar with the concept of distributive property as a quick multiplication strategy from Mathematics 5.

Have students verify all solutions to the linear equations. Verification helps develop increased understanding of the process involved. At this point, concrete materials and diagrams should be the focus when verifying solutions. To verify the previous solution symbolically, $x = 2$ will be substituted into the original equation. To model this with concrete materials, replace each variable tile with 2 positive unit tiles and then determine that both sides of the model have the same number of unit tiles, so balance has been maintained and the solution is correct.



Since both sides have the same value, the solution is demonstrated to be correct.

Students should now progress from balancing equations concretely to balancing equations symbolically. Students who have difficulty moving from models to the symbolic representation will benefit from more practice with models. Substitution can be used without models. Previously, with the aid of algebra tiles, it was determined that $x = 2$ for the equation $2x + 3 = 7$.

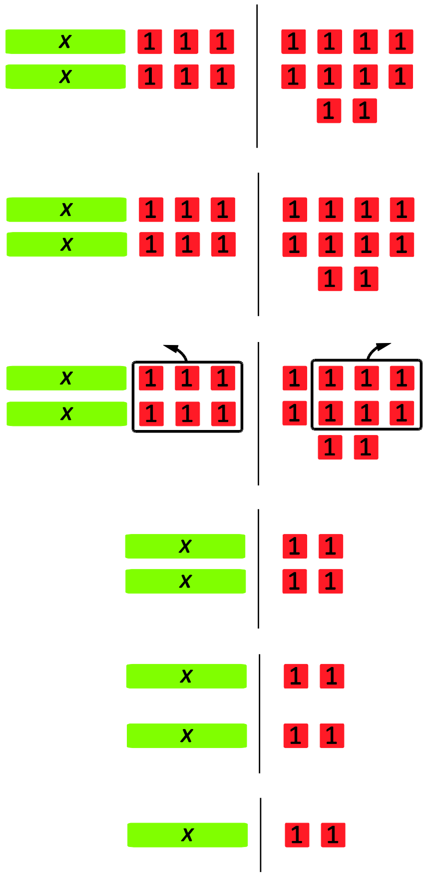
To verify this symbolically, substitute 2 into the original equation for x and evaluate.
$2x + 3 = 7$
$2(2) + 3 = 7$
$4 + 3 = 7$
$7 = 7$
Since both sides are equal, the equation is balanced and the solution is correct.

To solve a linear equation of the form $a\,(x + b) = c$ students will apply the distributive property. Students have used the distributive property with integers. This property is now extended to include algebraic expressions. Prior to solving equations, students should be able to explain the distributive property using diagrams or models, and use the distributive property to expand algebraic expressions. To reinforce the validity of the distributive property, it is useful to compare solutions obtained using the distributive property to solutions obtained using the order of operations.

Distributive Property	Order of Operations
4(5 + 2) (4)(5) + (4)(2) 20 + 8 28	4(5 + 2) 4(7) 28

A common error by students when using the distributive property is to distribute the negative sign over the first term only. Students may have to be reminded that an expression such as −(6 + 3) can also be written as −1(6 + 3) and each term in the brackets must be multiplied by −1. Having been exposed to simplifying expressions using the distributive property, students will then apply the property to solve equations. As with other types of linear equations, modelling equations of the form $a\,(x + b) = c$ should precede solving symbolically. An example of a model using algebra tiles to solve $2(x + 3) = 10$ follows:

Two groups of $(x + 3)$ equals 10



$$2(x + 3) = 10$$

$$2x + 6 = 10$$

$$2x + 6 - 6 = 10 - 6$$

$$2x = 4$$

$$\frac{2x}{2} = \frac{4}{2}$$

$$x = 2$$

Students should be encouraged to verify their own solutions to linear equations. It is also beneficial to provide them with worked solutions of linear equations to verify. Along with providing the correct answers, they should communicate about errors they find in solutions, including why errors might have occurred and how they can be corrected. This reinforces the importance of verifying solutions and recording solution steps, rather than only giving a final answer.

To solve problems, it is necessary for students to make connections to previous work with linear equations. Problem solving also requires communication, and can be done so through the application of a four-step process:

- understand the problem by identifying given information
- make a plan to solve the problem
- carry out the plan and record the solution
- verify that the solution is correct for the information given in the problem

Students should be able to solve problems such as the following:

- A grade 8 class held a charity car wash and earned \$368. They donated \$200 to an animal shelter, and shared the rest of the money equally among four other charities. How much money did each of the other four charities receive?
  - **Understand the Problem:** I can let *m* represent the amount of money each charity receives, since this is what I am trying to find. There are four charities. I know how much the animal shelter gets, and I also know the total amount raised.

- **Make a Plan:** I can multiply *m* by 4 and add it to the animal shelter amount to get the total. An equation to represent this is $4m + 200 = 368$, where *m* represents the amount each charity receives.
- **Carry out the Plan:** I can solve $4m + 200 = 368$ using strategies I learned previously.

$$
\begin{aligned}
&4m+200=368\\
&4m+200-200=368-200\\
&4m=168\\
&\frac{4m}{4}=\frac{168}{4}\\
&m=42
\end{aligned}
$$

Each charity will receive $42.

- **Verify the Solution:** If each charity receives $42, four of them together would receive $168. Add this to the animal shelter donation of $200 and the total is $368.

Using substitution, the solution can be verified algebraically.

$$
\begin{aligned}
&4m+200=368\\
&4(42)+200=368\\
&168+200=368\\
&368=368
\end{aligned}
$$

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Tasks such as the following could be used to determine students' prior knowledge.

- Solve each equation using algebra tiles, and by inspection. Verify each solution by substitution.
  - $c + 4 = 7$
  - $n - 3 = 6$

- Ask students to complete the following table.

Problem	Concrete Representation and Solution (When you have completed this column, show it to the teacher before moving on to the next two columns.)	Pictorial Representation and Solution	Symbolic Representation and Solution
$15 = n + 7$			
$t - 4 = 3$			
$12 = 4x$			
$\frac{p}{3}=2$			

Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Solve the following problems. Write the equation, solve it, and verify your answer.
  - The grade 8 students had a dance. The disc jockey charged $150 for setting up the music plus $3.00 per student who attended the dance. The disc jockey was paid $375. How many students attended the dance?
  - The high temperature today is 6°C higher than twice the high temperature yesterday. The high temperature today is 12°C. What was the high temperature yesterday?
- Tell students that Kim used the distributive property to solve the following equation: $12(x – 3) = 72$. Check her work to see if her solution is correct. If there is an error, correct it.

  $12(x – 3) = 72$
  $12x – 36 = 72$
  $12x – 36 – 36 = 72 – 36$
  $12x = 36$
  $x = \frac{36}{12}$
  $x = 3$

- Ask students which of the following produces the smallest value for *d*?

  $7d = 42$
  $\frac{d}{5} = –2 = –2$
  $3d + 4 = –5$
  $\frac{d}{4} + 12 = 36$
  $5(d + 4) = –15$

- Some cows and some chickens live on a farm. If the total number of legs is 38, and the total number of heads is 16, use algebra to find how many cows and how many chickens live on the farm.

- Explain each step in this solution to the equation $16 + 5m = 6$. Verify that the solution is correct.

  Step 1: $16 – 16 + 5m = 6 –16$
  Step 2: $5m = –10$
  Step 3: $m = –2$

- Sandy started with the equation $4p − 14 = −46$. She covered up the $4p$ and asked herself the question: What added to −14 gives −46?
  - What should her answer have been?
  - Using her answer above, Sandy wrote a new equation which was $4p = \square$. She then asked herself: What multiplied by 4 equals $\square$? What is the value of *p*?
- Solve $2(x − 4) = −20$ in two different ways. Record the processes you used.

- A taxicab company charges a basic rate of \$3.75 plus \$2.00 for every kilometre driven. If the total bill was \$33.75, use algebra to find how far the cab ride was.

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

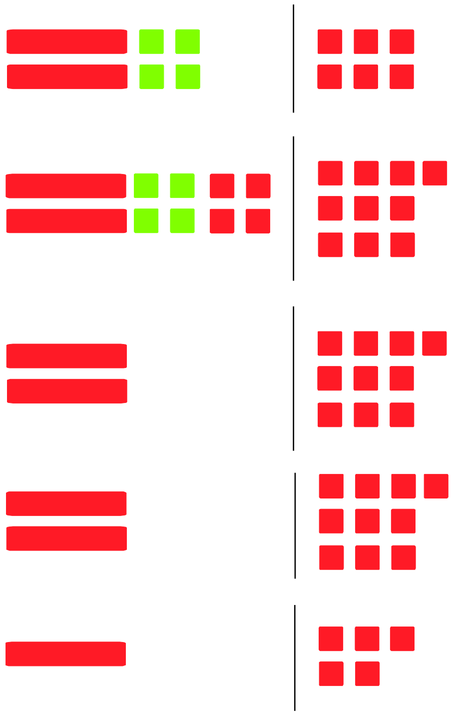
CHOOSING INSTRUCTIONAL STRATEGIES

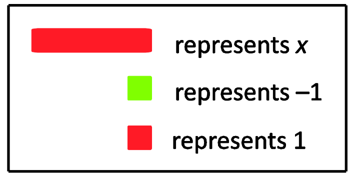
Consider the following strategies when planning daily lessons.

- Three friends decided to check their answers for a homework question. They each had a different solution to the linear equation $4(s - 3) = 288$ and wanted to determine whose answer was correct.

<u>Sam's Solution</u> $4(s-3)=288$ $4s-12=288$ $4s-12-12=288-12$ $\frac{4s}{4}=\frac{276}{4}$ $s=69$	<u>Leah's Solution</u> $4(s-3)=288$ $4s-12=288$ $4s-12+12=288+12$ $\frac{4s}{4}=\frac{300}{4}$ $s=75$	<u>Paul's Solution</u> $4(s-3)=288$ $4s-3=288$ $4s-3+3=288+3$ $\frac{4s}{4}=\frac{291}{4}$ $s=72.75$
<u>Sam's Verification</u> $4(s-3)=288$ $4(69-3)=288$ $4(66)=288$ $264\neq 288$ The solution is incorrect. By examining Sam's solution students should recognize that Sam should not have subtracted 12 in the third step, but added 12.	<u>Leah's Verification</u> $4(s-3)=288$ $4(75-3)=288$ $4(72)=288$ $288=288$	<u>Paul's Verification</u> $4(s-3)=288$ $4(72.75-3)=288$ $4(69.75)=288$ $279\neq 288$ The solution is incorrect. By examining Paul's solution students should recognize that he did not correctly apply the distributive property in step two.

- Use concrete materials and diagrams to demonstrate the idea of solving for "x" as a natural progression and lead the students to an understanding of the steps needed to isolate the variable. After exploring this progression, students will be able to solve for "x" in a linear equation and record the process.





$2x - 4 = 6$

$2x - 4 + 4 = 6 + 4$

$2x = 10$

$$\frac{2x}{2} = \frac{10}{2}$$

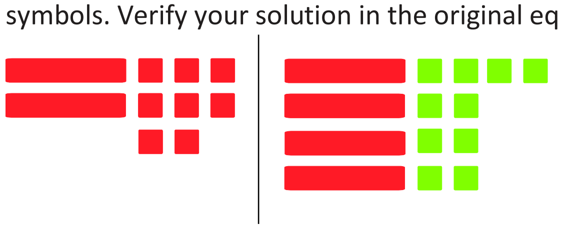
$x = 5$

- In the example above, it will not be intuitive to a student that if $-x = 5$, they can write the equivalent expression $x = -5$. This progression can be explained using the idea of opposites. Since x is the opposite of $-x$ and -5 is the opposite of 5. It will be a true statement that if $-x = 5$, then $x = -5$. The last line of the solution shows this symbolically and through modelling.
- Use an area model to expand expressions to explain the distributive property.
- Use interactive websites that allow students to explore solving linear equations, such as the algebra manipulatives for grades 6–8 from National Library of Virtual Manipulatives (http://nlvm.usu.edu/).

Suggested Learning Tasks

- Use the distributive property to write $7(c + 2)$ as a sum of terms. Sketch a diagram.

- Write and solve the equation modelled below, and write each step involved in the solution using symbols. Verify your solution in the original equation.



- Work with a partner to solve the following equations with algebra tiles. Take turns doing the following: Decide who will be the scribe and who will model the algebra tiles. The partner modelling with the tiles will tell the other person the steps to solve the equation. The scribe writes down the procedure algebraically.
  - $3x = -6$
  - $\dfrac{x}{3} = 4 + 2$
  - $6x = 4x - 4$

- Verify each of the following solutions. Identify and correct any errors.

  - $$x + 4 = 3$$
    $$x + 4 - 4 = 3 + 4$$
    $$x = 7$$

  - $$5 + 4x - 4x = 13 - 4x$$
    $$5 = 9x$$
    $$\frac{5}{9} = \frac{9x}{9}$$
    $$\frac{5}{9} = x$$

  - $$5 + 4x = 13$$
    $$56 = 8(x + 3)$$
    $$56 = 8x + 24$$
    $$\frac{32}{32} = \frac{8x}{32}$$
    $$x = \frac{8}{32}$$
    $$x = \frac{1}{4}$$

  - $$-2(x - 1) = -22$$
    $$-2x - 2 = -22$$
    $$-2x - 2 + 2 = -22 + 2$$
    $$-2x = -20$$
    $$\frac{-2x}{2} = \frac{-20}{2}$$
    $$x = 10$$

  - $$7x - 2 = -16$$
    $$7x - 2 + 2 = -16 + 2$$
    $$7x = -14$$
    $$x = -2$$

  - $$\frac{m}{6} + 3 = 11$$
    $$69(\frac{m}{6}) + 3 = 6(11)$$
    $$m + 3 = 66$$
    $$m + 3 - 3 = 66 - 3$$
    $$m = 63$$

- Determine methods of solving the following problems using a linear equation. Be prepared to present your solution methods to the class.
  - Joe is 16 years older than Bill. Sam is the same age as Bill. Their combined age is 79. How old are Joe, Bill, and Sam?
  - Your local high school sold advance tickets to their musical for $3.00 per ticket. Tickets purchased at the door were sold for $5.00 per ticket. How many tickets were sold in advance if 20 tickets were sold at the door and $340.00 were collected in total?
  - A baker is packaging cookies in identical boxes. She has filled seven boxes with all but 5 cookies in another box. She has packaged 51 cookies. How many cookies are in a full box?

SUGGESTED MODELS AND MANIPULATIVES

- algebra tiles*
- integer tiles*

* also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ balance ▪ constant ▪ equation ▪ equivalent ▪ evaluate ▪ expression ▪ formula ▪ identity property of addition ▪ identity property of multiplication ▪ inverse relationship ▪ one-step linear equation ▪ opposite operation ▪ substitution ▪ two-step linear equation ▪ variable ▪ zero property of addition	▪ balance ▪ constant ▪ equation ▪ equivalent ▪ evaluate ▪ expression ▪ formula ▪ one-step linear equation ▪ opposite operation ▪ substitution ▪ two-step linear equation ▪ variable

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 6: Linear Equations and Graphing
  - Section 6.1: Solving Equations Using Models
  - Section 6.2: Solving Equations using Algebra

  - Section 6.3: Solving Equations Involving Fractions
  - Section 6.4: The Distributive Property
  - Section 6.5: Solving Equations Involving the Distributive Property
  - Game: Make the Number
  - Unit Problem: Planning a Ski Trip
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006c), 287–294.

Digital

- *National Library of Virtual Manipulatives* (Utah State University 2015): http://nlvm.usu.edu (Algebra manipulatives for Grades 6–8.)

Measurement (M)

GCO: Students will be expected to use direct or indirect measurement to solve problems.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Long-term Planning

- Yearly plan available at Mathematics Learning Commons: Grades 7–9: http://nsvs.ednet.ns.ca/nsps/nsps26/course/view.php?id=3875.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO M01 Students will be expected to develop and apply the Pythagorean theorem to solve problems. [CN, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M01.01 Model and explain the Pythagorean theorem concretely, pictorially, or using technology.
M01.02 Explain, using examples, that the Pythagorean theorem applies only to right triangles.
M01.03 Determine whether or not a given triangle is a right triangle by applying the Pythagorean theorem.
M01.04 Determine the measure of the third side of a right triangle, given the measures of the other two sides, to solve a given problem.
M01.05 Solve a given problem that involves Pythagorean triples.

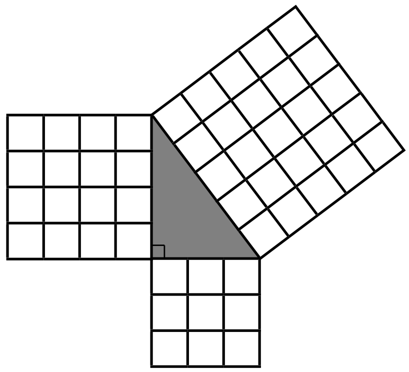
Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
M02 Students will be expected to develop and apply a formula for determining the area of triangles, parallelograms, and circles.	**M01** Students will be expected to develop and apply the Pythagorean theorem to solve problems.	—

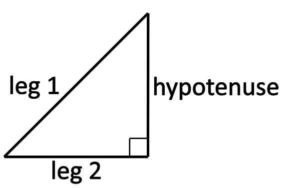
Background

Pythagoras of Samos, c. 560–c. 480 BC, was a Greek philosopher who is credited with providing the first proof of the Pythagorean theorem. The Egyptians and other ancient cultures used a 3-4-5 rule in construction. In Egypt, Pythagoras studied with engineers, known as “rope-stretchers,” who built the pyramids. They had a rope with 12 evenly spaced knots. If the rope was pegged to the ground in the dimensions 3-4-5, a right triangle would result. This enabled them to lay the foundations of their buildings accurately.

The Pythagorean theorem states that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides. In this diagram, the symbolic sentence for this relationship is 25 = 9 + 16.



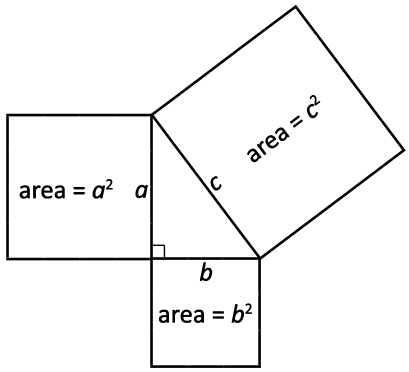
The Pythagorean theorem is used to find the unknown length of a side in a right triangle and therefore is usually thought of as a relationship among the lengths of the sides of a right triangle. For the diagram above, it would also be true to say that 52 = 32 + 42. The two sides that form the right angle are called “legs” and the side opposite the right angle is called the “hypotenuse.”



The notation used for this theorem is h^2=$(leg_2)^2$+$(leg_2)^2$.

Have students do investigations, where they explore and discover this relationship. Give groups of students a variety of right triangles that have whole number sides, such as the 3 cm, 4 cm, 5 cm triangle, the 6 cm, 8 cm, 10 cm triangle, or the 5 cm, 12 cm, 13 cm triangle (or ask students to draw such triangles). Have students cut out squares from centimetre grid paper so the sides of each square are the same as the side lengths for each triangle. Place the squares on the sides of the triangle as shown. Find the area of each square. Ask students what they notice. This same development can be done with square tiles and 1-inch graph paper. Give students a non-example where the area relationship will not work and therefore the triangle is not right-angled.

The conventional formula used for the Pythagorean theorem is c^2= a^2 + b^2. This formula should not be introduced until the investigations that will establish understanding are completed and students are able to use different variables when stating the Pythagorean theorem.

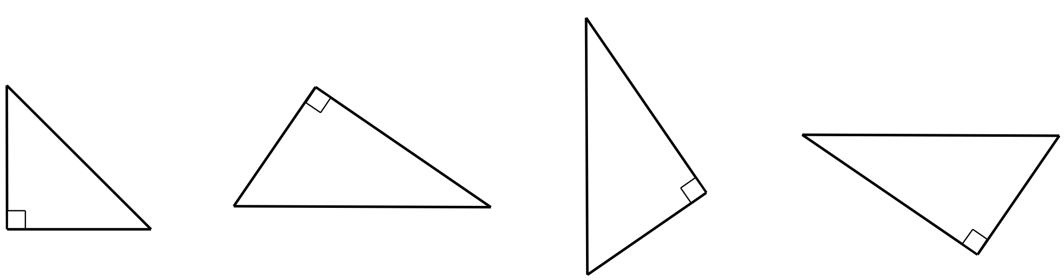


It is also important to note that sides of triangles are labelled with lower case letters, as the upper case letters are reserved for the vertices. It is common convention to label a triangle in a counter clockwise direction, using uppercase letters A, B, and C as vertices, and their opposite side lengths are named using corresponding lower case letters, a, b, and c.

