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Pre-calculus 12

CURRICULUM GUIDE

NOVA SCOTIA

Pre-calculus 12

Implementation Draft
March 2016

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Arlene Andreyck (Retired)
Cape Breton Victoria Regional School Board

Darryl Breen
Strait Regional School Board

Trudy Comeau
Tri-County Regional School Board

Jennifer Courish
Chignecto Central Regional School Board

Bob Crane
Mi'kmaw Kina'matnewey

Darlene MacKeen Hudson
Chignecto Central Regional School Board

Erick Lee
Halifax Regional School Board

Patsy Height Lewis (Retired)
Tri-County Regional School Board

David MacFarlane
Strait Regional School Board

Mark MacLeod
South Shore Regional School Board

Sonya O'Sullivan
Halifax Regional School Board

Mark Pettipas
Strait Regional School Board

Lynn Evans Phillips
South Shore Regional School Board

Marlene Urquhart
Cape-Breton Victoria Regional School Board

Tom Willis
Tri-County Regional School Board

Contents

Introduction

Background and Rationale

Mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society. It is essential that the mathematics curriculum reflect current research in mathematics instruction. To achieve this goal, Western and Northern Canadian Protocol's (WNCP) *The Common Curriculum Framework for Grades 10–12 Mathematics* (2008) has been adopted as the basis for the new mathematics curriculum in Nova Scotia.

The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan, and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and performance indicators agreed upon by the seven jurisdictions. The outcomes and performance indicators have been adapted for Nova Scotia. This document is based on both national and international research by the WNCP and the National Council of Teachers of Mathematics (NCTM).

There is an emphasis in the Nova Scotia curriculum on particular key concepts at each grade that will result in greater depth of understanding and, ultimately, stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

Purpose

This document provides sets of outcomes and performance indicators to be used as a mandated common base for defining mathematics curriculum expectations. This common base should result in consistent student outcomes in mathematics within the province of Nova Scotia. It should also enable easier transfer for students moving within the province or from any jurisdiction that has adopted the WNCP framework. This document is intended to clearly communicate to all education partners across the province the high expectations for students' mathematical learning.

Program Design and Components

Pathways

The *Common Curriculum Framework for Grades 10–12 Mathematics* (WNCP 2008), on which the Nova Scotia Mathematics 10–12 curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics* (WNCP 2006). In Nova Scotia, four pathways are available: Mathematics Essentials, Mathematics at Work, Mathematics, and Pre-calculus.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all four pathways are to provide prerequisite attitudes, knowledge, skills, and understandings for specific post-secondary programs or direct entry into the work force. All four pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour, and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of the Mathematics Essentials courses was designed in Nova Scotia to fill a specific need for Nova Scotia students. The content of each of the Mathematics at Work, Mathematics, and Pre-calculus pathways has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* (System Improvement Group, Alberta Education 2006) and on consultations with mathematics teachers.

Mathematics Essentials (Graduation)

This pathway is designed to provide students with the development of the skills and understandings required in the workplace, as well as those required for everyday life at home and in the community. Students will become better equipped to deal with mathematics in the real world and will become more confident in their mathematical abilities.

Mathematics at Work (Graduation)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, and statistics and probability.

Mathematics (Academic)

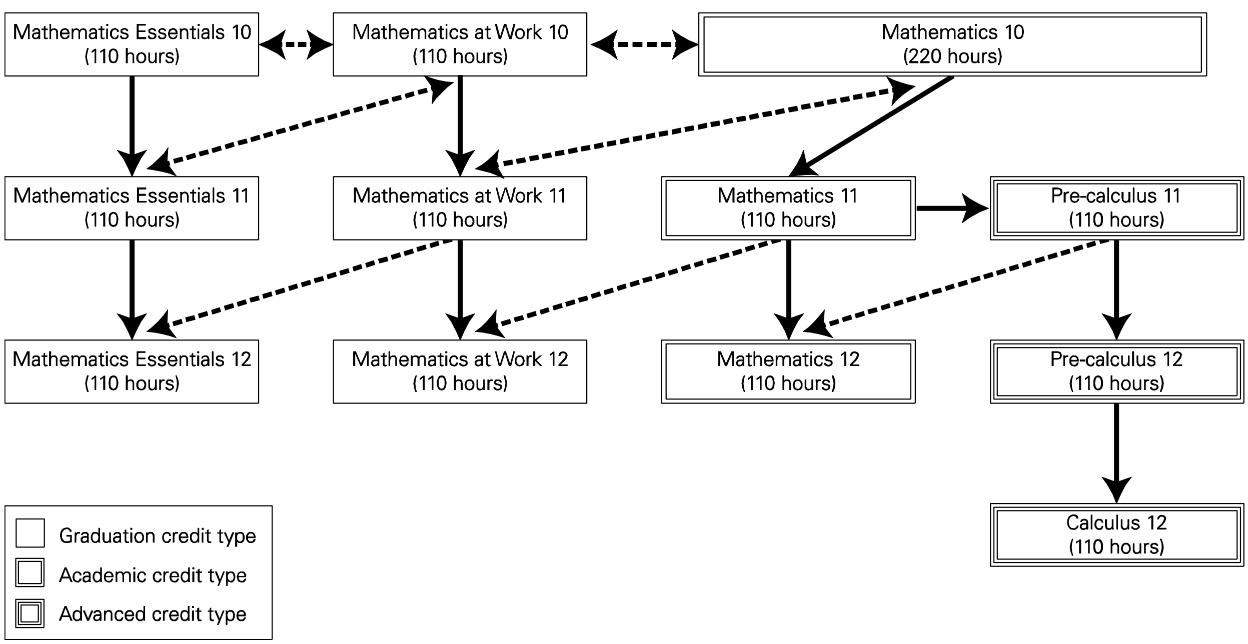
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that require an academic or pre-calculus mathematics credit. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, and statistics and probability. **Note:** After completion of Mathematics 11, students have the choice of an academic or pre-calculus pathway.