Whenever a triangle has a right angle and two known side lengths, the Pythagorean theorem should come to mind as the relationship to use to find the missing side. Provide students with experiences that involve finding the length of the hypotenuse as well as situations where the hypotenuse and one side are known and the other side is to be found.

Some students develop the misconception that there are different formulas for the Pythagorean theorem that can be applied to a right triangle. It should be made clear that $c^2 - b^2 = a^2$, for example, is a rearrangement of the equation $c^2 = a^2 + b^2$. Whether formula rearrangement is used first, or side lengths are substituted into the Pythagorean theorem immediately, the procedure reiterates the concept of preservation of equality that students were introduced to in Mathematics 7.

It is important to present diagrams of right triangles in various orientations.



Students should recognize the hypotenuse as being the side opposite the right angle, regardless of the orientation of the figure. They should also recognize that the hypotenuse is the longest side of the triangle. While the use of technology is permissible, students should be encouraged to attempt to find an unknown side without the use of calculators. This will help develop mental mathematics skills and number sense.

It is also true that if the Pythagorean theorem works for a given triangle, that triangle is a right triangle. This is known as the "converse of the Pythagorean theorem" and states that if the sides of a triangle have lengths a, b, and c such that $\boldsymbol{c^2 = a^2 + b^2}$, then the triangle is a right triangle. These side lengths are called a **Pythagoren triple**. Such a triple is commonly written a, b, c and a well-known example is 3, 4, 5. Students should be able to use the idea of Pythagorean triples to determine if three given side lengths are, or are not, the sides of a right triangle. The emphasis should be on properly being able to used Pythagorean triples and not on this idea being called a converse to a theorem.

If *a*, *b*, *c* is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k. For example, since 3, 4, 5 is a triple then so is 6, 8, 10 and so on. As mentioned, Egyptians and other ancient cultures used a 3-4-5 rule (a = 3, b = 4, c = 5) in construction to ensure buildings were square. The 3-4-5 rule allowed them a quick method of establishing a right angle. This is still used today in construction.

Right triangles with non-integer sides do not form Pythagorean triples. For instance, the triangle with sides a = 1, b = 1, and c = 2 is a right triangle, but $(1, 1, \sqrt{2})$ is not a Pythagorean triple because $\sqrt{2}$ is not a positive integer. Some Pythagorean triples with *c* < 100 are indicated below.

(3, 4, 5) (5, 12, 13) (7, 24, 25) (8, 15, 17)

(9, 40, 41) (11, 60, 61) (12, 35, 37)   (20, 21, 29)

Many animated proofs of the Pythagorean theorem are available online; some can be found in the Resources section at the end of this outcome.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Tasks such as the following could be used to determine students' prior knowledge.

- Provide students with a set of triangles (be sure to include a variety of different types and different orientations). Have them sort the triangles first according to the length of the sides (equilateral, isosceles, scalene) and explain their sorting rule. Repeat the task having students sort the triangles according to the measures of the angles (right, acute, obtuse) and explain their sorting rule.
- Have students draw a variety of squares on grid paper and find the area of the squares.

Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

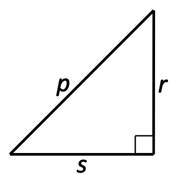
- Have students draw a 6 cm, 8 cm, 10 cm right triangle on grid paper. Have students explain the Pythagorean relationship and show the relationship symbolically.
- Have students solve problems such as the following:
  - For safety reasons, a construction company established the following rule: When placing a ladder against the side of a building, the distance of the base of the ladder from the wall should

be at least $\frac{1}{3}$ of the length of the ladder. Can an 8 m ladder reach a 7 m window when this rule is followed?

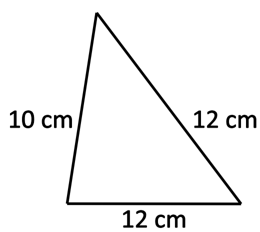
- An airplane is flying at an elevation of 5000 m or 5 km. The airport is 3 kilometres away from a point directly below the airplane on the ground. How far is the airplane from the airport?
- The dimensions of a rectangular frame are 10 cm by 24 cm. A carpenter wants to put a brace between two opposite corners of the frame. How long should the brace be?
- You have purchased a new entertainment centre wall unit. The space for the television is 60 cm by 80 cm. What is the largest television you can put in this space?
- A wheel chair ramp is 10.3 m long. It spans a horizontal distance of 9.7 m. What is the ramps vertical height of the ramp to the nearest tenth of a metre?

▪ Determine whether each of the following student's work is correct and explain your thinking.

- Corey wrote the Pythagorean relationship as $r^2 = p^2 + s^2$.



- Mia wrote the Pythagorean relationship as $12^2 = 8^2 + 10^2$.



▪ Explain how you can determine whether or not a triangle is a right triangle if you know that it has side lengths of 7 cm, 11 cm, and 15 cm.

FOLLOW-UP ON ASSESSMENT

Guiding Questions

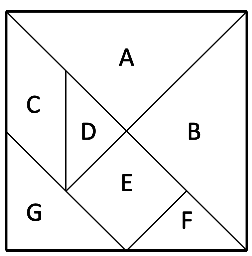
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

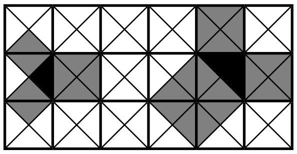
- Use concrete materials, such as geoboards, grid paper, dot paper, tangrams, etc., to establish the relationship between the hypotenuse and legs of a right triangle.
- Use technology such as The Geometer's Sketchpad (Key Curriculum Press 2015) and GeoGebra (International GeoGebra Institute 2015) to explore the relationships between the hypotenuse and legs of a right triangle.
- Provide students with a variety of problems that involve finding the hypotenuse, finding the measure of a missing side, and determining whether a triangle is a right triangle by applying the theorem.
- The Pythagorean theorem can be explored using Tangrams.



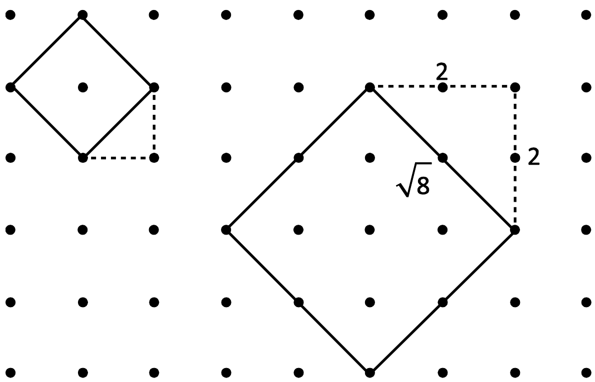
Have students place one of the small triangles in the centre of the paper and trace around it. Label the hypotenuse as h and the legs as leg_1 and leg_2. Use Tangram pieces to form a perfect square along each side of the triangle. Trace around the squares. The students should determine that two small triangles were used to make the square on $legs_1$ and leg_2, and four small triangles were needed to make the square on the hypotenuse. Discuss how the squares of leg_1 and leg_2 combined to make a square on the hypotenuse. Repeat this using the medium triangle, and then again using the large triangle. Have students compare the three drawings. Discuss the relationship of the areas of the squares along each leg of the right triangle to the area of the squares along the hypotenuse. The students should conclude that the sum of the areas of the squares on the legs is equal to the square of the hypotenuse.

SUGGESTED LEARNING TASKS

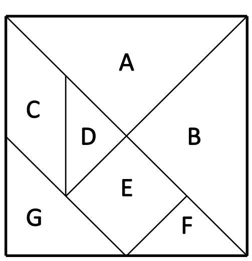
- To explore the Pythagorean theorem use a grid on which the diagonals of each square are drawn. Select any right triangle in the grid and find squares on the two shorter sides and a square on the hypotenuse.



The diagram below shows squares of 2 square units and of 8 square units. Using dot paper or geoboards, make squares of area 1 square unit, 4 square units, 5 square units, 9 square units, and 10 square units.



- Explore Pythagorean triples such as 3, 4, 5. Multiply each number by 2. Determine whether the resulting three numbers form a Pythagorean triple. Explore by multiplying by other whole numbers. Is there any whole number that does not make a Pythagorean triple when 3, 4, 5 are multiplied by it?
- Research and present a proof for Pythagorean theorem.
- Ross has a rectangular garden in his backyard. He measures one side of the garden as 7 m and the diagonal as 11 m. What is the length of the other side of his garden?
- The dimensions of a rectangular frame are 30 cm by 50 cm. A carpenter wants to put a diagonal brace between two opposite corners of the frame. How long should the brace be?
- Designate the side length of the square made of the seven Tangram pieces as 1 unit. Using the Pythagorean theorem, determine the lengths of all sides of each of the seven Tangram pieces.



- Determine whether each triangle with sides of given lengths is a right triangle.
  - 9 cm, 12 cm, 15 cm
  - 16 mm, 18 mm, 29 mm
  - 7 m, 9 m, 13 m
  - 6 cm, 7 cm, 13 cm
- Draw triangles other than right triangles. Measure the side lengths and check to see if the Pythagorean theorem works for these non-right triangles.

- Carpenters often use a 3-4-5 triangle to determine if corners are square (90°). Explain, in your own words, why this works.

SUGGESTED MODELS AND MANIPULATIVES

- colour tiles*
- diagonal grid paper
- dot paper
- Geoboards*
- grid paper
- tangrams*
- The Geometer's Sketchpad

* also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ hypotenuse ▪ leg ▪ Pythagorean theorem ▪ relationship ▪ side ▪ square root	▪ hypotenuse ▪ leg ▪ Pythagorean theorem ▪ relationship ▪ side ▪ square root

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 1: Square Roots and the Pythagorean Theorem
  - Section 1.5: The Pythagorean Theorem
  - Technology: Verifying the Pythagorean Theorem
  - Section 1.6: Exploring the Pythagorean Theorem
  - Section 1.7: Applying the Pythagorean Theorem
  - Unit problem: The Locker Problem
- *ProGuide* (CD [Word files]; NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD; NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), pp. 205–206

Mathematics for Elementary Teachers: A Contemporary Approach, Seventh Edition (Musser, Peterson, and Burger 2006), pp. 667–668

Developing Thinking in Geometry (Johnson-Wilder and Mason 2005), pp. 113–119

Digital

- "Proof without Words: Pythagorean Theorem," *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/Activities.aspx?grade=3
- SmartBoard users can locate an animated proof at www.smarttech.com
- The Geometer's Sketchpad (Key Curriculum 2013; NSSBB #: 50474, 50475, 51453)
- *Pythagorean Theorem Water Demo* (YouTube 2009): www.youtube.com/watch?v=CAkMUdeB06o.

SCO M02 Students will be expected to draw and construct nets for 3-D objects. [C, CN, PS, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M02.01 Match a given net to the 3-D object it represents.
M02.02 Construct a 3-D object from a given net.
M02.03 Draw nets for a given right cylinder, right rectangular prism, and right triangular prism, and verify by constructing the 3-D objects from the nets.
M02.04 Predict 3-D objects that can be created from a given net, and verify the prediction.

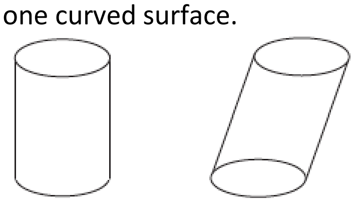
Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
–	**M02** Students will be expected to draw and construct nets for 3-D objects.	–

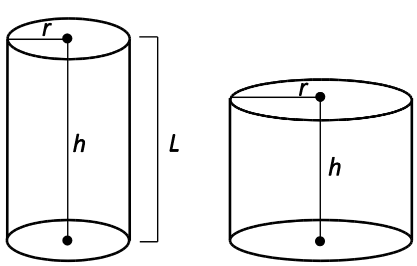
Background

In Mathematics 4, students constructed rectangular and triangular prisms from nets. Now students will study the use of nets to investigate and create 3-D objects, and draw or match a net to its corresponding 3-D object. Working with concrete models allows students to visualize the figures and encourages them to use reasoning when they explore related measurement concepts. Polydrons are excellent manipulatives for these activities (with the exception of the right cylinder).

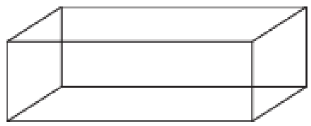
A cylinder is a geometric figure with two parallel and congruent flat surfaces, called **bases**, connected by one curved surface.



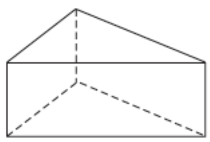
A **right cylinder** is a geometric figure with two parallel and congruent, flat circular bases connected by one curved surface. A right cylinder has a 90° angle where the base and height meet.



A prism has two congruent bases. The name of the prism is determined by the base. When all faces, excluding the two bases, are rectangles and perpendicular to the bases, the prism is called a right prism. A **right rectangular prism** is a prism whose six faces are rectangles; a prism with a rectangular base.



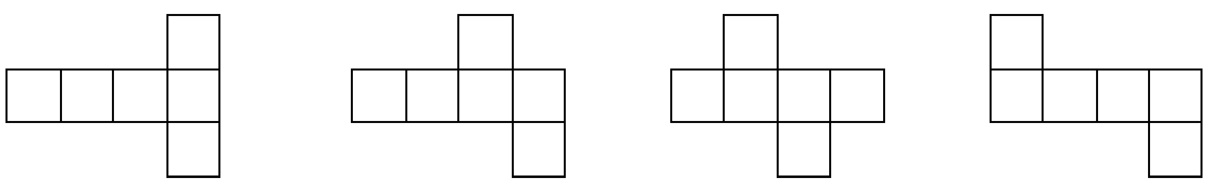
A **right triangular prism** is a prism with a triangular base whose faces meet the base at right angles.



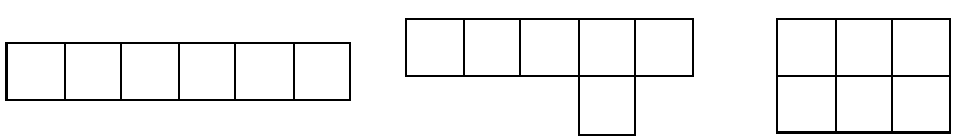
A **net** is a 2-D representation of a 3-D object that can be folded to recreate the object. A net shows all of the **faces** of an object. A net can be used to make a 3-D object called a **polyhedron**. Two faces meet at an **edge**. Three or more faces meet at a **vertex**. When students are making nets, they should focus on the faces, and how the faces fit together to form the shape. A polygon is a closed-plane shape that only has straight sides.

It is important for students to realize that there may be many different nets for a single 3-D object. Even though the faces do not change, they can be arranged in different ways and still fold to create the same 3-D object.

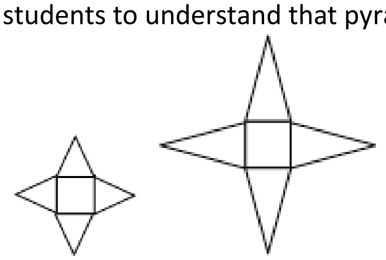
Note: It is not a different net if it is a reflection or rotation of one that is already constructed. Some of the nets for a cube follow. There are 11 in total. Students should be encouraged to find as many different nets as possible for a cube (a right rectangular prism).



Students cannot assume that because a cube has six square faces, any grouping of six squares will create a net. The following are not nets for a cube:



A regular pyramid has a regular polygon as its base. The other faces are triangles. It is important for students to understand that pyramids with different heights can be created on the same base.



Students should always be encouraged to visualize and predict before actually folding the nets to construct the 3-D objects.

An alternative to finding nets with polygons is to build the polyhedron and take it apart to find as many nets as possible.

Note: This outcome is closely related to Grade 8 M03.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

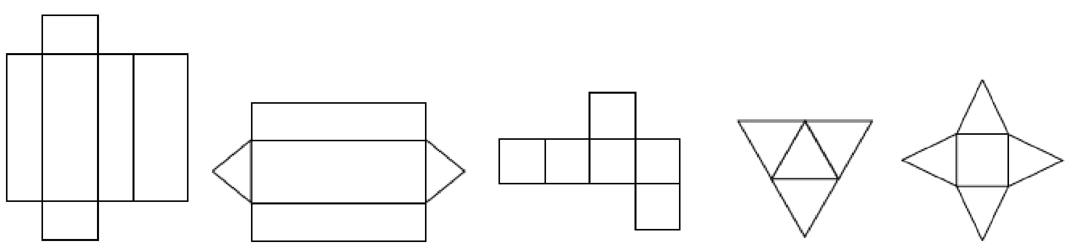
Tasks such as the following could be used to determine students' prior knowledge.

- Play "What Polygon Am I?" by providing the following clues:
  - I have four congruent sides.
  - I have one pair of sides that are both parallel and congruent.
  - My lines of symmetry go through the midpoints of my two opposite sides.

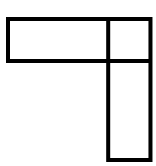
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

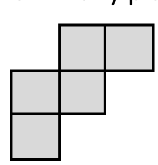
- Identify the object from its net:



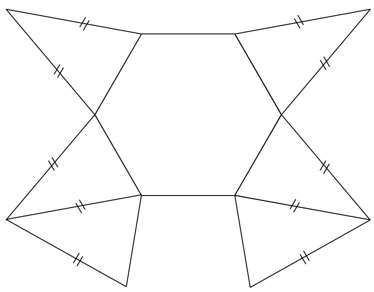
- Tell the students that this diagram is part of a net for a prism. Ask them to complete the net by drawing the additional faces that would be needed.



- Provide the students with a pentomino puzzle piece (a 2-D shape made by joining 5 squares along full sides) that would fold to make a box with no top, then add a square for the top of the box. In how many places can the square be added? (*Note: This can be cut from grid paper.*) For example:



- Have students draw all the possible nets for a triangular pyramid with all faces equilateral triangles. Repeat for a triangle pyramid with an equilateral base and three isosceles triangular faces. Ask: Did you get more nets for one of them? Why do you think this happened?
- Give students the diagram below (found in Appendix A). Ask students to predict if it is a net, to check their predictions by cutting it out, and to make any needed changes to create a true net if needed.



- Provide students with a prism or pyramid and some wrapping paper. Ask them to roll and trace a net for the shape, cut the net out, and to wrap the net to determine the shape created. Unwrap the net, cut off one face, and ask them for the possible places this face could be reattached to produce other nets. Use tape to reattach and check. *Extension: If centimetre graph paper is used for this activity, a good connection to surface area can be made.*

FOLLOW-UP ON ASSESSMENT

Guiding Questions

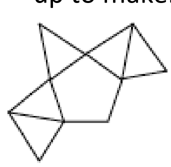
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

- Present the students with nets of a right prism and a pyramid that have the faces joined in a different way from the ones they have cut out before. Ask them to predict what shape it would fold up to make. Have them cut it out and fold to check their prediction. For example:



- Provide students with a square or rectangular right prism and an 11-pin x 11-pin geoboard. Ask them to use elastics to construct a net for the prism. Ask them to discuss how they might move one of the faces to make a new net for the same prism. Have them check by recording the new net on square dot paper and cutting it out.

- Have students explore a variety for methods of drawing nets. One method is to rolling and trace faces of a 3-D object, and then to cut out the net. Students could also create nets by wrapping 3-D objects with paper.
- Give students opportunities to investigate nets of pyramids, cylinders, and prisms, and draw nets for right cylinders, right rectangular prisms, and right triangular prisms.
- Provide copies of nets for students to cut out and fold up. They should be encouraged to unfold them and examine the 2-D shapes that are connected to make each net.
- Ensure that students focus on the faces and how the faces fit together to form the 3-D object. Students should be reminded that the pieces must be the correct size and to connect the shapes in the net. They may have all the pieces, but may still have difficulty drawing the net.

SUGGESTED LEARNING TASKS

- Cut along the edges of various shaped containers (cereal boxes, tennis ball canisters, potato chip cans, etc.) and unfold them to form a net. Predict what the net will look like before you cut it.
- Predict whether a net can be folded into a 3-D object. Using Polydrons, the students can build the nets and check to see that they fold into the 3-D objects.
- Find all of the nets for a square-based pyramid.

SUGGESTED MODELS AND MANIPULATIVES

- dot paper
- geoboards*
- grid paper
- pentominoes
- Polydrons

* also available in *Interactive Math Tools* (Pearson n.d.)

MATHEMATICAL LANGUAGE

Teacher	Student
▪ 2-D shapes ▪ 3-D objects ▪ area ▪ base ▪ cube ▪ edge ▪ face ▪ net ▪ polyhedron ▪ polyhedra ▪ regular pyramid ▪ right cylinder ▪ right rectangular prism ▪ right triangular prism ▪ regular prism ▪ rurface area ▪ vertex ▪ volume	▪ 2-D shapes ▪ 3-D objects ▪ area ▪ base ▪ cube ▪ edge ▪ face ▪ net ▪ polyhedron ▪ polyhedra ▪ regular pyramid ▪ right cylinder ▪ right rectangular prism ▪ right triangular prism ▪ regular prism ▪ rurface area ▪ vertex ▪ volume

Resources

Print

- *Math Makes Sense 8* (Baron et al. 2008; NSSBB #: TBD)
  - Unit 4: Measuring Prisms and Cylinders
    - Section 4.1: Exploring Nets
    - Section 4.2: Creating Objects from Nets
    - Section 4.3: Surface Area of a Right Rectangular Prism
    - Section 4.4: Surface Area of a Right Triangular Prism
    - Section 4.7: Surface Area of a Right Cylinder
  - *ProGuide* (CD [Word files]; NSSBB #: TBD)
    - Assessment Masters
    - Extra Practice Masters
    - Unit Tests
  - *ProGuide* (DVD; NSSBB #: TBD)
    - Projectable Student Book Pages
    - Modifiable Line Masters
- *Making Math Meaningful to Canadian Students, K–8* (Small 2008) pp. 305–306.
- *Developing Thinking in Geometry* (Johnson-Wilder and Mason 2005), pp. 98–99

Digital

- "Interactives: Geometry 3D Shapes," *Annenberg Learner* (Annenberg Foundation 2014): www.learner.org/interactives/geometry/platonic.html.

SCO M03 Students will be expected to determine the surface area of right rectangular prisms, right triangular prisms, and right cylinders to solve problems. [C, CN, PS, R, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M03.01 Explain, using examples, the relationship between the area of 2-D shapes and the surface area of a given 3-D object.
M03.02 Identify all the faces of a given prism, including right rectangular and right triangular prisms.
M03.03 Identify all the faces of a given right cylinder.
M03.04 Describe and apply strategies for determining the surface area of a given right rectangular and right triangular prism.
M03.05 Describe and apply strategies for determining the surface area of a given right cylinder.
M03.06 Solve a given problem involving surface area.

Scope and Sequence

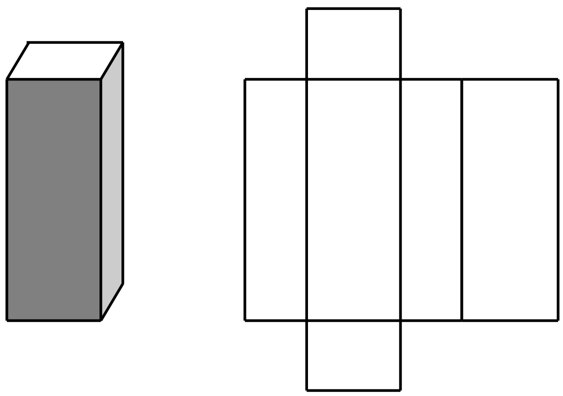
Mathematics 7	Mathematics 8	Mathematics 9
M02 Students will be expected to develop and apply a formula for determining the area of triangles, parallelograms, and circles.	**M03** Students will be expected to determine the surface area of right rectangular prisms, right triangular prisms, and right cylinders to solve problems.	**G01** Students will be expected to determine the surface area of composite 3-D objects to solve problems.

Background

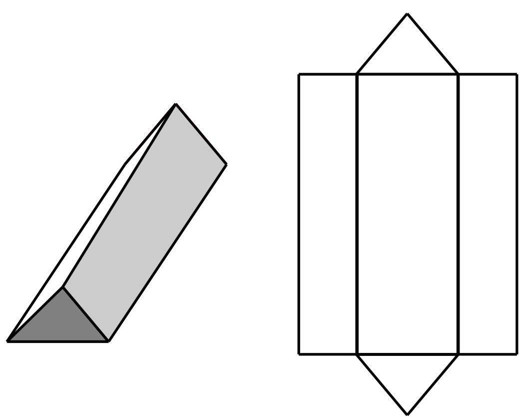
Surface area is the sum of the areas of all surfaces of a 3-D object. Surface area calculations should be a direct extension of previous work with **area** formulas and **nets**. A brief review of area of rectangles, triangles, and circles may be required. It is important for students to be able to visualize the net of a 3-D object to calculate the surface area of that object efficiently. It is important to use concrete materials, such as polydrons or boxes, to help students visualize the relationship between the **2-D net** and the **3-D object**. Explain that square units (e.g., cm^2) are used to measure area and surface area and need to be included.

In calculating surface area, students should start with objects such as cereal or cracker boxes for rectangular prisms and boxes from some types of chocolate bars or toys for triangular prisms. These objects can be cut open, and the net determined. Students should estimate the area of each face and total the areas to find the surface area. To calculate surface area, students must determine the dimensions of each part of the net and apply appropriate formulas to calculate each of the areas. Have students compare and discuss similarities and differences in their approaches. Teachers should facilitate discussions of the different methods, but encourage students to use the most efficient methods.

The surface area of a prism can be determined from its net, as the net shows all faces making up the object. Working from the net also allows for easy identification of congruent faces, which sometimes avoids the necessity of having to find the areas of each face individually. Some students may conclude that a rectangular prism has three pairs of congruent sides and therefore they can calculate the surface area using the formula SA = $2lw + 2lh + 2wh$. However, this formula should *not be* the focus. To ensure students have gained the conceptual understanding of surface area, other strategies should be explored before introducing the formula.

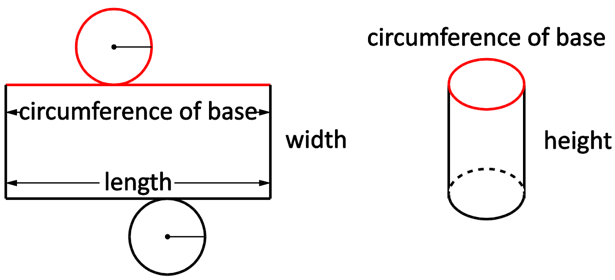


The net of a triangular prism shows the two triangular faces and three rectangular faces making up the prism. Students should recognize that the triangular bases of a **right triangular prism** are always congruent, and the three rectangular faces on the sides are congruent because they are attached to equal sides of the triangular bases.



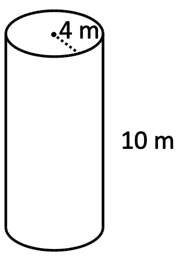
Note: Students must always include the units as part of the solution. Surface area is measured in square units.

A possible net for a cylinder is shown below. One dimension of the rectangle is the circumference of the circle, and the other is the height of the cylinder. It is important to note that a cylinder only has two faces (the flat circle surfaces), even though its net appears to have three. The curved surface in the cylinder is not a face. Students should discover that the width of the rectangle is actually the circumference of the circle and the length of the rectangle is the height of the cylinder.



One possible exploration would involve having students draw a net of a **right cylinder**. Ask students how to use the net to find the surface area. A sample discussion follows.

- The surfaces of a cylinder are the two faces (circles) and a curved surface, that opens to a rectangle. For the cylinder shown here, first calculate the area of the circles.



$$A_{circle} = \pi r^2$$
$$A_{circle} = \pi \times 4^2$$
$$A_{circle} = 16\pi$$
$$A_{circle} \doteq 50.24 m^2$$

This means the area of two circles is approximately 100.48 m^2.

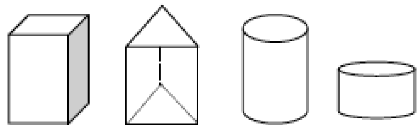
Note: It is acceptable to use 3.14 as an approximation of pi.

It may be easier to recognize that the other face is a rectangle if students use an object that can be unrolled. They should then be able to see that the length of the rectangle is actually the circumference of the circle and the width of the rectangle is the height of the cylinder. So the area becomes

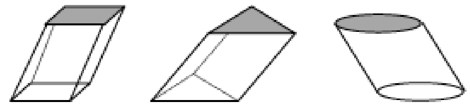
$$A = 2\pi r \times h$$
$$A = 2\pi 4 \times 10$$
$$A \doteq 251.2 m^2$$

The surface area is the total area of all faces, or 100.48 + 251.2 = 351.68m^2.
This leads to the development of the formula $SA=2\pi r^2+2\pi rh$.

Right rectangular prisms, right triangular prisms, and right cylinders are ones where the bases are aligned directly above each other as shown below and the bases are congruent.



right prisms and cylinders



oblique prisms and cylinders

Assessment, Teaching, and Learning

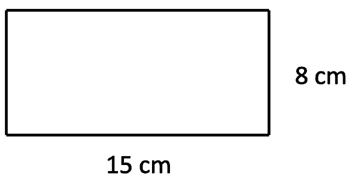
Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

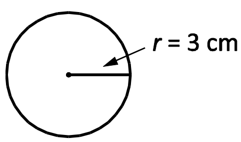
Tasks such as the following could be used to determine students' prior knowledge.

Calculate the following for each of these shapes

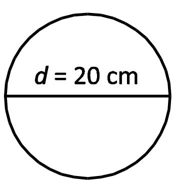
area



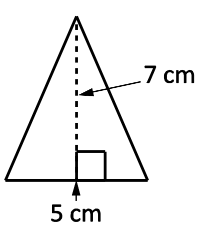
diameter



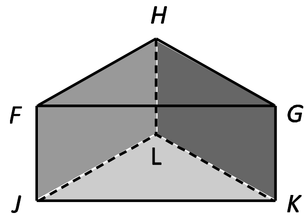
area

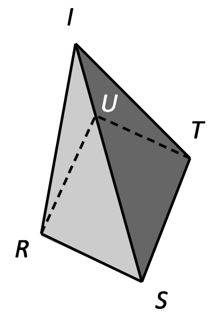


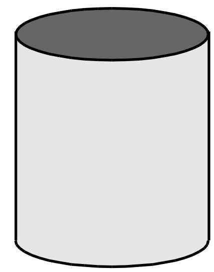
area

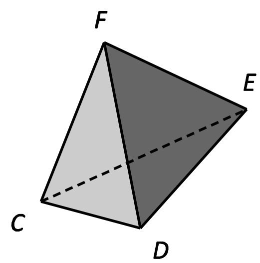


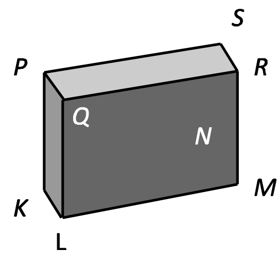
- Identify each object. Name the number of faces.







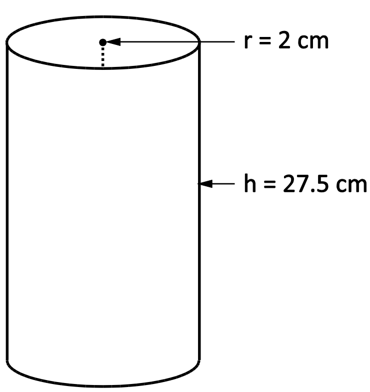




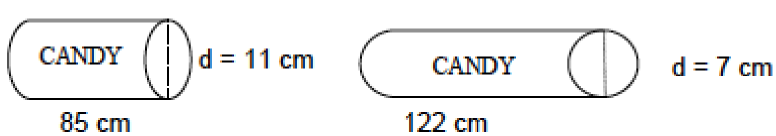
Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

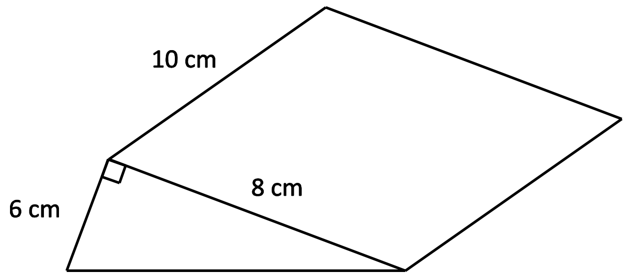
- How does drawing the net of a prism help you calculate its surface area?
- Have students calculate the surface area of the can below.



- Tell students that Jennifer and Jamie each bought a tube of candy. Both tubes cost the same amount. Which tube required more plastic to make?



- Find the surface area of the wedge of cheese shown.



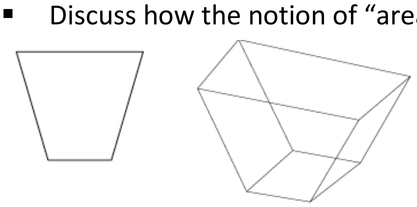
- Have students calculate the surface area of a box that contains a tablet (rectangular prism) to the nearest tenth of a square centimetre. Its plastic wrap covering the box measures 21.2 cm long, 14.1 cm wide, and 3.3 cm thick.
- Have students calculate the surface area of the pencil sharpener on Kay's desk. It is a right cylinder and has a diameter of 3.1 cm and a length of 5 cm.

Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

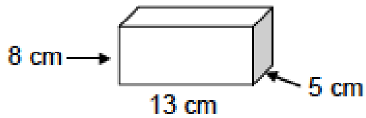
Consider the following strategies when planning daily lessons.

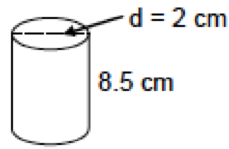
- Provide paper copies of nets for students who are having difficulty visualizing the parts of a 3-D object, for them to cut and fold or use Polydrons.
- Allow students to use Polydrons to build objects and unfold the Polydron pieces to find all of the possible nets for a given 3-D object.
- Use a variety of different shapes of boxes and containers for cutting and calculating surface area.
- Allow students the opportunity to place an object on grid paper and trace the faces.
- Provide students with the opportunity to build rectangular prisms from multi-linking cubes and determine the number of squares needed to cover the prism.
- Encourage students to estimate the surface area before calculating the exact answer to check the reasonableness of their calculations.
- Discuss with students why surface area is an important consideration for companies when they are deciding on the shape and sizes of their packages.
- Discuss how the notion of "area" differs for these two objects.



SUGGESTED LEARNING TASKS

- The owners of a cracker factory are trying to choose a box to hold their new flavour of cracker. They want a box that uses the least amount of cardboard. Which box should they choose? Use grid paper and calculators to help find the solution.





- Explain how two cylinders can have the same height, but different surface areas.
- Explore which of the twelve pentominoes could fold to make an “open” box. Why is the surface area the same for all of the boxes?
- Explain how calculating the surface area of a cylinder and calculating the surface area for a prism are alike and how are they different. Select a net of a right rectangular prism from a previous activity. Discuss these questions with the class:
  - How many faces does the prism have?
  - What shape are the faces?
  - Are any of the faces congruent? How do you know?
  - When the net is folded how many edges are there?
  - How many vertices are there?
  - Repeat this exercise for a right triangular prism and a right cylinder.
- Marie has 1 m^2 of paper to wrap a gift box 28 cm long, 24 cm wide, and 12 cm high. Does she have enough paper?
- A family is renovating their home and must redo the siding. Siding is on sale for $15.00 per square metre. How much will the siding cost to fully cover the outside walls of the house? Note: The dimensions of their house is 18 m long, 9 m wide, and 4 m high.
- Use the following question as a more challenging problem that can be worked on as a class and should be used after students have had some experience with the outcome. A cylindrical CD container has surface area of 225.0 cm^2. Each CD is 0.1 cm thick and 11.0 cm in diameter. How many CDs can the container hold? Explain, with the help of formulas, what you did to solve the problem.

SUGGESTED MODELS AND MANIPULATIVES

- boxes of various shapes and sizes
- grid paper
- linking cubes
- paper templates for folding
- Polydrons

Mathematical Language

Teacher	Student
▪ 2-D shapes ▪ 3-D objects ▪ area ▪ base ▪ cube ▪ edge ▪ face ▪ net ▪ polyhedron ▪ polyhedra ▪ right cylinder ▪ right rectangular prism ▪ right triangular prism ▪ regular prism ▪ surface area ▪ vertex	▪ 2-D shapes ▪ 3-D objects ▪ area ▪ base ▪ cube ▪ edge ▪ face ▪ net ▪ polyhedron ▪ polyhedra ▪ right cylinder ▪ right rectangular prism ▪ right triangular prism ▪ regular prism ▪ surface area ▪ vertex

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 4: Measuring Prisms and Cylinders
  - Section 4.3: Surface Area of a Right Rectangular Prism
  - Section 4.4: Surface Area of a Right Triangular Prism
  - Section 4.7: Surface Area of a Right Cylinder
  - Unit Problem: Prism Diorama
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Making Math Meaningful to Canadian Students, K–8 (Small 2008), 403, 428.

SCO M04 Students will be expected to develop and apply formulas for determining the volume of right rectangular prisms, right triangular prisms, and right cylinders. [C,CN,PS,R,V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

M04.01 Determine the volume of a given right prism, given the area of the base.
M04.02 Generalize and apply a rule for determining the volume of right cylinders.
M04.03 Explain the connection between the area of the base of a given right 3-D object and the formula for the volume of the object.
M04.04 Demonstrate that the orientation of a given 3-D object does not affect its volume.
M04.05 Apply a formula to solve a given problem involving the volume of a right cylinder or a right prism.

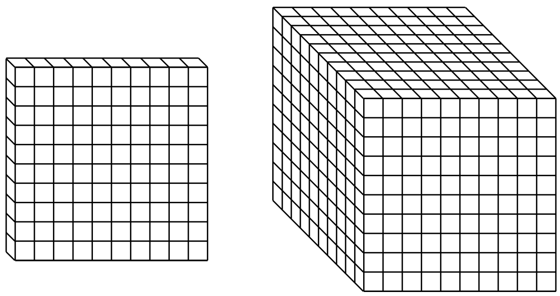
Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
M02 Students will be expected to develop and apply a formula for determining the area of triangles, parallelograms, and circles.	**M04** Students will be expected to develop and apply formulas for determining the volume of right rectangular prisms, right triangular prisms, and right cylinders.	–

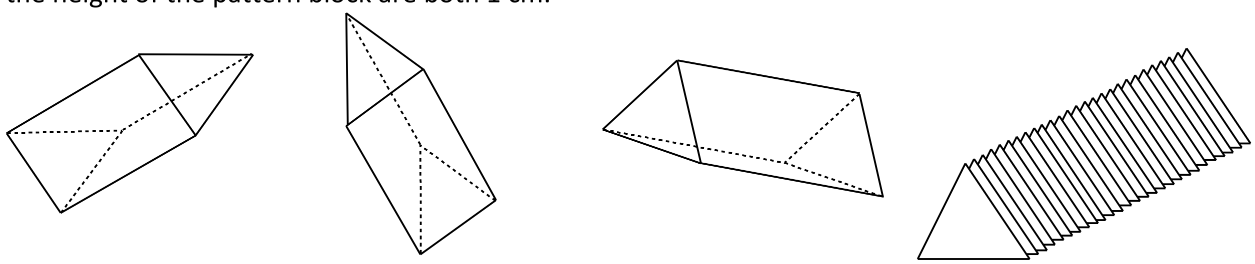
Background

The **volume** of an object is a measure that describes the amount of 3-D space that an object occupies. Students explored volume of rectangular prisms in Mathematics 5 and 6, and should recall that it is measured in cubic units. Connections should be made between the **area** of an object's **base** and calculating its volume. An object's volume should be thought of as the area of its base multiplied by its height ($\text{Area}_{\text{base}}$ x h). The key to determining the formula for the volume of any right prism or **right cylinder** is first determining the shape of the base. The focus should be on developing volume formulas in meaningful ways, rather than having students just memorize the formulas for the different 3-D objects.

Base-ten blocks can provide an effective means of developing the relationship between volume and the area of the base. Begin with a flat and discuss with the class the value of one flat (area of one hundred). Stack another flat on top and ask the students the value of this combination. As you stack the flats, count how many units you have altogether and discuss the idea of volume. Continue to stack the flats until you have made a large cube. Discuss the relationship between the stack of flats and a large cube. Students should make the link that the volume of a large cube is equal to the stack of ten flats (10 x 100).



This approach can be applied to right triangular prisms. Modelling can be done with the green pattern blocks, using one as the base and stacking them to make a triangular prism. The height of the flat and the height of the pattern block are both 1 cm.



Building cube models of prisms should guide students to the realization that rather than counting each cube to calculate the volume, they can multiply the number of cubes in each layer (the area of the base) by the number of layers (the height).

Students may have difficulty understanding the conservation of volume. 3-D objects should be placed in various orientations so students can see that the volume is not affected by orientation. A good demonstration is to bring in a soup can and ask students to identify the volume, (found on the label). Stand the soup can on its end and ask for the volume. Tip the can on its side and ask the class for the volume. Discuss why the volumes are the same in each case. They should conclude that volume does not change as a result of the cylinder's orientation, since the radius and height stay the same. Similarly, when a prism is placed on a different base, the dimensions do not change. So, the volume, or the space taken up by the prism, does not change. See sample class discussion prompts below. 3-D object manipulatives can be used to facilitate discussion.