Pre-calculus (Advanced)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, and permutations, combinations, and binomial theorem.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.



Instructional Focus

Each pathway in senior high mathematics pathways is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful.

Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems, and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Assessment

Ongoing assessment for learning is essential to effective teaching and learning. Research has shown that assessment for learning (formative assessment) practices produce significant and often substantial learning gains, close achievement gaps, and build students' ability to learn new skills (Black & Wiliam 1998; OECD 2006). Student involvement in assessment promotes learning. Timely and effective teacher feedback and student self-assessment allow students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes

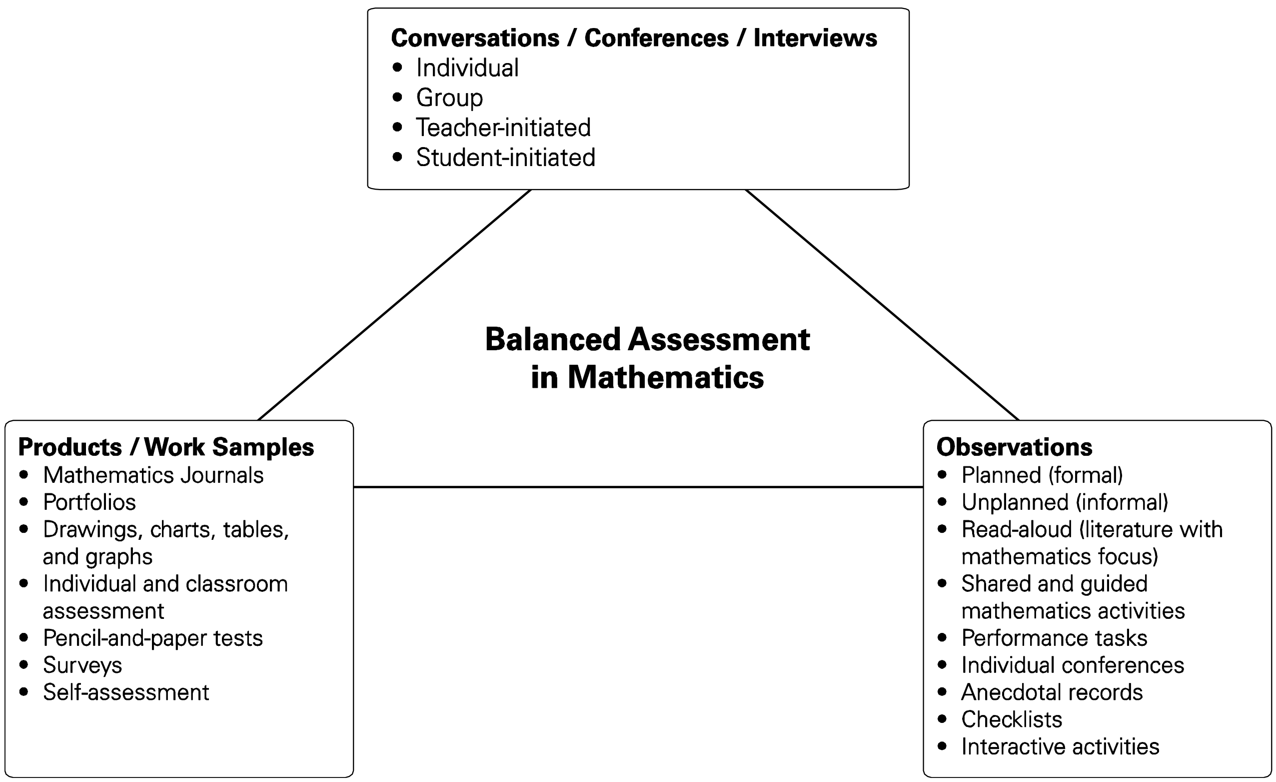
- providing clear goals, targets, and learning outcomes
- using exemplars, rubrics, and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning

(Davies 2000)

Assessment for learning practices act as the scaffolding for learning, which only then can be measured through assessment of learning (summative assessment). Assessment of learning tracks student progress, informs instructional programming, and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning, and produce achievement gains.

Assessment of student learning should

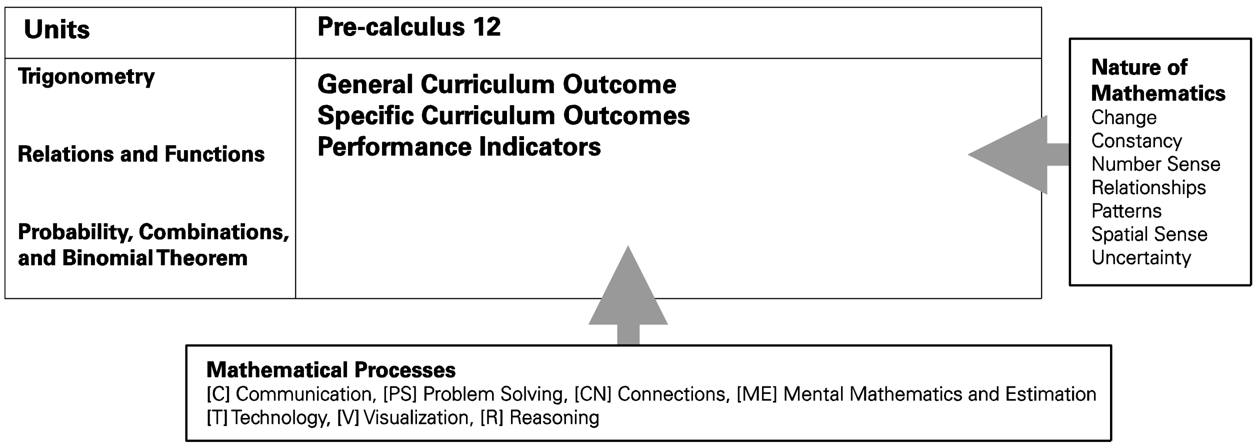
- align with curriculum outcomes
- clearly define criteria for success
- make explicit the expectations for students' performance
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction



Outcomes

Conceptual Framework for Mathematics 10–12

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



(Adapted with permission from Western and Northern Canadian Protocol, *The Common Curriculum Framework for K–9 Mathematics*, p. 5. All rights reserved.)

Structure of the Pre-calculus 12 Curriculum

Units

Pre-calculus 12 comprises three units:

- Trigonometry (T) (35–40 hours)
- Relations and Functions (RF) (60–65 hours)
- Permutations, Combinations, and Binomial Theorem (PCB) (10–15 hours)

Outcomes and Performance Indicators

The Nova Scotia curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes, and performance indicators.

General Curriculum Outcomes (GCOs)

General curriculum outcomes are overarching statements about what students are expected to learn in each strand/sub-strand. The GCO for each strand/sub-strand is the same throughout the pathway.

Trigonometry (T)

Students will be expected to develop trigonometric reasoning.

Relations and Functions (RF)

Students will be expected to develop algebraic and graphical reasoning through the study of relations.

Permutations, Combinations, and Binomial Theorem (PCB)

Students will be expected to develop algebraic and numeric reasoning that involves combinatorics.

Specific Curriculum Outcomes (SCOs) and Performance Indicators

Specific curriculum outcomes are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as expected for a given grade.

Performance indicators are samples of how students may demonstrate their performance of the goals of a specific curriculum outcome. The range of samples provided is meant to reflect the scope of the SCO. In the SCOs, the word **including** indicates that any ensuing items *must* be addressed to fully achieve the learning outcome. The phrase **such as** indicates that the ensuing items are provided for clarification only and are *not* requirements that must be addressed to fully achieve the learning outcome. The word **and** used in an outcome indicates that both ideas must be addressed to achieve the learning outcome, although not necessarily at the same time or in the same question.

TRIGONOMETRY (T)

T01 Students will be expected to demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V]

Performance Indicators

T01.01 Sketch, in standard position, an angle (positive or negative) when the measure is given in degrees.