- What is the base of each 3-D shape?
- How would you find the volume of each shape?
- Why would the volumes be the same?

Students should make connections between calculating the volume of a prism and calculating the volume of a cylinder. A sample introductory discussion follows.

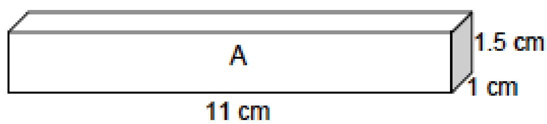
- Anne had a soup can and some unit cubes. She filled the can with unit cubes and counted them all.
  - Do you think the number of cubes in her can is smaller, larger, or equal to the actual volume? Explain.
- Anne then decided to find the volume using a different method. She traced the bottom of the soup can onto centimetre grid paper and counted the number of squares inside the circle.
  - What information does this give her? What other information does she need in order to find the volume of the soup can?
- Considering Anne's explorations, can you determine a rule for finding the volume of a cylinder.

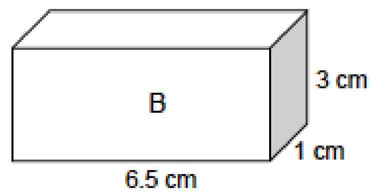
Once students have determined that calculating the volume involves multiplying the area of the base by the height, they should conclude that since the base is a circle, the formula is $V_{cylinder} = \pi r^2 h$.

It may be necessary to remind students of the relationship between volume and capacity that they learned in Mathematics 5 (i.e., 1 cm^3 = 1 mL). They could use this knowledge to compare the volume of cans with the capacity shown on the label.

Estimation and calculation of volume should be done in a variety of real-world situations. For example, it may be useful to find out how many cans or smaller packages will fit into a larger box, or to estimate the volume of a package when the dimensions are not accurately known. Often, for rough estimations, cylinders can be treated as if they were rectangular prisms. A rough estimation of volume would be found by multiplying the length × width × height, where the diameter of the circular base is treated as both length and width. All dimensions could be rounded to help with calculating mentally.

Some students may use only one dimension to estimate volume, but this can provide inaccurate conclusions. For example, a student may say that prism A has more volume than prism B because prism A is longer. However, prism B has a greater volume.





After students have developed strategies and formulas for calculating the volume of rectangular prisms, triangular prisms, and cylinders they should apply what they have learned to solve a variety of problems involving volume. They should be encouraged to draw models to help them visualize the shapes described in the problems.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

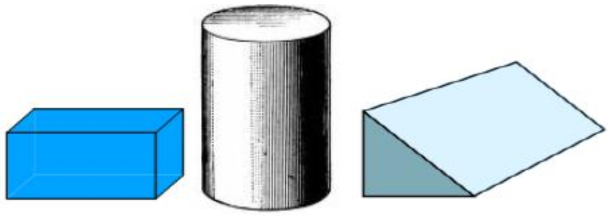
Tasks such as the following could be used to determine students' prior knowledge.

- Ask students to estimate the volume of the classroom in cubic metres and explain how their estimate was determined.
- Provide students with the dimensions of a real-world container that is a rectangular prism. Ask students to find the perimeter and area of each face. Students should also determine the volume for the prism. Ask students to determine the possible dimensions if the object needed to hold twice as much.
- Explain, using numbers, pictures, and/or words, why a rectangular prism that is 5 cm × 3 cm, with a height of 4 cm must have a volume of 60 cm^3.

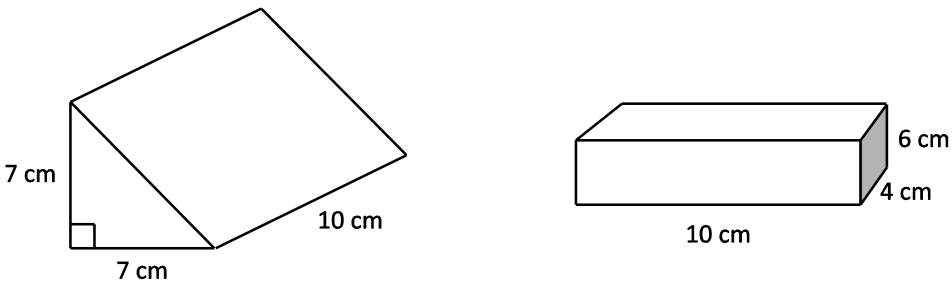
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

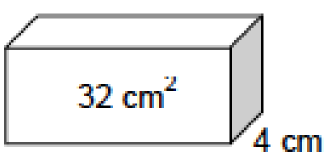
- What formula could you develop to find the volume of each of these right 3-D objects?



- Ask students to design a number of different rectangular boxes for a household product. Each design must have a volume of 1200 cm^3. Have the students select their favourite designs and justify their choices.
- Tell students that each piece of cheese cost $5.00. Which is the better deal?

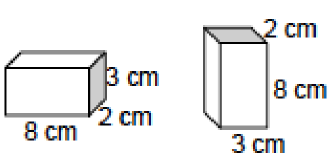


- Tell students that the class is having a fundraiser by selling popcorn and that the students are making their own containers to save on expenses.
  - If you have sheets of cardboard with dimensions 27 cm by 43 cm, would you have a greater volume if you folded the sheets to make cylindrical containers with a height of 27 cm or with a height of 43 cm? (A circular base will be added once the cardboard sheet is used for the sides.)
  - Justify the decision mathematically.
- Ask students how they would use the information presented to determine the volume of the box. Calculate the volume.

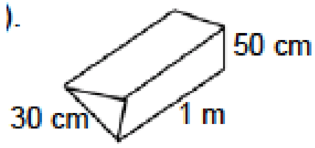


- Ask students which cylinder would hold more water and to explain their answer.

  Cylinder A: height 7.0 cm, diameter 5.0 cm
  Cylinder B: height 5.0 cm, diameter 7.0 cm
- Ask students how they know that the volume of these two prisms is the same.



- Have students find the volume of a cube that has a surface area of 96 cm^2.
- Have students find the volume of this prism (the base is a right triangle):



- Ask students how they can find the volume of any right 3-D object.

Follow-up on Assessment

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

Choosing Instructional Strategies

Consider the following strategies when planning daily lessons.

- Start discussions of volume using informal measurement methods, such as linking cubes. Show and discuss the centimetre cube. Explain that just as square units are used to measure area and surface area, cubic units are used to measure volume.
- Have students use centimetre cubes (standard measurement) or linking cubes (non-standard measurement) to help them visualize the volume of solids.
- Bring in small boxes of various shapes and sizes and have students use centimetre cubes to determine the volume of each box.
- Provide students with relevant contexts for determining volume.

Suggested Learning Tasks

- Create open-topped boxes from a sheet of centimetre grid paper by cutting away squares from the four corners and folding the sides up. Experiment to determine the dimensions of a box with the greatest volume given the same size grid paper. They will have to decide between flat, wide boxes, or tall, narrow boxes.
- With linking cubes construct rectangular prisms with the following dimensions: $3 \times 5 \times 2$ and $6 \times 5 \times 2$. Find the volume of each. How could you have anticipated that the second volume would be twice the first? How do you think a $6 \times 5 \times 4$ prism would compare to one a $3 \times 5 \times 2$ prism?
- An aquarium has the following dimensions: length 80 cm, width 35 cm, height 50 cm. You must fill the aquarium up to 4 cm from the top. How much water will you put in the aquarium?
- Bring in various shaped boxes or cans. Determine how you can estimate and find the volume of the containers. What would be your formula?
- A triangular prism has a volume of 128 cm^3. Its height is 8 cm. What is the area of its base?
- Predict and explore whether the volume of a cylinder created by rolling a sheet of paper lengthwise or by its width will be the same volume or a different volume. If the volumes are different, which will have the greatest volume? Discuss why this would be an important fact for companies to know.

- A tube of cookie dough has a volume of 785 cm^3 and a diameter of 10 cm. Each cookie will be 1 cm thick. How many cookies can Nicole make? Explore this problem with a partner.
- Students, in groups of 3 or 4, are to build as many different rectangular prisms as they can, using the multi-link cubes provided. Students are to record the surface area and volume for each rectangular prism they construct. What happens to the surface area as the prism becomes taller rather than cube like?
- Calculate the volume of a cylinder with a radius of 14 cm and a height of 12 cm.

Suggested Models and Manipulatives

- centimetre cubes
- grid paper
- multi-linking cubes
- Polydrons
- various shaped boxes and/or cans

Mathematical Language

Teacher	Student
▪ 2-D shapes ▪ 3-D objects ▪ area ▪ base ▪ cube ▪ edge ▪ face ▪ net ▪ polyhedron ▪ polyhedra ▪ right cylinder ▪ right rectangular prism ▪ right triangular prism ▪ regular prism ▪ vertex ▪ volume	▪ 2-D shapes ▪ 3-D objects ▪ area ▪ base ▪ cube ▪ edge ▪ face ▪ net ▪ polyhedron ▪ polyhedra ▪ right cylinder ▪ right rectangular prism ▪ right triangular prism ▪ regular prism ▪ vertex ▪ volume

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 4: Measuring Prisms and Cylinders
  - Section 4.5: Volume of a Right Rectangular Prism
  - Game: Largest Box Problem

  - Section 4.6: Volume of a Right Triangular Prism
  - Section 4.8: Volume of a Right Cylinder
  - Unit Problem: Prism Diorama
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 257–259.

Geometry (G)

GCO: Students will be expected to describe the characteristics of 3-D objects and 2-D shapes and analyze the relationships among them.

GCO: Students will be expected to describe and analyze position and motion of objects and shapes.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Long-term Planning

- Yearly plan available at Mathematics Learning Commons: Grades 7–9: http://nsvs.ednet.ns.ca/nsps/nsps26/course/view.php?id=3875.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO G01 Students will be expected to draw and interpret top, front, and side views of 3-D objects composed of right rectangular prisms. [C, CN, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

G01.01 Draw and label the top, front, and side views for a given 3-D object drawn on isometric dot paper.
G01.02 Compare different views of a given 3-D object to the object.
G01.03 Predict the top, front, and side views that will result from a described rotation (limited to multiples of 90°), and verify predictions.
G01.04 Draw and label the top, front, and side views that result from a given rotation (limited to multiples of 90°).
G01.05 Build a 3-D block object given the top, front, and side views, with or without the use of technology.
G01.06 Sketch and label the top, front, and side views of a 3-D object in the environment, with or without the use of technology.

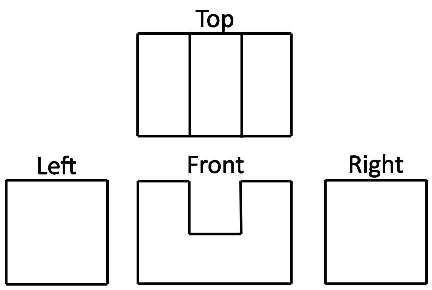
Scope and Sequence

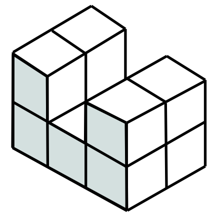
Mathematics 7	Mathematics 8	Mathematics 9
G03 Students will be expected to perform and describe transformations (translations, rotations, or reflections) of a 2-D shape in all four quadrants of a Cartesian plane (limited to integral number vertices).	**G01** Students will be expected to draw and interpret top, front, and side views of 3-D objects composed of right rectangular prisms.	**G03** Students will be expected to draw and interpret scale diagrams of 2-D shapes. **G04** Students will be expected to demonstrate an understanding of line and rotational symmetry.

Background

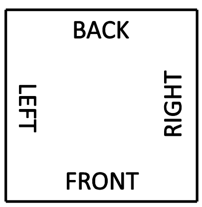
Observing 3-dimensional figures in various positions and representing these 3-D figures through 2-dimensional drawings helps students develop visualization and spatial reasoning. It is important that students are able to interpret information from 2-D pictures of the world, as well as to represent real-world information in 2-D.

Students should be able to interpret a series of 2-D views of a 3-D object and use multi linking cubes to construct an object that adheres to the views. **Orthographic views** are 2-D drawings used to represent or describe a 3-D object. They typically show the *left*, *front*, *right*, and *top* views of the object. When drawing views, internal line segments are drawn only where the depth or thickness of the object changes. These drawings are often done on square dot paper.



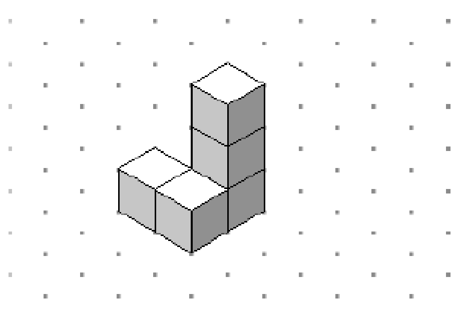


It is essential for students to use multi linking cubes as they explore these concepts. Students need opportunities to build the 3-D models represented by the **orthographic** views and to draw the orthographic views (top, front, and sides) of 3-D models. Students need to be able to view the models from different angles. It is useful to use a mat plan, as below, when working with physical models. Students build the model and place it on the mat. To see each view, students can turn the mat or look at the object from different vantage points. It is also useful for students to look at the object at eye level. They can then draw the views on square dot paper as they see them.



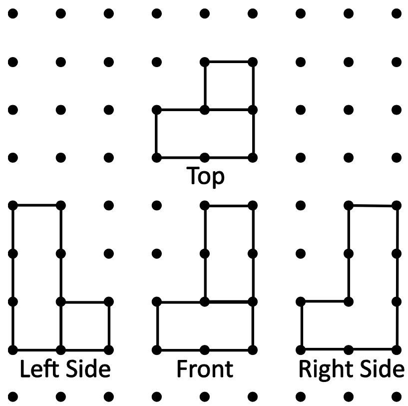
Teachers should discuss the relationship between each of the views and the actual object. Depending on the orthographic drawings, it may be possible to create more than one object that matches them. Students can explore the minimum and maximum number of cubes that can be used to model a given drawing.

Once students are able to identify and sketch these views from a physical model, move on to a 3-D image drawn on **isometric** paper. It is useful to have the faces of these drawings shaded to create the three-dimensional look. Consider the following drawing on isometric paper:



Note: This drawing was created by the following website: [*http://illuminations.nctm.org/ActivityDetail.aspx?ID=125*](http://illuminations.nctm.org/ActivityDetail.aspx?ID=125)

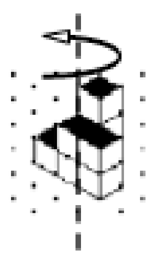
Here are the views:



Teachers should discuss the relationship between each of the views and the actual object. When drawing views, internal line segments are drawn only where the depth or thickness of the object changes.

Students should recognize that, when they are given only one view of an isometric drawing, all the cubes may not be visible because some are hidden. Students should be given opportunities to create structures from a given isometric drawing. Generally, not all students will make the same structure, and should discover that one drawing may represent more than one 3-D object.

Visualization of the movement of 3-D objects is an important skill. It is very useful, not only in careers such as art, design, architecture, and engineering, but also in arranging or moving furniture and packing. The purpose of this outcome is to provide students with some experiences in visualizing and recording the movement of 3-D objects. The focus on **rotations** should be on those performed around the **vertical axis** (object is rotated horizontally) and limited to multiples of 90°. Students should sketch their prediction of the view they think will result from performing a given rotation. After rotating the structure, students are expected to create the new orthographic drawings and compare the different views. Isometric drawings of the views can also be created. Students should be able to apply these skills to draw views of 3-D objects in their environment.



Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

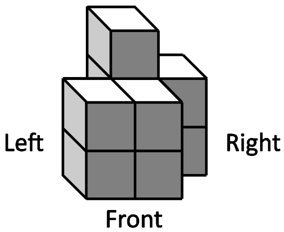
Tasks such as the following could be used to determine students' prior knowledge.

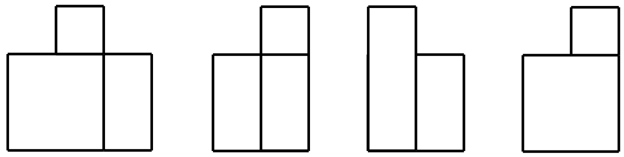
- Using dot paper, have students draw pentaminoes and rotate them about a given point, 90° clockwise, and 90°counter clockwise

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

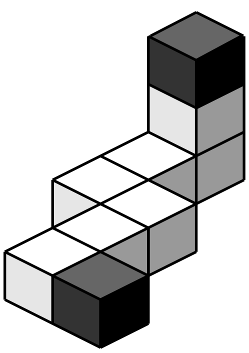
- Provide students with this picture of a 3-D object drawn from its front-left corner. Ask them which one is the correct orthographic view from the right.



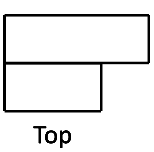


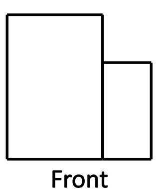
  - Build the 3-D object that satisfies all four views.
  - Draw and label the object on isometric dot paper or on a computer.

- Build the structure below with linking cubes. Draw the top, front, right, and left views of the structure (orthographic drawings). If you take away the black cubes, which views would look different? How would they be different?

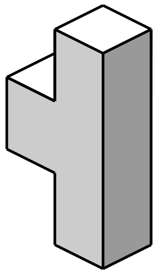


- Using some multi-link cubes and the orthographic drawings below.





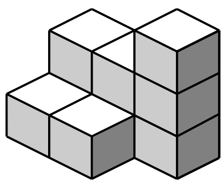
  - Draw and label an isometric drawing of the structure on isometric dot paper.
  - What is the maximum number of cubes used for a structure that will satisfy the views above?
  - What is the minimum number of cubes used for a structure that will satisfy the views above?
  - What do all of the structures have in common?
- Start with a 3-D object, such as the one shown, and rotate the object using 90° rotations **clockwise** and **counter-clockwise** to determine how many distinct drawings can be created. Predict the number of distinct drawings that can be made before beginning this task.



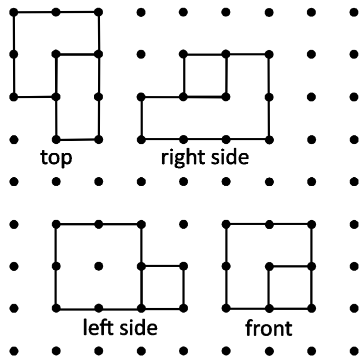
- Choose an object of interest (e.g., a building) and draw several views of the object.

- Match the views with objects:

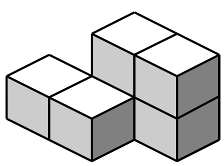
a)



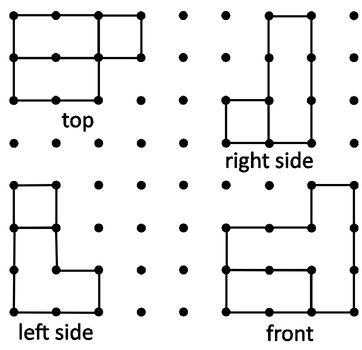
i)



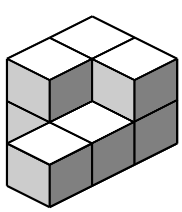
b)



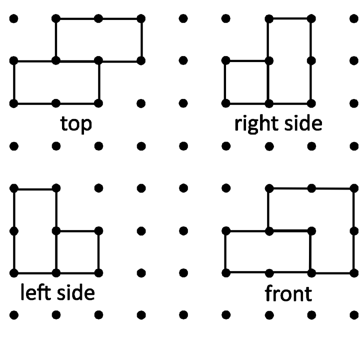
ii)



c)



iii)



Planning for Instruction

Choosing Instructional Strategies

Consider the following strategies when planning daily lessons.

- Use multi linking cubes as the basic building blocks for 3-D objects.
- Begin by showing students how to draw a single cube. Show students how to draw a single cube. Start by drawing a Y as in figure 1. Complete the cube by drawing the edges that can be seen from a corner view as illustrated in figure 2. Do not include the edges that we do not see, otherwise it may be confused as a hexagon with all diagonals drawn in as shown in figure 3.

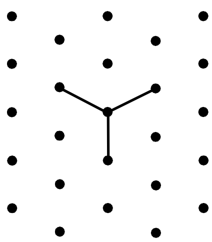


figure 1

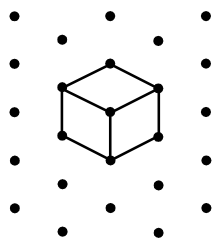


figure 2

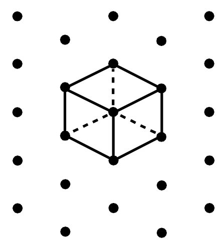
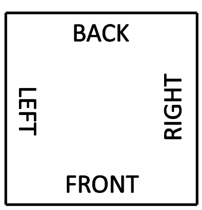


figure 3

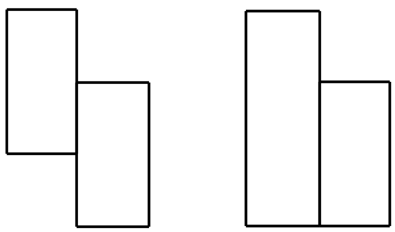
- Use mats to help students with 2-D drawings of 3-D objects. A square of plain paper appropriately marked with directions would be a simple mat for this purpose. This is particularly useful for drawing the orthographic views on square dot paper, and for rotating the object. Using models that students build from multi-link cubes, students place the model on the mat. To see each view, students can turn the mat or look at the object from different vantage points.



- Have students compare structures so they learn that there can sometimes be more than one structure that fulfills the information in a set of plans. Have students explore such questions as the following: What is the minimum number of cubes that can be used to fulfill the plans provided? What is the maximum? How many different objects can be built to fulfill the plans?
- Use interactive websites, such as NCTM's Illuminations website, to explore isometric drawings. (http://illuminations.nctm.org/Activity.aspx?id=4182).
- When discussing views for rotated objects, explore questions such as the following using both physical models of the objects and student drawings of the views
  - What happens to the object/views when the object is rotated 90° clockwise or counter clockwise?
  - What happens to the object/views when the object is rotated 90° clockwise or 270° counter clockwise?
  - What happens to the object/views when the object is rotated 180° clockwise or counter clockwise?

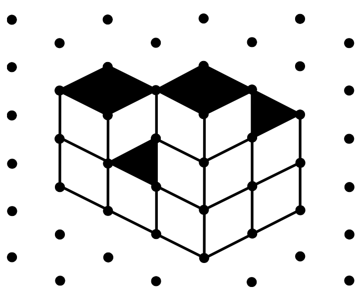
SUGGESTED LEARNING TASKS

- Use the orthographic plans below to
  - find as many different objects as possible that satisfy the plans and make isometric drawings
  - find the minimum number of cubes needed to construct the structure
  - find the maximum number of cubes needed to construct the structure



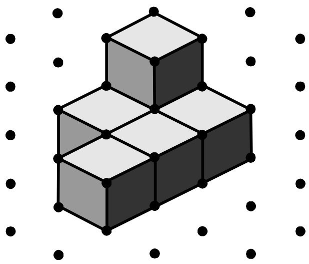
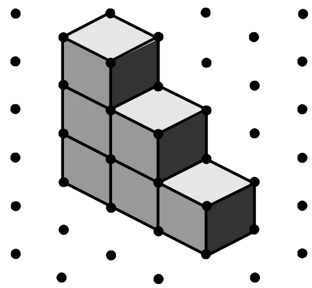
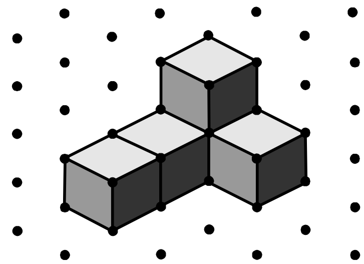
Top Front

- Answer the following questions about this sometric drawing.



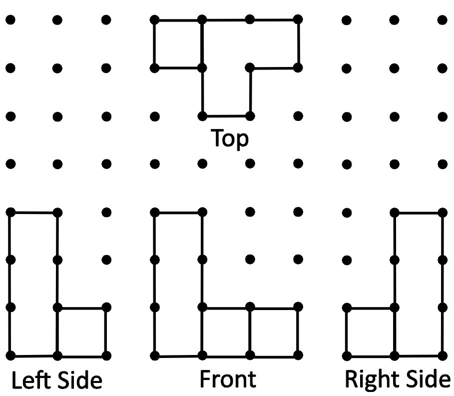
  - many different shapes are possible?
  - What is the minimum number of cubes needed to construct the structure?
  - What is the maximum number of cubes needed to construct the structure?

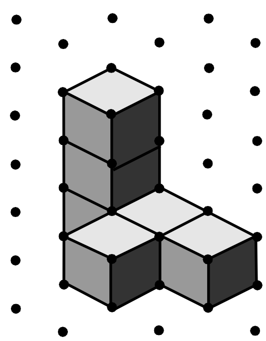
- Investigate how many different 3-D objects can be constructed using only 2 multi linking cubes; 3 linking cubes; 4 linking cubes, etc.
- Create a small 3-D structure with multi-link cubes. With the front of the object facing you, rotate it 90° clockwise around the **vertical axis** and sketch the new view of the object. Now rotate it another 90° clockwise and sketch it again. Rotate it one more time 90° clockwise and produce a third sketch. Compare all of the sketches. *(This activity could be extended to exploring rotations of 90° around the horizontal axis.)*
- Build a shape using a specified number of multi linking cubes.
- Make a sketch of the views of the top, front, and side views on square dot paper.
- Exchange your models and views with another student and verify each other's work.
- *Full class option*: Display the views prepared by a student and compare to all the built models. Students determine which model is illustrated in the views.
- Provide students with the isometric drawings below. Draw the top, front, left side, and right side views on square dot paper.

- Repeat the above exercise for a rotation of 90° clockwise.
- A classmate insists that you need all four views of an object to create a physical model. Is this correct? Explain why or why not.
- Research buildings made with right rectangular prisms. Choose an object of interest to you and draw the views of this object for display. Some examples: Habitat 67 in Montreal, Quebec; Turning Torso in Malmo, Sweden; The Cube, Birmingham, UK.

Give students the views of a 3-D object, such as those shown below, and ask them to use multi linking cubes, The Geometer's Sketchpad (Key Curriculum Press 2015), or NCTM's Illuminations website (http://illuminations.nctm.org/Activity.aspx?id=4182) to create the actual object.





Suggested Models and Manipulatives

- isometric dot paper
- linking cubes
- mat with labels to aid with orientation
- square dot paper

Mathematical Language

Teacher	Student
▪ 2-D shape ▪ 3-D object ▪ clockwise ▪ counter clockwise ▪ isometric drawings ▪ orthographic plans	▪ 2-D shape ▪ 3-D object ▪ clockwise ▪ counter clockwise ▪ isometric drawings ▪ orthographic plans

▪ vertical axis ▪ views(front, top, left, right) ▪ rotation	▪ vertical axis ▪ views(front, top, left, right) ▪ rotation

Resources/Notes

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 8: Geometry
  - Section 8.1: Sketching Views of Objects
  - Technology: Using a Computer to Draw Views of Objects
  - Section 8.2: Drawing Views of Rotated Objects
  - Section 8.3: Building Objects from Their Views
  - Technology: Using a Computer to Construct Objects from their Views
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Making Math Meaningful to Canadian Students K–8, Second Edition (Small 2013), 362–363.
Teaching Student-Centered Mathematics, Grades K–3, Volume One (Van de Walle and Lovin 2006), 224–225.

Digital

- The Geometer's Sketchpad (Key Curriculum 2013; NSSBB #: 50474, 50475, 51453)
- "BCLN - Math 08 - Drawing 3D Objects," *YouTube* (YouTube 2014): www.youtube.com/watch?feature=player_embedded&v=nwGLLxJqt-Y
- "Isometric Dot Paper (1 cm)," *Learning Resources and Technology* (Nova Scotia Department of Education and Early Childhood Development 2015): http://lrt.ednet.ns.ca/PD/BLM/pdf_files/dot_paper/iso_dot_1cm.pdf
- "Square Dot Paper (1 cm)," *Learning Resources and Technology* (Nova Scotia Department of Education and Early Childhood Development 2015): http://lrt.ednet.ns.ca/PD/BLM/pdf_files/dot_paper/sq_dot_1cm.pdf
- "Isometric Drawing Tool," Illuminations: Resources for Teaching Math (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/Activity.aspx?id=4182

SCO G02 Students will be expected to demonstrate an understanding of the congruence of polygons under a transformation. [CN, R, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

G02.01 Determine the coordinates of the vertices of an image following a given combination of transformations of the original figure.

G02.02 Draw the original figure and determine the coordinates of its vertices, given the coordinates of the image's vertices and a description of the transformation (translation, rotation, reflection).

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
G03 Students will be expected to perform and describe transformations (translations, rotations, or reflections) of a 2-D shape in all four quadrants of a Cartesian plane (limited to integral vertices).	**G02** Students will be expected to demonstrate an understanding of the congruence of polygons under a transformation.	**G04** Students will be expected to demonstrate an understanding of line and rotation symmetry.

Background

From their work in geometry in Mathematics 5, 6, and 7, students have knowledge of classification of shapes by their properties, symmetry of shapes, and transformation of shapes. Students have studied translations, reflections, and rotations, and composition of transformations, and are aware that transformations result in images that are **congruent** to the original object (**pre-image**). The focus in Mathematics 8 is on understanding the congruence of polygons and on describing and analyzing position and motion of shapes. Shapes can be moved in a plane. These movements can be described in terms of **translations**, **reflections**, and **rotations**.

Polygons are congruent if they meet the following criteria:

- same number of sides
- all corresponding sides have the same length
- all corresponding angles have the same measure

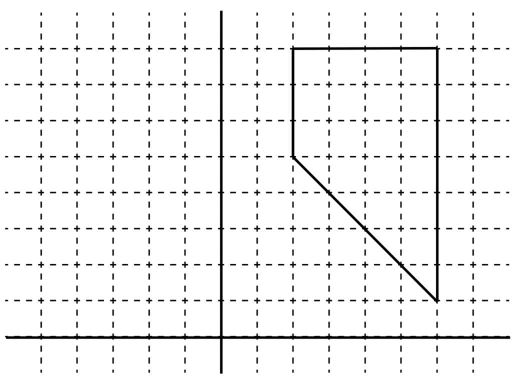
Corresponding sides are sides that have the same relative position in two polygons. Corresponding angles are angles that have the same relative position in two polygons.

Congruency is denoted by the symbol ≅; i.e., "quadrilateral ABCD is congruent to quadrilateral $A'B'C'D'$", which is shown by $ABCD \cong A'B'C'D'$. When comparing two polygons, begin at corresponding vertices, name the vertices in order, and go in the same direction to name and compare corresponding parts. A' is read as A **prime** and corresponds to vertex A; B' is read as B prime and corresponds to vertex B; C' is read as C prime and corresponds to vertex C; and D' is read as D prime and corresponds to vertex D. Subsequent transformations are written as **double prime** (*A′′*), **triple prime** (*A′′′*), etc. Coordinate systems can be used to describe the precise location of a shape in a plane. The coordinate view of a shape is also useful in understanding the property of transformation (changes in position) of shapes.

Students should be encouraged to use proper conventions when labeling axes, vertices, coordinates, etc. Accuracy in drawings is important. Students should draw polygons on the coordinate plane, label the vertices, and perform transformations of the polygons, label the image vertices, and make comparisons.

For example:

- Give students a quadrilateral ABCD on a grid and ask them to record the coordinates for the four vertices.



  - Tell them to translate vertex A to the right 5 and up 3, and name the image point A' (read A prime) and record its coordinates.
  - Ask students to record the coordinates for the corresponding vertices:
    (___ , ___) (coordinates for B')
    (___ , ___) (coordinates for C')
    (___ , ___) (coordinates for D')

  - Ask them to write in words how they could determine the coordinates for B' and C' without looking at the graph. Ask students what happens to the angles when transformations are applied, and what happens to the sides when transformations are applied. Ask if they believe this to always be true. Students should discover that congruence is preserved under certain transformations. To do the converse, give students the coordinates of another polygon. Ask them what the coordinates of its pre-image are if this has undergone a translation left 4 and up 2. Have them draw the pre-image.
  - Continue to have students look at how coordinates can be used to examine transformations by having them draw and label a concave pentagon, with coordinates A(2, 2), B(3, 5), C(5, 6), D(6, 4), *E*(4, 5). Have them reflect *ABCDE* in the *y*-axis. Label the vertices *A′B′C′D′E′*. Have students then reflect *A′B′C′D′E′* in the *x*-axis and label the vertices *A′′B′′C′′D′′E′′*. Have students reflect *A′′B′′C′′D′′E′′* in the *y*-axis and label the vertices *A′′B′′C′′D′′E′′*. Discuss how ABCDE is related to *A′B′C′D′E′* and what other transformations are possible to go from the pre-image to any of the

pentagons in the other quadrants. Have students examine the coordinates of each transformation and make comparisons.

Similar activities should take place that allow students to discover what happens for other transformations (rotations and reflections). Combinations of transformations should also be explored in the same manner.

Properties of transformations were developed informally in previous grades. Now, students should consider the congruence of the pre-image and the image. In discussing the properties of transformations, students should consider if the transformation of the image

- has side lengths and angle measures the same as the pre-image
- is both similar to and congruent to the given pre-image
- has the same orientation as the given pre-image
- appears to have remained stationary with respect to the given pre-image

Specific considerations for each of the transformations are as follows:

- **Translations:** Each point in the pre-image moves the same distance and in the same direction to create the image. The orientation of the pre-image and its image remain the same; only the location on the plane changes. Demonstrate translations on a coordinate grid by drawing the pre-image onto grid paper of the same size, cutting it out, and physically sliding the pre-image on the grid. Another method is to count the horizontal and vertical moves of each vertex. A slide arrow indicates the direction of a translation. Translations are also commonly described using coordinate notation with square brackets (e.g., 3 left, 2 up is [–3, 2]).
- **Reflections**: The points on the shape and the matching points on the image are equal distances from a line of reflection. The line of reflection may be inside or outside the shape. Doing a reflection of the pre-image in the line of reflection creates a mirror image in a new location and with a different orientation. Demonstrate reflections by physically flipping a copy of the pre-image in the line of reflection, by placing a Mira along the reflection line, or by counting the perpendicular distance of each vertex from the mirror line. The line of reflection is indicated by marking a mirror line on the grid.
- **Rotations**: The points of the pre-image are rotated the same number of degrees or fraction of a turn clockwise (cw) or counter-clockwise (ccw) around a point termed the centre of rotation. The centre of rotation may be any point on, inside, or outside the figure. The orientation of the image does not change. The change in location of the image varies greatly, depending on the location of the centre of rotation and the direction and angle of rotation. Demonstrate rotations by physically rotating the pre-image the specified number of degrees around the centre of rotation, or by tracing the pre-image and the centre of rotation onto tracing paper, and then matching the centre of rotation and physically rotating the tracing paper. Rotating the side of the pre-image rather than just one point may help reduce accidental sliding during the rotation. Another technique to locate a rotation requires the use of a protractor to measure the angle and a compass to copy the line length. Specify rotations with a curved arrow that indicates the direction of the rotation, and write the number of degrees or the fraction of a turn for the rotation. Rotations are commonly described using a degree measure and direction (e.g., 90° ccw).

Assessment, Teaching, and Learning

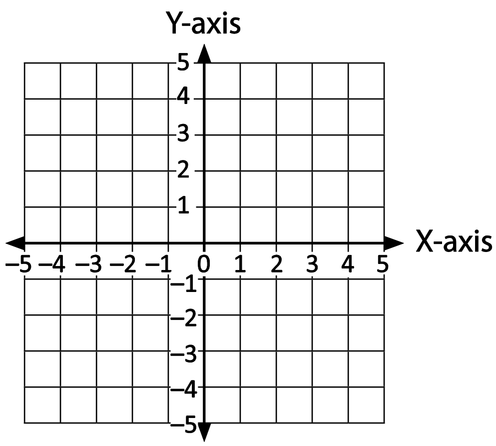
Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

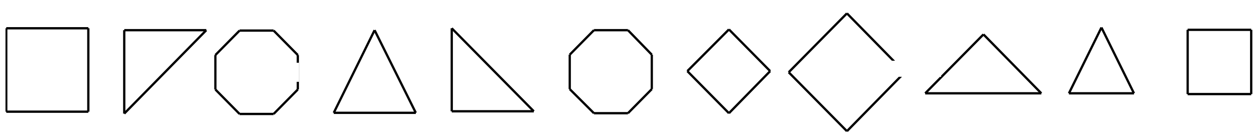
Tasks such as the following could be used to determine students' prior knowledge.

- Ask students to plot and label the following points as A, B, and C on the coordinate plane.

  (2, –1)
  (2, 4)
  (–4, –1)
  Add A, B, and C.
  What is the distance from Point A to Point B?
  What is the distance from Point A to Point C?

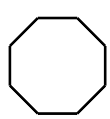


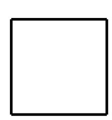
- Provide diagrams of 2-D shapes, some of which are congruent, such as:



Ask the students to:

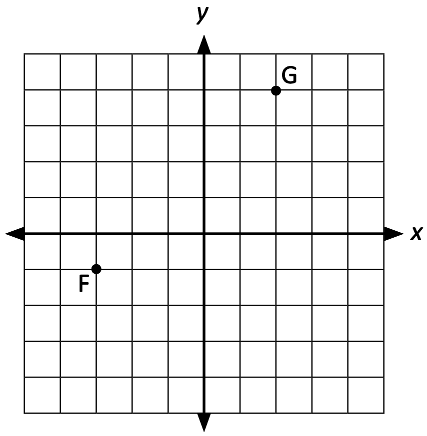
- put a check mark on shapes that are congruent to
- put an X on shapes congruent to





- shade in shapes that are congruent to
- explain the strategies they used to determine if the shapes were congruent

- Ask students to consider points F and G shown below. Reflect G in the *y*-axis to get a new point, G'. What are the coordinates of G'? Reflect F in the *x*-axis to get a new point F'. what are the coordinates of F'?



Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Create a design using your choice(s) of transformation. Explain how your design was formed from the combination of transformations you chose. Share your results with a partner. Share your results with a partner. (*This could be done in pairs or as individuals. This could be done as a take home assignment and/or used as an assessment tool for the outcome.*)
- Quadrilateral ABCD has coordinates A(1, 2), B(1, 6), C(4, 7), and D(3, 1). Reflect the quadrilateral in the *x*-axis and then translate the quadrilateral A'B'C'D' to the left 2 units and up 3 units. What are the coordinates of quadrilateral A"B"C"D"?
- Δ ABC was rotated 90° clockwise about the origin resulting in Δ A’B’C’, A’(3.2), B’(6.4), C’(8.1). Determine the vertices of Δ ABC.
- Construct Δ RST on a coordinate plane with vertices R(–4, 4), S(–6, 2), and T(–3, 2). Trace and cut out a copy of RST and label it R'S'T'. Translate R'S'T' four spaces to the left.
  - What are the new coordinates of this triangle?
  - Compare with the vertices of the original triangle.
  - Find the area of the image and the pre-image. What do you notice?

Planning for Instruction

CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

- Provide hands-on activities such as the following:.
  - Encourage the use of manipulatives such as tracing paper, patty paper, and graph paper to explore the meaning of congruence of polygons.
  - Use pentomino pieces or tangram pieces to help visualize transformations
  - Have students use different coloured pencils to do the transformations.
  - Have students use a document camera or graph paper transparency to share their results in a class discussion.

- Have the students use a graphic organizer such as a Frayer Model to reflect on their learning of the congruence of polygons.
- Have students share their ideas and justify their thinking by explaining why they used a particular strategy to make a transformation.

SUGGESTED LEARNING TASKS

- Have the students demonstrate their understanding of transformations as follows:
  - Draw a shape of their choice in Quadrant I of the Cartesian plane.
  - Demonstrate a combination of transformations:
    - All three transformations—translation, reflection, and rotation—must be demonstrated.
    - Each drawing must contain a combination of at least two different transformations.
    - All drawings must have vertices properly labelled and the coordinates of each vertex given.
    - The drawings must be accompanied with a written explanation justifying the end result of the combination of transformations.
  - For translations:
  - Have students draw a quadrilateral in the first quadrant of a Cartesian plane. Label the coordinates of the vertices ABCD. Suggest that students use values of *y* less than 7.
  - Ask students to translate (slide) the figure to the right 7 units and up 2 units. Label the vertices and find the coordinates for the position of the figure A'B'C'D'.
  - Have the students share their results.
  - For reflections
  - Have the students reflect quadrilateral A'B'C'D' (from above) in the line $y = 8$. Label the vertices of the image A"B"C"D" and find the coordinates of the vertices.
  - For reflections and rotations
  - On a new Cartesian plane, have the students draw another quadrilateral using the coordinates of the vertices: A (2, 1), B (2, 3), C (4, 4) and D (4, 0). Reflect figure ABCD in the line $y = 6$. Label the vertices A'B'C'D'.
  - Have the students rotate figure A'B'C'D' 90° clockwise about the origin. Label the vertices A"B"C"D" and find the coordinates of the vertices.
  - Have students share their results.
  - Combination of transformations
  - Have the students work in pairs.
  - Have students start with a new polygon—a design of their choice—and perform a combination of transformations—at least two. For example a translation of right 2 and up 6, and then rotate about point (6, 8).
  - Other combination transformation suggestions:
    - Translate 2 units to the right and, rotate clockwise about the point (6, 6).
    - Reflect in the line $x = 3$ and rotate 180°.
    - Reflect in the line $y = 6$, translate right 1 and down 1 unit and rotate clockwise 90°.
    - If using a reflection, students might try to have one side of their image touch one side of the pre-image.
    - If using a rotation, students might try to have one vertex of the image touch the vertex of the pre-image.