T01.02 Describe the relationship among different systems of angle measurement, with emphasis on radians and degrees.

T01.03 Sketch, in standard position, an angle with a measure of one radian.

T01.04 Sketch, in standard position, an angle with a measure expressed in the form $k\pi$ radians, where $k \in \mathrm{Q}$.

T01.05 Express the measure of an angle in radians (exact value or decimal approximation), given its measure in degrees.
T01.06 Express the measure of an angle in degrees, given its measure in radians (exact value or decimal approximation).
T01.07 Determine the measures, in degrees or radians, of all angles in a given domain that are coterminal with a given angle in standard position.
T01.08 Determine the general form of the measures, in degrees or radians, of all angles that are coterminal with a given angle in standard position.
T01.09 Explain the relationship between the radian measure of an angle in standard position and the length of the arc cut on a circle of radius *r*, and solve problems based upon that relationship.

T02 Students will be expected to develop and apply the equation of the unit circle. [CN, R, V]

Performance Indicators
T02.01 Derive the equation of the unit circle from the Pythagorean theorem.
T02.02 Describe the six trigonometric ratios, using a point P (*x*, *y*) that is the intersection of the terminal arm of an angle and the unit circle.
T02.03 Generalize the equation of a circle with centre (0, 0) and radius *r*.

T03 Students will be expected to solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V]

Performance Indicators
T03.01 Determine, with technology, the approximate value of a trigonometric ratio for any angle with a measure expressed in either degrees or radians.
T03.02 Determine, using a unit circle or reference triangle, the exact value of a trigonometric ratio for angles expressed in degrees that are multiples of 0°, 30°, 45°, 60°, or 90°, or for angles expressed in radians that are multiples of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, or $\frac{\pi}{2}$ and explain the strategy.
T03.03 Determine, with or without technology, the measures, in degrees or radians, of the angles in a specified domain, given the value of a trigonometric ratio.
T03.04 Explain how to determine the exact values of the six trigonometric ratios, given the coordinates of a point on the terminal arm of an angle in standard position.
T03.05 Determine the measures of the angles in a specified domain in degrees or radians, given a point on the terminal arm of an angle in standard position.
T03.06 Determine the exact values of the other trigonometric ratios, given the value of one trigonometric ratio in a specified domain.
T03.07 Sketch a diagram to represent a problem that involves trigonometric ratios.
T03.08 Solve a problem, using trigonometric ratios.

T04 Students will be expected to graph and analyze the trigonometric functions sine, cosine, and tangent to solve problems. [CN, PS, T, V]

Performance Indicators
T04.01 Sketch, with or without technology, the graph of $y = \sin x$, $y = \cos x$, or $y = \tan x$.
T04.02 Determine the characteristics (amplitude, asymptotes, domain, period, range, and zeros) of the graph of $y = \sin x$, $y = \cos x$, or $y = \tan x$.
T04.03 Determine how varying the value of a affects the graphs of $y = a \sin x$ and $y = a \cos x$.
T04.04 Determine how varying the value of d affects the graphs of $y = \sin x + d$ and $y = \cos x + d$.
T04.05 Determine how varying the value of c affects the graphs of $y = \sin (x + c)$ and $y = \cos (x + c)$.

T04.06 Determine how varying the value of b affects the graphs of $y = \sin bx$ and $y = \cos bx$.
T04.07 Sketch, without technology, graphs of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$, using transformations, and explain the strategies.
T04.08 Determine the characteristics (amplitude, asymptotes, domain, period, phase shift, range and zeros) of the graph of a trigonometric function of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$.
T04.09 Determine the values of $a, b, c,$ and d for functions of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$ that correspond to a given graph, and write the equation of the function.
T04.10 Determine a trigonometric function that models a situation to solve a problem.
T04.11 Explain how the characteristics of the graph of a trigonometric function relate to the conditions in a problem situation.
T04.12 Solve a problem by analyzing the graph of a trigonometric function.

T05 Students will be expected to solve, algebraically and graphically, first- and second-degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]

Performance Indicators

T05.01 Verify, with or without technology, that a given value is a solution to a trigonometric equation.
T05.02 Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form, when possible.
T05.03 Determine, using technology, the approximate solution of a trigonometric equation in a restricted domain.
T05.04 Relate the general solution of a trigonometric equation to the zeros of the corresponding trigonometric function (restricted to sine and cosine functions).
T05.05 Determine, using technology, the general solution of a given trigonometric equation.
T05.06 Identify and correct errors in a solution for a trigonometric equation.

T06 Students will be expected to prove trigonometric identities, using

- reciprocal identities
- quotient identities
- Pythagorean identities
- sum or difference identities (restricted to sine, cosine, and tangent)
- double-angle identities (restricted to sine, cosine, and tangent)

[R, T, V]

Performance Indicators

T06.01 Explain the difference between a trigonometric identity and a trigonometric equation.
T06.02 Verify a trigonometric identity numerically for a given value in either degrees or radians.
T06.03 Explain why verifying that the two sides of a trigonometric identity are equal for given values is insufficient to conclude that the identity is valid.
T06.04 Determine, graphically, the potential validity of a trigonometric identity, using technology.
T06.05 Determine the non-permissible values of a trigonometric identity.
T06.06 Prove, algebraically, that a trigonometric identity is valid.
T06.07 Determine, using the sum, difference, and double-angle identities, the exact value of a trigonometric ratio.

Relations and Functions (RF)

RF01 Students will be expected to demonstrate an understanding of operations on, and compositions of, functions. [CN, R, T, V]

Performance Indicators

RF01.01 Sketch the graph of a function that is the sum, difference, product, or quotient of two functions, given their graphs.

RF01.02 Write the equation of a function that is the sum, difference, product, or quotient of two or more functions, given their equations.

RF01.03 Determine the domain and range of a function that is the sum, difference, product, or quotient of two functions.

RF01.04 Write a function $h(x)$ as the sum, difference, product, or quotient of two or more functions.

RF01.05 Determine the value of the composition of functions when evaluated at a point, including $f[f(a)]$, $f[g(a)]$, and $g[f(a)]$.

RF01.06 Determine, given the equations of two functions $f(x)$ and $g(x)$, the equation of the composite function $f[f(x)]$, $f[g(x)]$, and $g[f(x)]$, and explain any restrictions.

RF01.07 Sketch, given the equations of two functions $f(x)$ and $g(x)$, the graph of the composite function $f[f(x)]$, $f[g(x)]$, and $g[f(x)]$.

RF01.08 Write a function $h(x)$ as the composition of two or more functions.

RF01.09 Write a function $h(x)$ by combining two or more functions through operations on, and compositions of, functions

RF02 Students will be expected to demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. [C, CN, R, V]

Performance Indicators

RF02.01 Compare the graphs of a set of functions of the form $y-k=f(x)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of k.

RF02.02 Compare the graphs of a set of functions of the form $y=f(x-h)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of h.

RF02.03 Compare the graphs of a set of functions of the form $y-k=f(x-h)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effects of h and k.

RF02.04 Sketch the graph of $y-k=f(x)$, $y=f(x-h)$, or $y-k=f(x-h)$ for given values of h and k, given a sketch of the function $y=f(x)$, where the equation of $y=f(x)$ is not given.

RF02.05 Write the equation of a function whose graph is a vertical and/or horizontal translation of the graph of the function $y=f(x)$.

RF03 Students will be expected to demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. [C, CN, R, V]

Performance Indicators

RF03.01 Compare the graphs of a set of functions of the form $y=af(x)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of a.

RF03.02 Compare the graphs of a set of functions of the form $y=f(bx)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of b.

RF03.03 Compare the graphs of a set of functions of the form $y=af(bx)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effects of a and b.

RF03.04 Sketch the graph of $y = af(x)$, $y = f(bx)$, or $y = af(bx)$ for given values of *a* and *b*, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

RF03.05 Write the equation of a function, given its graph which is a vertical and/or horizontal stretch of the graph of the function $y = f(x)$.

RF04 Students will be expected to apply translations and stretches to the graphs and equations of functions. [C, CN, R, V]

Performance Indicators

RF04.01 Sketch the graph of the function $y - k = af[b(x - h)]$ for given values of *a*, *b*, *h*, and *k*, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

RF04.02 Write the equation of a function, given its graph that is a translation and/or stretch of the graph of the function $y = f(x)$.

RF05 Students will be expected to demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the *x*-axis, *y*-axis, and line *y* = *x*. [C, CN, R, V]

Performance Indicators

RF05.01 Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection in the *x*-axis, the *y*-axis, or the line *y* = *x*.

RF05.02 Sketch the reflection of the graph of a function $y = f(x)$ in the *x*-axis, the *y*-axis, or the line *y* = *x*, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

RF05.03 Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$ in the *x*-axis, the *y*-axis, or the line *y* = *x*.

RF05.04 Sketch the graphs of the functions $y = -f(x)$, $y = f(x)$, and $x = f(\mathrm{y})$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$, is not given.

RF05.05 Write the equation of a function, given its graph that is a reflection of the graph of the function $y = f(x)$ in the *x*-axis, the *y*-axis, or the line *y* = *x*.

RF06 Students will be expected to demonstrate an understanding of inverses of relations. [C, CN, R, V]

Performance Indicators

RF06.01 Explain how the graph of the line *y* = *x* can be used to sketch the inverse of a relation.