> Have the students share their results with other members of the class.
> Have students explore the relationship between a 180° rotation and reflections in two perpendicular lines.

- The exploration of translations, reflections, and slides can also be done using dynamic geometry programs, such as GeoGebra (International GeoGebra Institute 2015) and The Geometer's Sketchpad (Key Curriculum Press 2015).

SUGGESTED MODELS AND MANIPULATIVES

- compasses
- geometry software
- graph paper
- MIRA
- patty paper
- protractors

MATHEMATICAL LANGUAGE

Teacher	Student
▪ centre of rotation	▪ centre of rotation
▪ congruent	▪ congruent
▪ corresponding	▪ corresponding
▪ double prime	▪ double prime
▪ image	▪ image
▪ pre-image	▪ pre-image
▪ preservation of congruence	
▪ prime	▪ prime
▪ reflection	▪ reflection
▪ rotation	▪ rotation
▪ translation	▪ translation
▪ triple prime	▪ triple prime

Resources/Notes

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 8: Geometry
  - Section 8.4 A: Transformations on a Coordinate Grid (Nova Scotia companion document only; found on Mathematics Learning Commons, Grades 7–9)
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Planning Guide: Grade 8 Congruence of Polygons, Shape and Space (Transformations), Specific Outcome 6 (Learn Alberta 2008) (Can be found at www.learnalberta.ca/content/mepg8/html/pg8_congruenceofpolygons/pdf/pg8_congruenceofpolygons.pdf)

The Alberta K–9 Mathematics Program of Studies with Achievement Indicators (Alberta Education 2007).
About Teaching Mathematics: A K–8 Resource, Second Edition (Burns 2000), pp. 344–349.
Van de Walle, John A. and LouAnn H. Lovin. *Teaching Student-Centered Mathematics, Grades 5–8,* Volume Three. Boston, MA: Pearson Education, Inc., 2006, pp 179-198, 212, 214-219.

Digital

- "Archimedes' Puzzle," *Illuminations: Resources for Teachers* (National Council of Teachers of Mathematics 2008): http://illuminations.nctm.org/Lesson.aspx?id=2523
- *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/
- GeoGebra (International GeoGebra Institute 2015): www.geogebra.org/cms/en
- The Geometer's Sketchpad (Key Curriculum 2013; NSSBB #: 50474, 50475, 51453)

Statistics and Probability (SP)

GCO: Students will be expected to collect, display, and analyze data to solve problems.

GCO: Students will be expected to use experimental or theoretical probabilities to represent and solve problems involving uncertainty.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Long-term Planning

- Yearly plan available at Mathematics Learning Commons: Grades 7–9: http://nsvs.ednet.ns.ca/nsps/nsps26/course/view.php?id=3875.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO SP01 Students will be expected to critique ways in which data is presented. [C, R, T, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

SP01.01 Compare information provided for the same data set by a given set of graphs, including circle graphs, line graphs, bar graphs, and pictographs, to determine the strengths and limitations of each graph.
SP01.02 Identify the advantages and disadvantages of different graphs, including circle graphs, line graphs, bar graphs, and pictographs, in representing a given set of data.
SP01.03 Justify the choice of a graphical representation for a given situation and its corresponding data set.
SP01.04 Explain how the format of a given graph, such as the size of the intervals, the width of the bars, and the visual representation, may lead to misinterpretation of the data.
SP01.05 Explain how a given formatting choice could misrepresent the data.
SP01.06 Identify conclusions that are inconsistent with a given data set or graph, and explain the misinterpretation.

Scope and Sequence

Mathematics 7	Mathematics 8	Mathematics 9
SP03 Students will be expected to construct, label, and interpret circle graphs to solve problems.	**SP01** Students will be expected to critique ways in which data is presented.	**SP03** Students will be expected to develop and implement a project plan for the collection, display, and analysis of data by ▪ formulating a question for investigation ▪ choosing a data collection method that includes social considerations ▪ selecting a population or a sample ▪ collecting the data ▪ displaying the collected data in an appropriate manner ▪ drawing conclusions to answer the question

Background

Students will examine and create various types of graphs that are used in data management: **circle graphs**, **bar graphs**, **pictographs**, **line graphs**, and **double bar graphs**. They will make informed decisions about which graph best represents a data set and will also learn how to justify that decision. In many cases, one type of graph is better than another when representing a data set, as it provides an informative image can impact how one might interpret the data. Students will differentiate between graphs that are accurate and graphs that are misleading. They will also learn to recognize false conclusions that misleading graphs try to represent. Evaluating the arguments of others enhances students' understanding of statistics. This is particularly important since advertising, forecasting, and public policy are frequently based on data analysis. The media is full of representations of data to support statistical claims. These real-world examples can be used to stimulate discussion.

As an introduction to this outcome, do a survey with the class on favourite colour or sport, TV show, etc. Divide the class into groups and assign each group a different graph to represent the data collected on large chart paper. This opportunity could also be used to review how to construct each type of graph. This could be done as a demonstration. The focus of this outcome is not on the ability to construct a graph.

Have each group present their graph and discuss what they found easy about creating the graph and any challenges they encountered.

Questions for class discussion could include:

- Which of the graphs display the data in a manner that is easier to interpret? Why? (Students should discover that there could be more than one graph depending on data.)
- Which of the graphs are not as easy to interpret? Why not?
- Which graph could you use to identify the percent of students who like blue/hockey, etc.? Why?
- Which graph could you use to find the total number of students surveyed?
- How are the graphs similar?
- How are the graphs different?
- Which graph best represents the data collected?

Discussions should lead to the following conclusions:

- The suitability of a graph will depend on the type of information collected/given.
- The choice of graph will depend on what you want the graph to display.

It is important for students to be asked to evaluate various situations to determine and debate why a particular graph is best suited to a specific type of data, or to a given context. Students should be able to discuss this in terms of **continuous** versus **discrete data** sets. For example, given a bar graph and a line graph, students should determine which is most appropriate to display the amount of water flowing into a container and justify their choice.

Students should also be aware of the characteristics of a good graph: accurately shows the facts; complements or demonstrates arguments presented in the text; has a title and labels; shows data without misrepresenting the data; clearly shows any trends or differences in the data. Student should also be able to identify conclusions that are inconsistent with the data and provide a rationale.

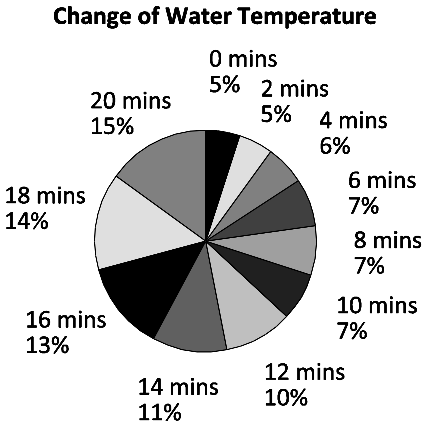
Note: Sample advantages and disadvantages are provided below for each type of graph. This table is not intended to be all-inclusive and can be added to:

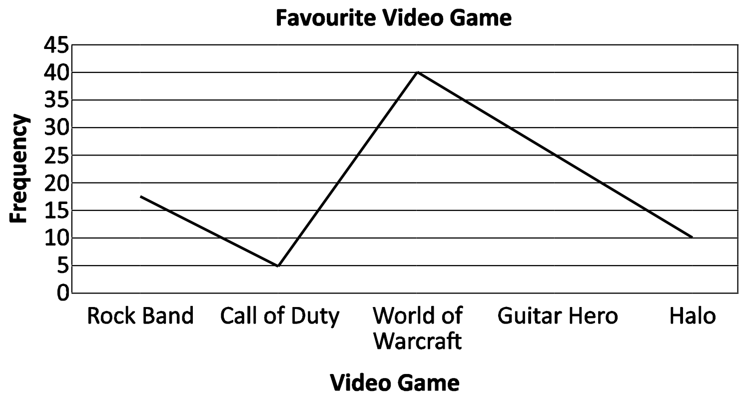
Type of Graph	Advantages	Disadvantages
Circle (Discrete Data)	Compares parts of the data relative to the whole - data is displayed as a percent (shows proportion) - size of sector can be easily compared to other sectors to make conclusions	Does not show actual amount or number for each category - more difficult to draw by hand (time and accuracy) - data must have obvious part-to-whole relationship

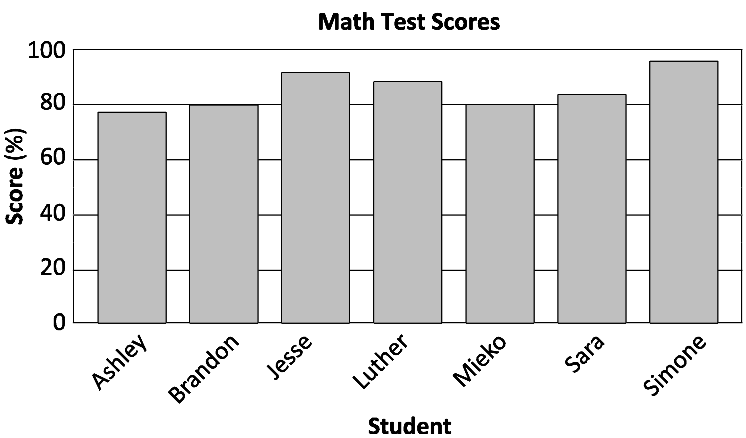
		- Too many categories can make it look crowded
Line (Continuous/Non-Discrete Data)	Shows change over a period of time - can see trends easily - can be used to estimate values between data points and beyond data points (interpolation/extrapolation) - easy to draw by hand	Limited to data collected over time (continuous data) - may be difficult to read accurately depending on scale used - comparisons between categories are not identified as quickly
Bar (Discrete Data)	Shows numbers of items in specific categories - easy to compare data, especially when the data set is large - easy to draw by hand	May be difficult to read accurately depending on scale used - trends are identifiable, but not for purposes of interpolating and extrapolating.
Double Bar (Discrete Data)	Contains two sets of data that show numbers of items in categories - useful for comparing one set of data to another - easy to draw by draw and interpret	May be difficult to read accurately depending on scale used - trends are identifiable but not for purposes of interpolating and extrapolating
Pictograph (Discrete Data)	Symbols are visually appealing - useful for small data values - symbol quickly shows the subject of graph - easy to compare among similar situations	Need to be able to divide the symbol into smaller pieces - choice of symbol and accuracy when drawing may be challenging

In previous grades, students have constructed and interpreted pictographs, bar graphs, line graphs, and circle graphs. The focus in Mathematics 8 is to interpret or justify the use of a type of graph and its validity in a certain context and *note* the ability to construct the graph.

Provide examples of various graphs that are used correctly and incorrectly to display data. This will lead to class discussions on choices of graphical representations. Some examples are provided below.







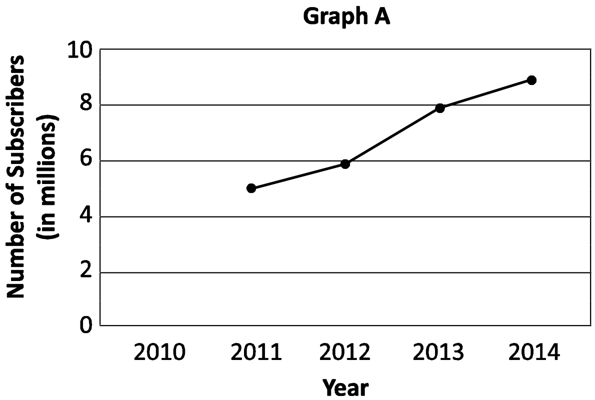
Questions for discussion:

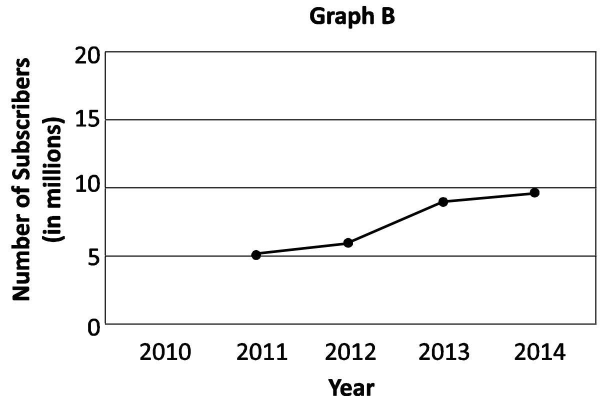
- What conclusions can you make based on each graph?
- What trends do you notice in each graph?
- What do you think of the use of each type of graph for the data it represents?
- If you feel a graph is unsuitable, what graph would you have used to best represent the data? Why?

Although a graph may be drawn correctly, changes to the scale or spread of the data will affect how the information is interpreted.

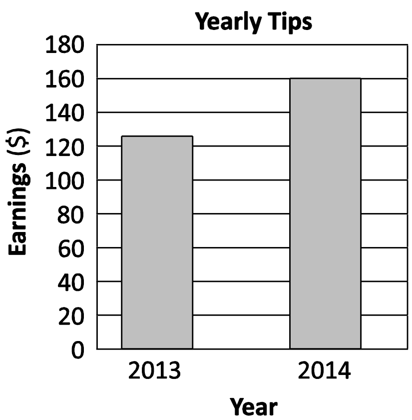
Consider this example.

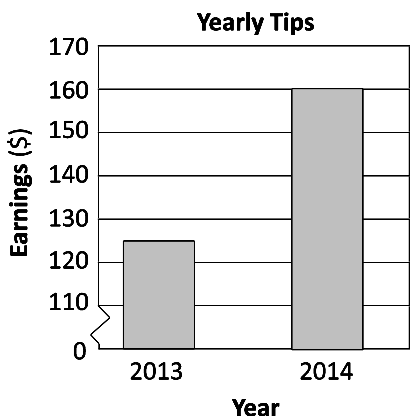
- In Graph A, the increase in subscribers appears to be greater than the increase shown in Graph B. However, both graphs actually show the same increase, which is approximately double the number of subscribers.





- In the second of the two graphs below, the change in scale may lead to the misinterpretation that there was a much greater increase in tips from 2013 to 2014. This discussion could lead to conversations about how you would graph the data depending on the intended audience.



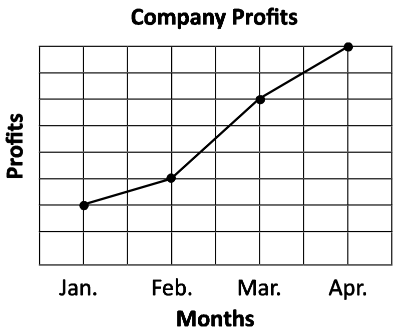


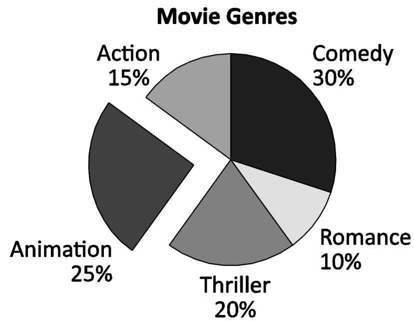
Sometimes the creator of a graph will intentionally misrepresent the data by incorrectly constructing the graph. This is usually done to emphasize or draw the reader's attention to an intended interpretation.

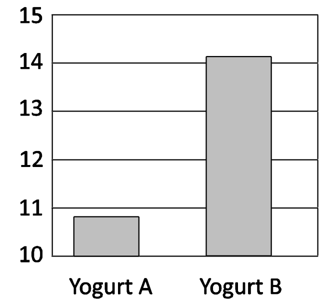
The following are some ways in which graphs can be misrepresented:

- starting the scale at a number other than zero
- using bars of different widths
- absence of a scale
- larger symbols used for a particular category on a pictograph
- no key given for pictograph symbol
- sections of a circle graph pulled away from the other sectors to emphasize it

Some examples are shown below:







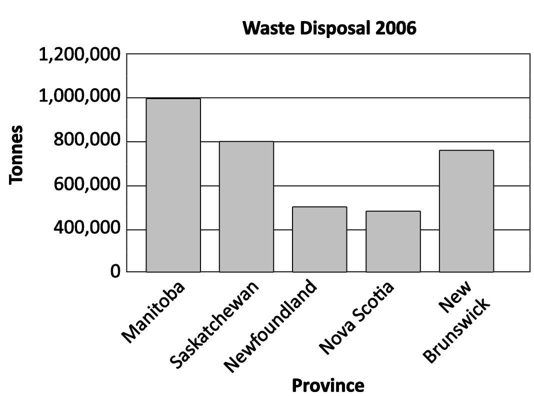
Assessment, Teaching, and Learning

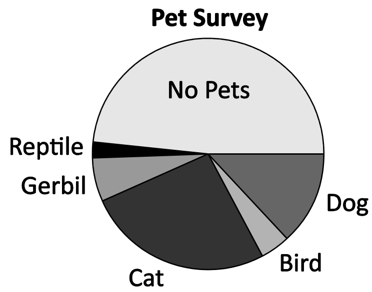
Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Tasks such as the following could be used to determine students' prior knowledge.

- Have students read the following scenarios and indicate whether the statements are true or false.
  - Have students fill out the Start column at the beginning of your work with data.
  - Have students fill out the End column at the end of your work with data, as a post-assessment.
  - If students select false, have them explain why in the space provided.

Scenario	True or False Start	True or False End	Explanation
All students in the class have identified their favourite search engine. The data should be displayed with a broken line graph.			
A circle graph must be filled up entirely, with no empty sectors.			
25% of a circle is 60 degrees.			
The intervals on the bar graph below are properly spaced. 			

<table>
<tr><td>In the following circle graph representing 100 people, about 25 of them have a cat.

</td><td></td><td></td><td></td></tr>
</table>

- Have students create a pictograph, bar graph, line graph, and circle graph for each set of data presented below. (Collect and save these data representations for future use. Tell students that, through exploration, they will be able to determine the strengths and limitations of different graphs, identify advantages and disadvantages of graphs, and justify their choice of graphical representation for a given situation.)

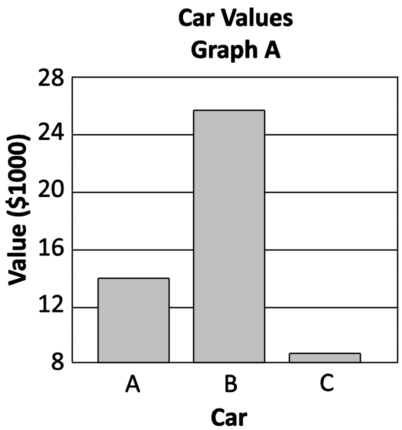
  - The following are average masses of some North American animals when they are full-grown.
    - Arctic wolf: 80 kg
    - black bear: 135 kg
    - bobcat: 9 kg
    - grizzly bear: 450 kg
    - mule deer: 90 kg
  - On a typical week day, Pierre spends his time doing the following:
    - watching tv and using the computer: 10%
    - at school: 25%
    - doing homework: 5%
    - sleeping: 33.3%
    - hanging out with friends: 10%
    - reading: 5%
    - eating: 5%
    - other: 6.6%

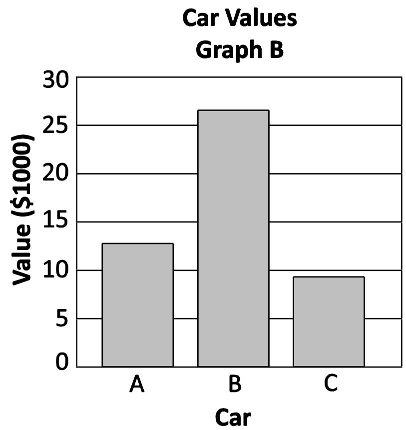
Whole-Class/Group/Individual Assessment Tasks

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

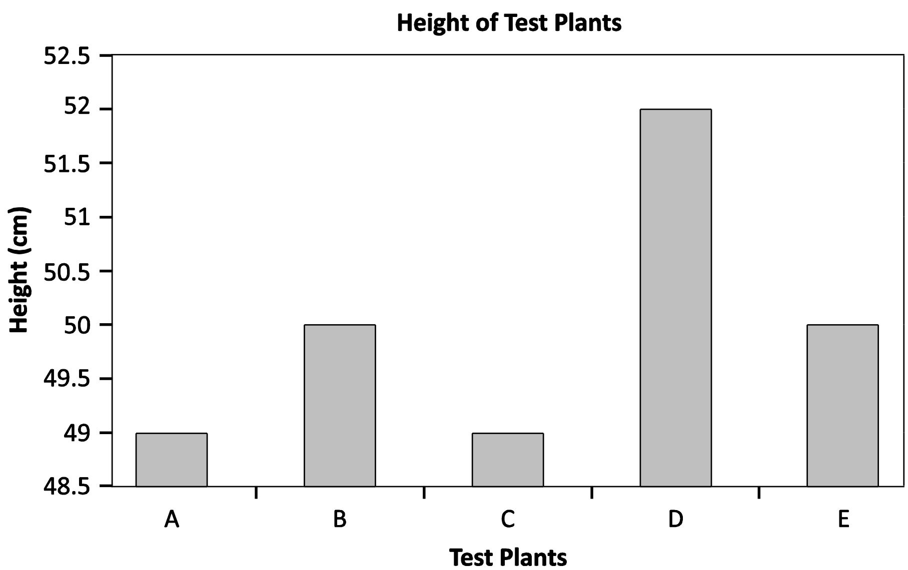
- What graph would you use to represent the data below? Explain your choice.
  - the cost of car insurance over the past 20 years
  - prices of different brands of athletic shoes
  - the favourite after school activities of one grade compared to another grade
  - the percentage of favorite ice-cream flavours of grade 8 stduents
- For the following set of graphs:

- How is the data in each graph misrepresented?
- Why would the creator of each graph choose to portray the information this way?
- Explain how the interpretation of Graph A would be different from the interpretation of Graph B.