RF06.02 Explain how the transformation $(x, y) \Rightarrow (y, x)$ can be used to sketch the inverse of a relation.

RF06.03 Sketch the graph of the inverse relation, given the graph of a relation.

RF06.04 Determine if a relation and its inverse are functions.

RF06.05 Determine restrictions on the domain of a function in order for its inverse to be a function.

RF06.06 Determine the equation and sketch the graph of the inverse relation, given the equation of a linear or quadratic relation.

RF06.07 Explain the relationship between the domains and ranges of a relation and its inverse.

RF06.08 Determine, algebraically or graphically, if two functions are inverses of each other.

RF07 Students will be expected to demonstrate an understanding of logarithms. [CN, ME, R]

Performance Indicators

RF07.01 Explain the relationship between logarithms and exponents.
RF07.02 Express a logarithmic expression as an exponential expression and vice versa.
RF07.03 Determine, without technology, the exact value of a logarithm, such as $\log_2 8$ and ln e.
RF07.04 Estimate the value of a logarithm, using benchmarks, and explain the reasoning (e.g., since $\log_2 8 = 3$ and $\log_2 16 = 4$, $\log_2 9$ is approximately equal to 3.1).

RF08 Students will be expected to demonstrate an understanding of the product, quotient, and power laws of logarithms. [C, CN, R, T]

Performance Indicators

RF08.01 Develop and generalize the laws for logarithms, using numeric examples and exponent laws.
RF08.02 Derive each law of logarithms.
RF08.03 Determine, using the laws of logarithms, an equivalent expression for a logarithmic expression.
RF08.04 Determine, with technology, the approximate value of a logarithmic expression, such as $\log_2 9$ and ln 10.

RF09 Students will be expected to graph and analyze exponential and logarithmic functions. [C, CN, T, V]

Performance Indicators

RF09.01 Sketch, with or without technology, a graph of an exponential function of the form $y = a^x$, $a > 0$.
RF09.02 Identify the characteristics of the graph of an exponential function of the form $y = a^x$, $a > 0$, including the domain, range, horizontal asymptote and intercepts, and explain the significance of the horizontal asymptote.
RF09.03 Sketch the graph of an exponential function by applying a set of transformations to the graph of $y = a^x$, $a > 0$, and state the characteristics of the graph.
RF09.04 Sketch, with or without technology, the graph of a logarithmic function of the form $y = \log_b x$, $b > 1$.
RF09.05 Identify the characteristics of the graph of a logarithmic function of the form $y = \log_b x$, $b > 1$, including the domain, range, vertical asymptote and intercepts, and explain the significance of the vertical asymptote.
RF09.06 Sketch the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_b x$, $b > 1$, and state the characteristics of the graph.
RF09.07 Demonstrate, graphically, that a logarithmic function and an exponential function with the same base are inverses of each other.

RF10 Students will be expected to solve problems that involve exponential and logarithmic equations. [C, CN, PS, R]

Performance Indicators

RF10.01 Determine the solution of an exponential equation in which the bases are powers of one another.
RF10.02 Determine the solution of an exponential equation in which the bases are not powers of one another, using a variety of strategies.
RF10.03 Determine the solution of a logarithmic equation, and verify the solution.
RF10.04 Explain why a value obtained in solving a logarithmic equation may be extraneous.

RF10.05 Solve a problem that involves exponential growth or decay.
RF10.06 Solve a problem that involves the application of exponential equations to loans, mortgages, and investments.
RF10.07 Solve a problem that involves logarithmic scales, such as the Richter scale and the pH scale.
RF10.08 Solve a problem by modelling a situation with an exponential or a logarithmic equation.

RF11 Students will be expected to demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). [C, CN, ME]

Performance Indicators

RF11.01 Explain how long division of a polynomial expression by a binomial expression of the form $x-a,\ a \in Z$ is related to synthetic division.
RF11.02 Divide a polynomial expression by a binomial expression of the form $x-a,\ a \in Z,$ using long division or synthetic division.
RF11.03 Explain the relationship between the linear factors of a polynomial expression and the zeros of the corresponding polynomial function.
RF11.04 Explain the relationship between the remainder when a polynomial expression is divided by $x-a,\ a \in Z,$ and the value of the polynomial expression at $x = a$ (remainder theorem).
RF11.05 Explain and apply the factor theorem to express a polynomial expression as a product of factors.

RF12 Students will be expected to graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). [C, CN, T, V]

Performance Indicators

RF12.01 Identify the polynomial functions in a set of functions, and explain the reasoning.
RF12.02 Explain the role of the constant term and leading coefficient in the equation of a polynomial function with respect to the graph of the function.
RF12.03 Generalize rules for graphing polynomial functions of odd or even degree.
RF12.04 Explain the relationship among the zeros of a polynomial function, the roots of the corresponding polynomial equation, and the *x*-intercepts of the graph of the polynomial function.
RF12.05 Explain how the multiplicity of a zero of a polynomial function affects the graph.
RF12.06 Sketch, with or without technology, the graph of a polynomial function.
RF12.07 Solve a problem by modelling a given situation with a polynomial function and analyzing the graph of the function.

RF13 Students will be expected to graph and analyze radical functions (limited to functions involving one radical). [CN, R, T, V]

Performance Indicators

RF13.01 Sketch the graph of the function $y=\sqrt{x}$, using a table of values, and state the domain and range.
RF13.02 Sketch the graph of the function $y-k=a\sqrt{b(x-h)}$ by applying transformations to the graph of the function $y=\sqrt{x}$, and state the domain and range.
RF13.03 Sketch the graph of the function $y=\sqrt{f(x)}$, given the graph of the function $y=f(x)$, and explain the strategies used.

RF13.04 Compare the domain and range of the function $y=\sqrt{x}$ to the domain and range of the function $y=f(x)$, and explain why the domains and ranges may differ.

RF13.05 Describe the relationship between the roots of a radical equation and the *x*-intercepts of the graph of the corresponding radical function.

RF13.06 Determine, graphically, an approximate solution of a radical equation.

RF14 Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials). [CN, R, T, V]

Performance Indicators

RF14.01 Graph, with or without technology, a rational function.

RF14.02 Analyze the graphs of a set of rational functions to identify common characteristics.

RF14.03 Explain the behaviour of the graph of a rational function for values of the variable near a non-permissible value.

RF14.04 Determine if the graph of a rational function will have an asymptote or a hole for a non-permissible value.

RF14.05 Match a set of rational functions to their graphs, and explain the reasoning.

RF14.06 Describe the relationship between the roots of a rational equation and the *x*-intercepts of the graph of the corresponding rational function.

RF14.07 Determine, graphically, an approximate solution of a rational equation.

Permutations, Combinations, and Binomial Theorem (PCB)

PCB01 Students will be expected to apply the fundamental counting principle to solve problems. [C, PS, R, V]

Performance Indicators

PCB01.01 Count the total number of possible choices that can be made, using graphic organizers such as lists and tree diagrams.

PCB01.02 Explain, using examples, why the total number of possible choices is found by multiplying rather than adding the number of ways the individual choices can be made.

PCB01.03 Solve a simple counting problem by applying the fundamental counting principle.

PCB02 Students will be expected to determine the number of permutations of *n* elements taken *r* at a time to solve problems. [C, PS, R, V]

Performance Indicators

PCB02.01 Count, using graphic organizers such as lists and tree diagrams, the number of ways of arranging the elements of a set in a row.

PCB02.02 Determine, in factorial notation, the number of permutations of *n* different elements taken *n* at a time to solve a problem.

PCB02.03 Determine, using a variety of strategies, the number of permutations of *n* different elements taken *r* at a time to solve a problem.

PCB02.04 Explain why *n* must be greater than or equal to *r* in the notation ${}_nP_r$.

PCB02.05 Solve an equation that involves ${}_nP_r$ notation.

PCB02.06 Explain, using examples, the effect on the total number of permutations when two or more elements are identical.

PCB03 Students will be expected to determine the number of combinations of n different elements taken r at a time to solve problems. [C, PS, R, V]

Performance Indicators

PCB03.01 Explain, using examples, the difference between a permutation and a combination.

PCB03.02 Determine the number of ways that a subset of k elements can be selected from a set of n different elements.

PCB03.03 Determine the number of combinations of n different elements taken r at a time to solve a problem.

PCB03.04 Explain why n must be greater than or equal to r in the notation ${}_nC_r$ or $\binom{n}{r}$.

PCB03.05 Explain, using examples, why ${}_nC_r = {}_nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$.

PCB03.06 Solve an equation that involves ${}_nC_r$ or $\binom{n}{r}$. notation, such as ${}_nC_2 = 15$ or $\binom{n}{2} = 15$.

PCB04 Students will be expected to expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]

Performance Indicators

PCB04.01 Explain the patterns found in the expanded form of $(x+y)^n$, $n \leq 4$, and $n \in \mathrm{N}$, by multiplying n factors of $(x+y)$.

PCB04.02 Explain how to determine the subsequent row in Pascal's triangle, given any row.

PCB04.03 Relate the coefficients of the terms in the expansion of $(x+y)^n$ to the $(n+1)$ row in Pascal's triangle.