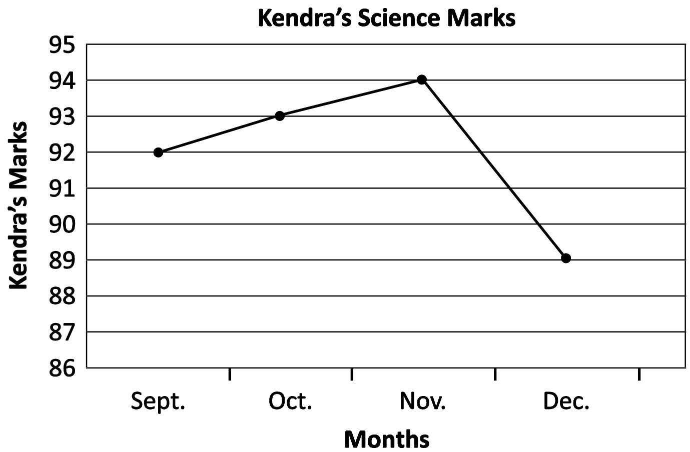




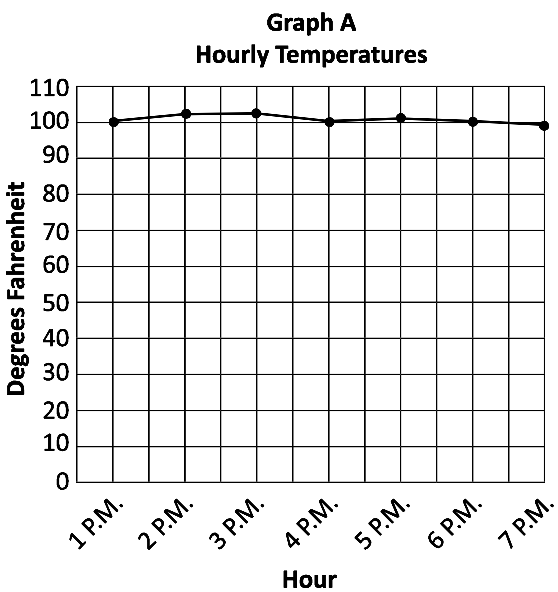
- Tell students that the following graph shows that Test Plant D grew much taller than the other plants. How is this information misleading?

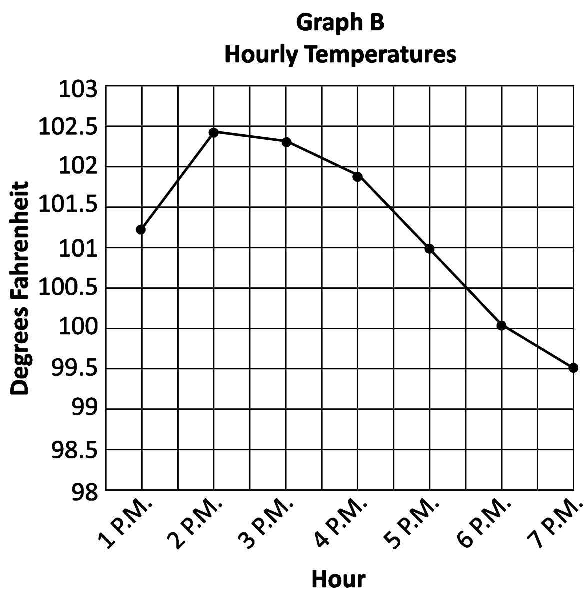


- Review graphs from various sources.
  - What are some advantages for using the graph that was chosen?
  - What are some disadvantages for using the graph that was chosen?
  - What other graphs could have been used?
- What type of graph would you use to represent the data below and explain your choice.
  - the average monthly temperatures for New Brunswick and Nova Scotia for the past year
  - prices of different brands of athletic shoes
  - the percentage of grade 8 students involved in various after school activities
  - the favourite type of cell phone for teens
- This graph shows that Kendra received a much lower grade in science class during December. Do you think Kendra should be worried by what appears to be such a large drop in her grades? Explain your reasoning.



- Explain what you know about different types of graphs and how they can be used.
- Make a table showing advantages and disadvantages of circle graphs, line graphs, bar graphs, double bar graphs, and pictographs.
- For the following set of graphs:
  - How is the data in each graph misrepresented?
  - Why would the creator of each graph choose to portray the information this way?
  - Explain how the interpretation for Graph A would be different from the interpretation for Graph B.



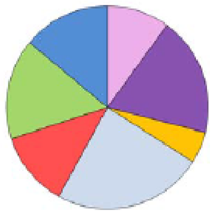


Planning for Instruction

Choosing Instructional Strategies

Consider the following strategies when planning daily lessons.

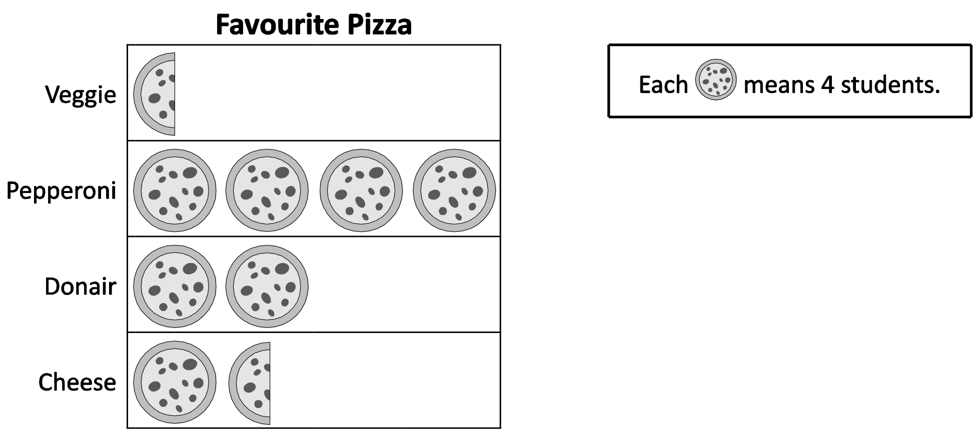
- Provide an untitled and unlabelled graph and ask students to come up with different sets of data that might realistically be represented by the graph.



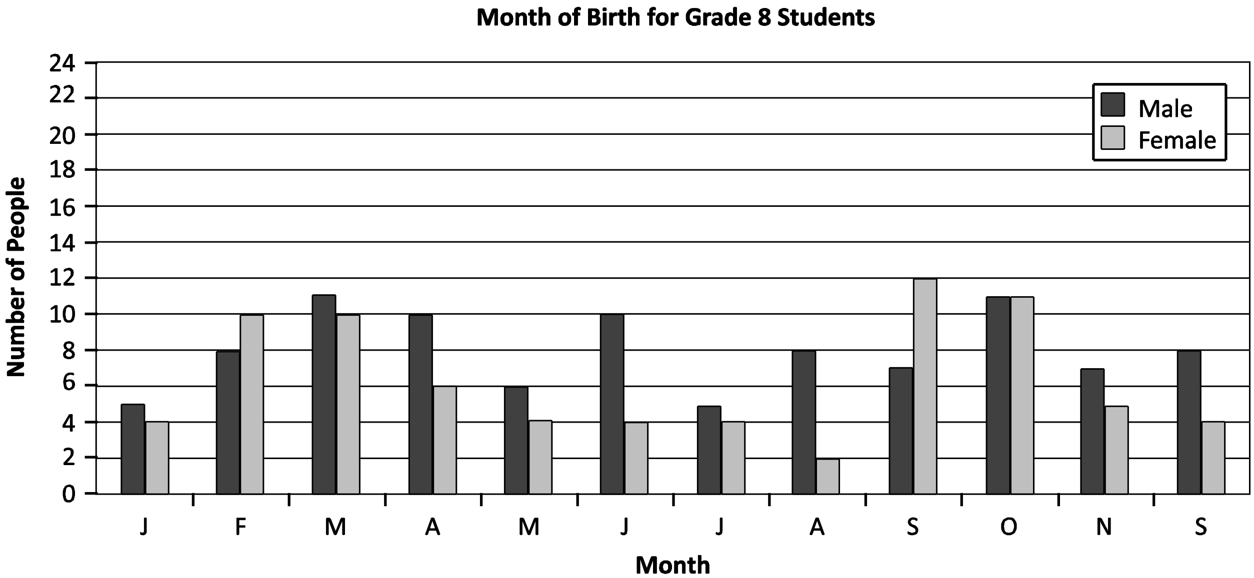
- Have the class collect a set of data. Instruct each group to display the data on a different type of graph. Discuss the pros and cons of each display for the set of data.
- Make a table showing advantages and disadvantages of circle graphs, line graphs, bar graphs, double bar graphs, and pictographs.
- Discuss whether a graph misrepresents the data. What is its purpose and who is the intended audience? Is the graph effective for its purpose and audience?

Suggested Learning Tasks

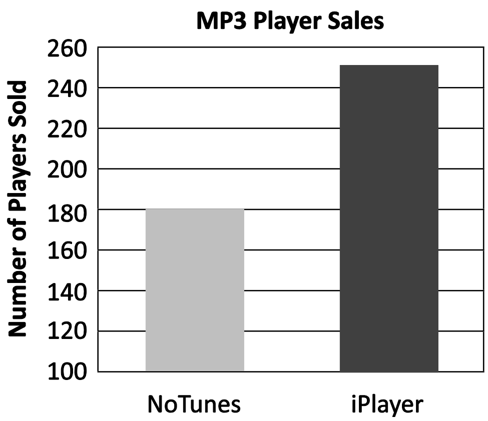
- Have students bring in graphs that misrepresent data from newspapers, internet, or other sources. Discuss why they think the graph misrepresents the data. Who is the intended audience? Is the graph effective for its purpose?
- Use the data displayed on the pictograph to have students create a different type of graph that you believe would be another suitable representation of "Favourite Pizza."



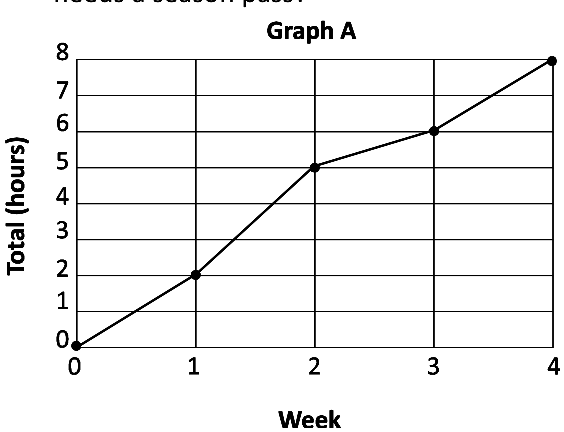
- Look at the double bar graph provided to summarize the information presented and indicate some of the pros and cons of displaying the data using this particular type of graph.

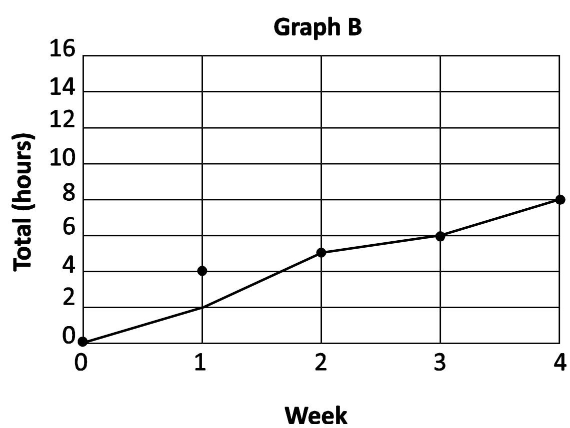


- Why is the following statement is incorrect? "Sales of iPlayer were about double the sales of NoTunes." Discuss what could be changed on the graph, or added to it to make it less misleading.



- The two graphs below show Simone's hours of skiing during the month of February. Which graph would be best used to convince her parents that she has increased her skiing time so much that she needs a season pass?





- Do a survey with your class on time spent doing homework. Have students draw two of the same type of graph, each with a specific audience in mind.
  - What is the difference between each graph?

- What interpretation can be made from each graph?
- Who is the intended audience for each graph? How does this change how the data is displayed?

Suggested Models and Manipulatives

- computer spreadsheet applications (e.g., Microsoft Excel)
- graphs from various sources
- computer, tablet
- websites for data sources (e.g., Statistics Canada; www.statcan.gc.ca)
- websites for interactives (e.g., Shodor; www.shodor.org\interactive)

Mathematical Language

Teacher	Student
bar graph	bar graph
circle graph	circle graph
continuous	continuous
discrete	discrete
distort	distort
double bar graph	double bar graph
double line graph	double line graph
horizontal axis	horizontal axis
interval	interval
line graph	line graph
pictograph	pictograph
scale	scale
trend	trend
vertical axis	vertical axis

Resources

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 7: Data Analysis and Probability
  - Section 7.1: Choosing an Appropriate Graph
  - Technology: Using Spreadsheets to Record and Graph Data
  - Section 7.2: Misrepresenting Data
  - Technology: Using Spreadsheets to Investigate Formatting
  - Unit Problem: Promoting Your Cereal
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
- Unit Tests
- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006), 324.

Math Matters: Understanding the Math You Teach, Second Edition (Chapin and Johnson, 2006) 94–300.

Digital

- *Statistics Canada* (Government of Canada 2015): www.statcan.gc.ca
- *Shodor* (Shodor 2015): www.shodor.org\interactive

SCO SP02 Students will be expected to solve problems involving the probability of independent events. [C, CN, PS, T]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

SP02.01 Determine the probability of two given independent events, and verify the probability using a different strategy.
SP02.02 Generalize and apply a rule for determining the probability of independent events.
SP02.03 Solve a given problem that involves determining the probability of independent events.

Scope and Sequence

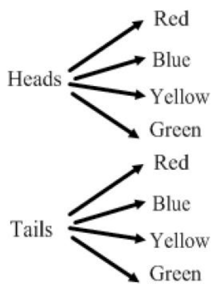
Mathematics 7	Mathematics 8	Mathematics 9
SP06 Students will be expected to conduct a probability experiment to compare the theoretical probability (determined using a tree diagram, table, or other graphic organizer) and experimental probability of two independent events.	**SP02** Students will be expected to solve problems involving the probability of independent events.	**SP04** Students will be expected to demonstrate an understanding of the role of probability in society.

Background

The concept of probability was introduced in Mathematics 5. Students should understand the difference between **experimental** and **theoretical probability**, as this was introduced in Mathematics 6. From their work in Mathematics 7, students should be familiar with the construction of tables (limited to two events) and tree diagrams (two or more events) for determining the sample space of all possible outcomes for an event. Students should be also be adept at expressing probability outcomes as fractions, decimals, and percents. Conversion between fractions, decimals, and percents is necessary since students will be required to represent a probability using these three forms.

In Mathematics 8, probability questions will be limited to those involving **independent events**. Two events are considered to be independent if the result of one does not depend on or influence the result of another. Tossing heads on a coin and rolling a 5 on a number cube are independent events (the outcome of one event has no effect on the outcome of another). Although the focus is on independent events, it is still important for students to understand the difference between independent and dependent events. Selecting a heart from a deck of cards, not replacing the card in the deck, and then selecting another heart would be an example of dependent events; the outcome of the second event is affected by the outcome of the first.

If you toss a coin and spin a spinner, what is the probability of getting heads and red? (Teachers with Smart Boards or Mimios can use the applications in the gallery). Ask students to explain the strategies they would use to organize the possible outcomes for these independent events. Possible suggestions could be: tree diagram, table, line plot, or other graphic organizer. A tree diagram is shown below:



	Red	Blue	Yellow	Green
Head	RH	BH	YH	GH
Tail	RT	BT	YT	GT

P(Heads and Red) = $\frac{1}{8}$ (because there are 8 possible outcomes and one of them is H,R)

Students should be given ample opportunity to investigate situations that will help them generalize and develop the rule for finding the probability of two independent events. They should then be able to apply the rule to find the total number of possible outcomes (sample space).

P(Event 1 and Event 2) = P(Event 1) $\times$ P(Event 2)

P(Event 1 and Event 2 and Event 3) = P(Event 1) $\times$ P(Event 2) $\times$ P(Event 3)

The probability that an event will occur is denoted as P(E) and is found by $\frac{\text{\# of favourable outcomes}}{\text{total \# of outcomes}}$

A possible investigation follows.

- Using a die and a 3-section spinner, determine the probability of rolling a number less than 6 and spinning blue.

- Have students determine the probability of rolling a number less than 6.

  $P(<6)=\frac{5}{6}$

- Have students determine the probability of spinning blue.

  $P(blue)=\frac{1}{3}$

- Have students construct a sample space using a graphic organizer (tree diagram or table).

	Red	Blue	Green
1	R1	B1	G1
2	R2	B2	G2
3	R3	B3	G3
4	R4	B4	G4
5	R5	B5	G5
6	R6	B6	G6

– Using the sample space, have students determine the probability of rolling a number less than 6 and spinning blue.

$P(< 6 \textit{ and blue}) = \frac{5}{18}$

– Consider the probabilities of P(< 6) and P(blue) and P(< 6 and blue). Encourage students to make a link between these three fractions. They should be able to conclude that

$P(< 6 \textit{ and blue}) = \mathrm{P}(<6) \times \mathrm{P}(\text{blue})$

– Using the same die and spinner, ask students to verify the rule by finding probabilities such as

> P(prime number and red)

> P(3and green)

> P(odd and red)

Students should now be able to use a graphic organizer or a rule to find the probability of independent events, and apply this to problem-solving situations.

A sample problem follows.

- A school uniform allows students the choice to wear blue or black pants and white or grey shirts. What is the probability that a student is wearing grey pants with a white shirt?

  $P(\textit{grey} \text{ and white}) = \mathrm{P}(\text{grey}) \times \mathrm{P}(\text{white})$

  $P(\textit{grey} \text{ and white}) = \frac{1}{2} \times \frac{1}{2}$

  $P(\textit{grey} \text{ and white}) = \frac{1}{4}$

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Tasks such as the following could be used to determine students’ prior knowledge.

- Emma and her mom are playing a number cube game and it is Emma’s turn to roll the number cube. If she can roll a 5, she will win the game. Emma knows that the chances of rolling a 5 are pretty small—in fact only 1 in 6. Explain, using words and diagrams, whether Emma is correct.

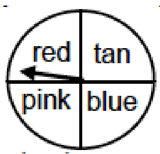
- Describe whether the following events (A and B) are dependent or independent and explain your thinking:

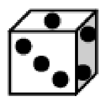
A. Mrs. Brown's first child was a boy. B. Mrs. Brown's second child will be a boy.	A. Allison got an A in her last math test. B. Allison will get an A in her next math test.
A. It snowed last night. B. Jon will be late for school this morning.	A. Matthew tossed a head with his last coin toss. B. Matthew will toss a head in his next coin toss.
A. Leif swam 2 hours every day for the last ten months. B. Leif 's swimming times have improved.	

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following **sample tasks** (that can be adapted) for either assessment for learning (formative) or assessment of learning (summative).

- Compare the theoretical and experimental probability for spinning red on the spinner below and rolling a prime number on a number cube.





- Tell students that the probability of two independent events is $\frac{5}{12}$.

  If one of the events is tossing heads on a coin, what could the other event be?

- Solve the following problems:
  - A large basket of fruit contains 3 oranges, 2 apples, and 5 bananas. If a piece of fruit is chosen at random what is the probability of getting an orange or a banana? Express your answer in fraction, decimal, and percent form.
  - At the cafeteria you can choose: milk, water, or juice to drink; a ham or turkey sandwich; and apple, cherry, or pumpkin pie for dessert. What is the probability that a student will have a turkey sandwich with milk and cherry pie?
  - The cafeteria offers chicken burgers, pizza, or veggie wraps as entrees and fruit slushies, water, or milk as beverages. What is the probability that your friend chooses the following:
    - P(pizza and milk)?
    - P(chickenburger and water)?
    - P(veggie wraps and not milk)?
  - In the game of Monopoly, you must roll doubles on the dice in order to get out of jail. What is the probability of rolling doubles on your next turn?
  - Both you and your friend have a bag of fruit snacks. Each bag has 3 grape, 4 strawberry, 3 orange, and 2 lemon snacks. What is the probability of you picking an orange fruit snack and your friend picking a grape fruit snack?

Planning for Instruction

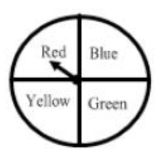
CHOOSING INSTRUCTIONAL STRATEGIES

Consider the following strategies when planning daily lessons.

- Explore both experimental and theoretical probabilities through a variety of situations and materials.
- Review the approaches for determining a sample space (tree diagrams and tables) and extend this to establish the fact that the space can be determined by simply multiplying the outcomes together.
  - For example: A menu offers a lunch special of either a hot dog or a hamburger with a choice of an apple, orange, or banana for dessert. How many different meal combinations could be ordered?
- Do not explicitly present a rule for determining the probability of two or more independent events to the students. Rather, through the use of tree diagrams and/or tables students should be given the opportunity to discover a rule themselves. It is also important to realize that the idea of multiplying individual probabilities can be extended to include more than just two events.
- Use the SMARTBoard Gallery (SMART Technologies), Mimio Tools (Mimio), or websites such as the *National Library of Virtual Manipulatives* (http://nlvm.usu.edu/en/nav/vlibrary.html) to access digital manipulatives that can be used for simulation purposes (e.g., coin toss, spinner, number cubes, playing cards, or random number generator).

SUGGESTED LEARNING TASKS

- Work with a partner and give each group two number cubes. Player A will score a point if the sum of the two number cubes is even and Player B will score a point if the sum is odd. The player that scores 10 points first wins. Once finished your game, use a graphic organizer such as a tree diagram or a table to find the outcomes for the independent events.
  - Is the game fair? If not, how could you make the game fair?
  - What is the probability of getting a sum of 2, 8, 11, etc.?
  - Repeat the task using the product of the dice to receive points.
- Develop a new game that is fair using the probability of independent events.
- Prove that the rule $P(Event\ 1 \text{ and } Event\ 2) = P(Event\ 1) \times P(Event\ 2)$ works in the following situations:



  - P(heads and yellow)
  - P(tails and green or blue)
  - P(heads and not red)

  - P(blue and 6 and heads)
- Invite each student in the class to flip three coins 5 times each to simulate the genders of the children in families with three children. Use heads to indicate a girl, and tails to indicate a boy. Combine class results for all outcomes.
  - What is the experimental probability of getting 3 girls?
  - What is the theoretical probability of getting 3 girls?
  - Compare the experimental and theoretical probabilities. Why are the two values different?

- Tell students that Keith wrote a different number from one to ten on each of ten small pieces of paper and put them in a bag. He drew one number from the bag. At the same time, he tossed a coin. Using three different methods show another student how to determine the total number of possible outcomes.

SUGGESTED MODELS AND MANIPULATIVES

- coins
- number cards
- number cubes
- playing cards
- polyhedral dice
- spinners

MATHEMATICAL LANGUAGE

Teacher	Student
▪ certain ▪ experimental probability ▪ impossible ▪ independent events ▪ less likely ▪ likely ▪ more likely ▪ outcome ▪ probability ▪ probable ▪ simulation ▪ theoretical probability	▪ certain ▪ experimental probability ▪ impossible ▪ independent events ▪ less likely ▪ likely ▪ more likely ▪ outcome ▪ probability ▪ probable ▪ simulation ▪ theoretical probability

Resources/Notes

Print

Math Makes Sense 8 (Baron et al. 2008; NSSBB #: 2001642)

- Unit 7: Data Analysis and Probability
  - Section 7.3: Probability of Independent Events
  - Game: Empty the Rectangles
  - Section 7.4: Solving Problems Involving Independent Events
  - Technology: Using Technology to Investigate Probability
  - Unit Problem: Promoting Your Cereal
- *ProGuide* (CD; Word Files) (NSSBB #: 2001643)
  - Assessment Masters
  - Extra Practice Masters
  - Unit Tests

- *ProGuide* (DVD) (NSSBB #: 2001643)
  - Projectable Student Book Pages
  - Modifiable Line Masters

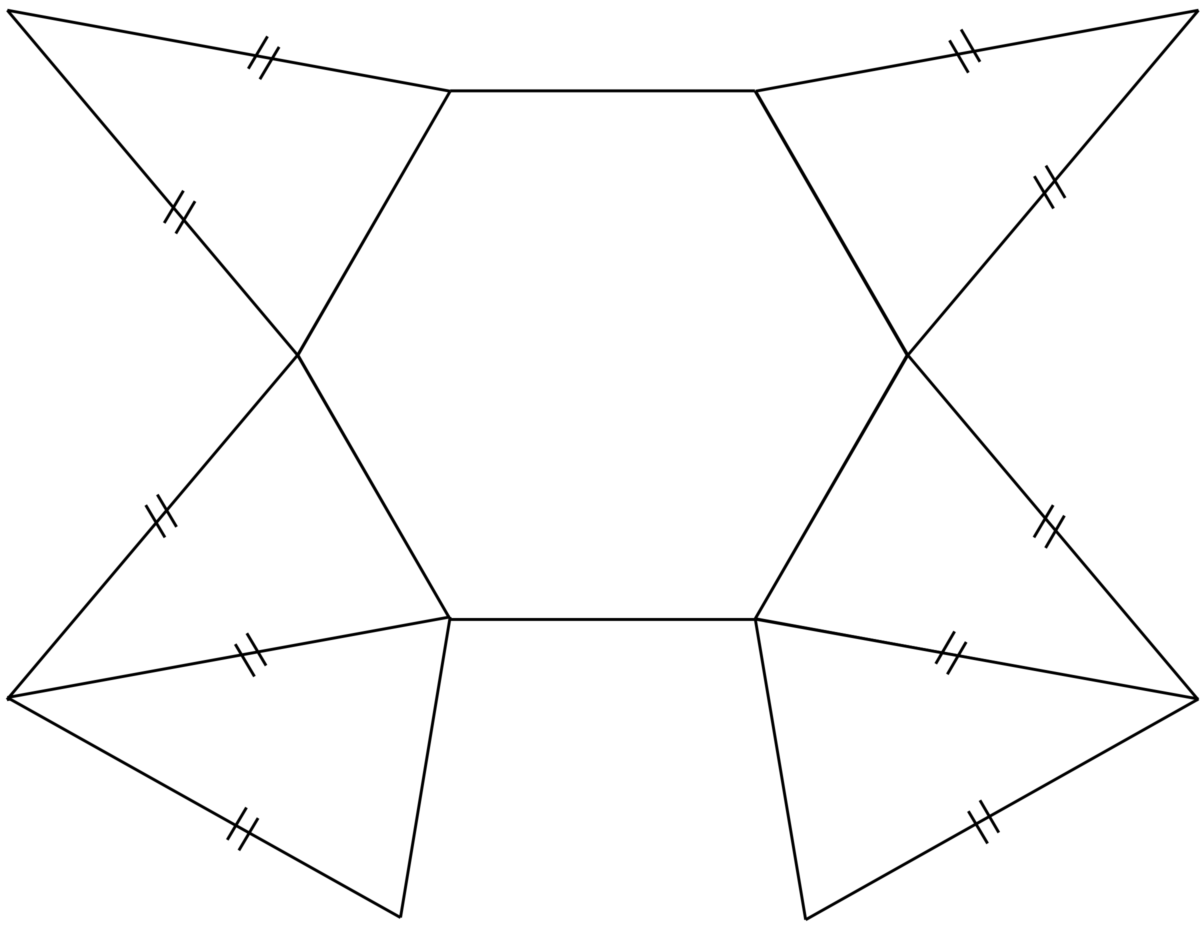
Teaching Student-Centered Mathematics, Grades 5–8, Volume Three (Van de Walle and Lovin 2006) 341–342.

Digital

- “Virtual Dice,” *BGFL* (Birmingham City Council 2015): www.bgfl.org/bgfl/custom/resources_ftp/client_ftp/ks1/maths/dice/index.htm
- “Virtual Spinner [Unnamed],” *Math Playground* (MathPlayground.com 2015): www.mathplayground.com/probability.html
- “Adjustable Spinner,” *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/adjustablespinner
- “Spinner [Unnamed], *National Library of Virtual Manipulatives* (Utah State University 2015): http://nlvm.usu.edu/en/nav/frames_asid_186_g_1_t_1.html?open=activities
- *SMARTBoard Gallery* (SMART Technologies 2015): http://smarttech.com
- *Mimio Tools* (Mimio 2015): http://mimio.com
- *National Library of Virtual Manipulatives* (Utah State University 2015): http://nlvm.usu.edu/en/nav/vlibrary.html

Appendices

Appendix A: Net



References

ABCya.com LLC. 2015. "Virtual Manipulatives: Fractions, Decimals, Percents." *abcya.com.* http://media.abcya.com/games/fraction_tiles/flash/fraction_tiles.swf.

Alberta Education. 2003. "Math Interactives: Rate/Ratio/Proportion." *LearnAlberta.ca.* www.learnalberta.ca/content/mejhm/index.html?ID1=AB.MATH.JR.NUMB&ID2=AB.MATH.JR.NUMB.RATE&lesson=html/video_interactives/rateRatioProportions/rateRatioProportionsInteractive.html.

——. 2007. *The Alberta K–9 Mathematics Program of Studies with Achievement Indicators*. Edmonton, AB: Alberta Education.

American Association for the Advancement of Science. 1993. *Benchmark for Science Literacy*. New York, NY: Oxford University Press.

Annenberg Foundation. 2014. "Interactives: Geometry 3D Shapes." *Annenberg Learner.* www.learner.org/interactives/geometry/platonic.html.

Armstrong, T. 1999. *Seven Kinds of Smart: Identifying and Developing Your Many Intelligences*. New York, NY: Plume.

Baron, Lorraine, Trevor Brown, Garry Davis, Sharon Jeroski, Susan Ludwig, Elizabeth Milne, Kanwal Neel, John Pusic, Robert Sidley, and David Sufrin. 2008. *Mathematics Makes Sense 8*. Toronto, ON: Pearson Education Canada. (NSSBB #: 2001642)

——. 2008. *Mathematics Makes Sense 8 ProGuide* [with CD and DVD]. Toronto, ON: Pearson Education Canada. (NSSBB #: 2001643)

Birmingham City Council. 2015. "Virtual Dice," *BGFL.* www.bgfl.org/bgfl/custom/resources_ftp/client_ftp/ks1/maths/dice/index.htm.

Black, Paul, and Dylan Wiliam. 1998. "Inside the Black Box: Raising Standards Through Classroom Assessment." *Phi Delta Kappan* 80, No. 2 (October 1998), 139–144, 146–148.

British Columbia Ministry of Education. 2000. *The Primary Program: A Framework for Teaching*. Victoria, BC: Province of British Columbia.

Burns, Marilyn. 2000. *About Teaching Mathematics: A K–8 Resource*, 2nd ed. Sausalito, CA: Math Solutions Publications.

Caine, Renate Numella, and Geoffrey Caine. 1991. *Making Connections: Teaching and the Human Brain*. Reston, VA: Association for Supervision and Curriculum Development.

Chapin, Suzanne H., and Art Johnson. 2006. *Math Matters: Understanding the Math You Teach*, 2nd ed. Sausalito, CA: Math Solutions.

Davies, Anne. 2000. *Making Classroom Assessment Work*. Courtenay, BC: Classroom Connections International, Inc.

Erickson, Sheldon. 2000. *Proportional Reasoning*. Fresno, CA: AIMS Education Foundation. (NSSBB #: 23187)

Frankenstein, Marilyn. 1995. "Equity in Mathematics Education: Class in the World outside the Class." *New Directions for Equity in Mathematics Education*. Cambridge, MA: Cambridge University Press.

GeoGebra. 2015. *GeoGebra*. International GeoGebra Institute. www.geogebra.org/cms/en.

Government of Canada. 2015. *Statistics Canada*. www.statcan.gc.ca.

Gurl, Theresa, Alice Artzt, Alan Sultan, and Frances Curcio. 2013. *Implementing the Common Core Statement Standards through Mathematical Problem Solving: Grades 6–8*. Reston, VA: National Council of Teachers of Mathematics.

Gutstein, Eric. 2003. "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School." *Journal for Research in Mathematics Education* 34, No. 1. Reston, VA: National Council of Teachers of Mathematics.

Herzig, Abbe. 2005. "Connecting Research to Teaching: Goals for Achieving Diversity in Mathematics Classrooms." *Mathematics Teacher*, Volume 99, No. 4. Reston, VA: National Council of Teachers of Mathematics.

Hope, Jack A., Larry Leutzinger, Barbara Reys, and Robert Reys. 1988. *Mental Math in the Primary Grades*. Palo Alto, CA: Dale Seymour Publications.

Hume, Karen. 2011. *Tuned Out: Engaging the 21st Century Learner*. Don Mills, ON: Pearson Education Canada.

Johnston-Wilder, Sue, and John Mason. 2005. *Developing Thinking in Geometry*. London, UK: Sage Publications Ltd.

Key Curriculum Press. 2015. The Geometer's Sketchpad. Columbus, ON: McGraw-Hill Education. Computer Software. (NSSBB #: 50474, 50475, 51453)

Ladson-Billings, Gloria. 1997. "It Doesn't Add Up: African American Students' Mathematics Achievement." *Journal for Research in Mathematics Education* 28, No. 6. Reston, VA: National Council of Teachers of Mathematics.

Learn Alberta. 2008. *Planning Guide: Grade 8 Congruence of Polygons, Shape and Space (Transformations), Specific Outcome 6.* Edmonton, AB: Alberta Education.

Math Playground LLC. 2014. "Thinking Blocks: Model Your Math Problems," *Math Playground* [iPad apps]. http://thinkingblocks.com/index.html.

MathPlayground.com. 2014. "Thinking Blocks: Model and Solve Word Problems, Ratio and Proportion Practice," *Math Playground*. www.mathplayground.com/tb_ratios/thinking_blocks_ratios.html.

——. 2015. "Virtual Spinner [Unnamed]," *Math Playground*. www.mathplayground.com/probability.html.

McGraw-Hill Education. 2015. "Interactive Hundredths Grid [unnamed]." *McGraw-Hill Education*. www.glencoe.com/sites/common_assets/mathematics/ebook_assets/vmf/VMF-Interface.html.

——. 2015. "Two-Colour Counters [unnamed]." *McGraw-Hill Education*. www.glencoe.com/sites/common_assets/mathematics/ebook_assets/vmf/VMF-Interface.html.

Mimio. 2015. MimioStudio. Cambridge MA: Mimio. Computer Software.

Musser, Gary L., Blake E. Peterson, and William F. Burger. 2006. *Mathematics for Elementary Teachers: A Contemporary Approach*, 7th ed. Milton: Queensland, Australia.

National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

——. 2005. "Computation, Calculators, and Common Sense: A Position of the National Council of Teachers of Mathematics" (position paper, May 2005). Reston, VA: National Council of Teachers of Mathematics.

——. 2015. *Illuminations: Resources for Teaching Math.* http://illuminations.nctm.org/.

——. 2015. "Isometric Drawing Tool," *Illuminations: Resources for Teaching Math*. http://illuminations.nctm.org/Activity.aspx?id=4182.

——. 2015. "Adjustable Spinner," *Illuminations: Resources for Teaching Math.* http://illuminations.nctm.org/adjustablespinner.

——. 2015. "Equivalent Fractions." *Illuminations: Resources for Teaching Math.* http://illuminations.nctm.org/Activity.aspx?id=3510.

——. 2015. "Proof without Words: Pythagorean Theorem," *Illuminations: Resources for Teaching Math*. http://illuminations.nctm.org/Activities.aspx?grade=3.

New Brunswick Department of Education. 2009. *Mathematics Grade 8 Curriculum*. Fredericton, NB: Province of New Brunswick.

Newfoundland and Labrador Department of Education. 2009. *Mathematics: Grade 8*, *Interim Edition*. St. John's, NL: Government of Newfoundland and Labrador.

Nova Scotia Department of Education. 2002. *Time to Learn Strategy, Guidelines for Instructional Time: Grades Primary—6*. Halifax, NS: Province of Nova Scotia. www.ednet.ns.ca/files/ps-policies/instructional_time_guidelines_p-6.pdf.

——. 2002. *Time to Learn Strategy: Instructional Time and Semestering*. Halifax, NS: Province of Nova Scotia. www.ednet.ns.ca/files/ps-policies/semestering.pdf.

——. 2010. *Gifted Education and Talent Development*. Halifax, NS: Province of Nova Scotia.

Nova Scotia Department of Education and Early Childhood Development. 2015. "Isometric Dot Paper (1 cm)," *Learning Resources and Technology*. http://lrt.ednet.ns.ca/PD/BLM/pdf_files/dot_paper/iso_dot_1cm.pdf.

——. 2015. "Square Dot Paper (1 cm)," *Learning Resources and Technology*. http://lrt.ednet.ns.ca/PD/BLM/pdf_files/dot_paper/iso_dot_1cm.pdf.

OECD Centre for Educational Research and Innovation. 2006. *Formative Assessment: Improving Learning in Secondary Classrooms*. Paris, France: Organization for Economic Co-operation and Development (OECD) Publishing.

Rubenstein, Rheta N. 2001. "Mental Mathematics beyond the Middle School: Why? What? How?" *Mathematics Teacher*, September 2001, Vol. 94, No. 6. Reston, VA: National Council of Teachers of Mathematics.

Shaw, J. M., and Cliatt, M. F. P. 1989. "Developing Measurement Sense." In P.R. Trafton (Ed.), *New Directions for Elementary School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Shodor. 2015. *Shodor*. www.shodor.org\interactive.

Small, Marian. 2008. *Making Math Meaningful to Canadian Students, K–8*. Toronto, ON: Nelson Education Ltd.

——. 2013. *Making Math Meaningful to Canadian Students, K–8*, 2nd ed. Toronto, ON: Nelson Education Ltd.

SMART Technologies. 2015. SMART Board. http://smarttech.com.

Steen, L.A. (ed.). 1990. *On the Shoulders of Giants: New Approaches to Numeracy*. Washington, DC: National Research Council.

Tate, William F. 1995. "Returning to the Root: A Culturally Relevant Approach to Mathematics Pedagogy." *Theory into Practice* 34, Issue 3. Florence, KY: Taylor & Francis.

Utah State University. 2015. "Spinner [Unnamed]." *National Library of Virtual Manipulatives*. http://nlvm.usu .edu/en/nav/frames_asid_186_g_1_t_1.html?open=activities.

——. 2015. *National Library of Virtual Manipulatives*. http://nlvm.usu.edu.

Van de Walle, John A., and LouAnn H. Lovin. 2006a. *Teaching Student-Centered Mathematics, Grades K–3,* Volume One. Boston, MA: Pearson Education, Inc.

——. 2006b. *Teaching Student-Centered Mathematics, Grades 3–5,* Volume Two. Boston, MA: Pearson Education, Inc.

——. 2006c. *Teaching Student-Centered Mathematics, Grades 5–8,* Volume Three. Boston, MA: Pearson Education, Inc.

Western and Northern Canadian Protocol (WNCP) for Collaboration in Education. 2006. *The Common Curriculum Framework for K–9 Mathematics*. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.

YouTube. 2009. "BCLN - Math 08 - Drawing 3D Objects." *YouTube*. www.youtube.com/watch?feature=player_embedded&v=nwGLLxJqt-Y.

——. "Pythagorean Theorem Water Demo." *YouTube*. www.youtube.com/watch?v=CAkMUdeB06o.