PCB04.04 Explain, using examples, how the coefficients of the terms in the expansion of $(x+y)^n$ are determined by combinations.

PCB04.05 Expand, using the binomial theorem, $(x+y)^n$.

PCB04.06 Determine a specific term in the expansion of $(x+y)^n$.

Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to

- communicate in order to learn and express their understanding of mathematics (Communication [C])
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines (Connections [CN])
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation [ME])
- develop and apply new mathematical knowledge through problem solving (Problem Solving [PS])
- develop mathematical reasoning (Reasoning [R])

- select and use technologies as tools for learning and solving problems (Technology [T])
- develop visualization skills to assist in processing information, making connections, and solving problems (Visualization [V])

The Nova Scotia curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning. The key to these process standards is presented in a box, as shown below, with each specific outcome within the units.

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics.

Students also need to communicate their learning using mathematical terminology. Communication can help students make connections between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, written and symbolic—of mathematical ideas. Students must communicate *daily* about their mathematics learning. This enables them to reflect, to validate, and to clarify their thinking and provides teachers with insight into students' interpretations of mathematical meanings and ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts.

When students encounter new situations and respond to questions of the type, How would you ... ? or How could you ... ?, the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

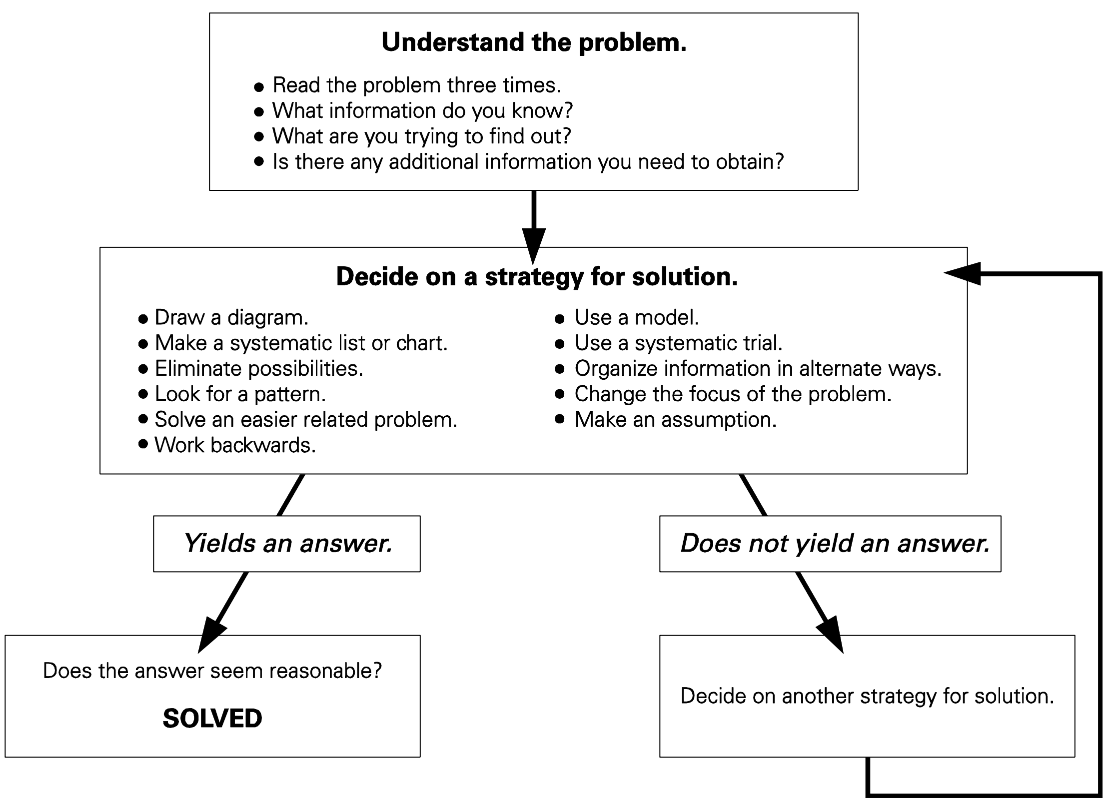
Both conceptual understanding and student engagement are fundamental in molding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive- and deductive-reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for, and engage in finding, a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

A possible flow chart to share with students is as follows:



Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to one another or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

“Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding. … *Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*.” (Caine and Caine 1991, 5).

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates learning. Concepts and skills should also be connected to everyday situations and other curricular areas. For example, when developing literacy skills students learn to make text-to-world, text-to-text, and text-to-self connections. Students can also make connections to make mathematics come alive through math-to-world, math-to-math, and math-to-self connections.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external aids.

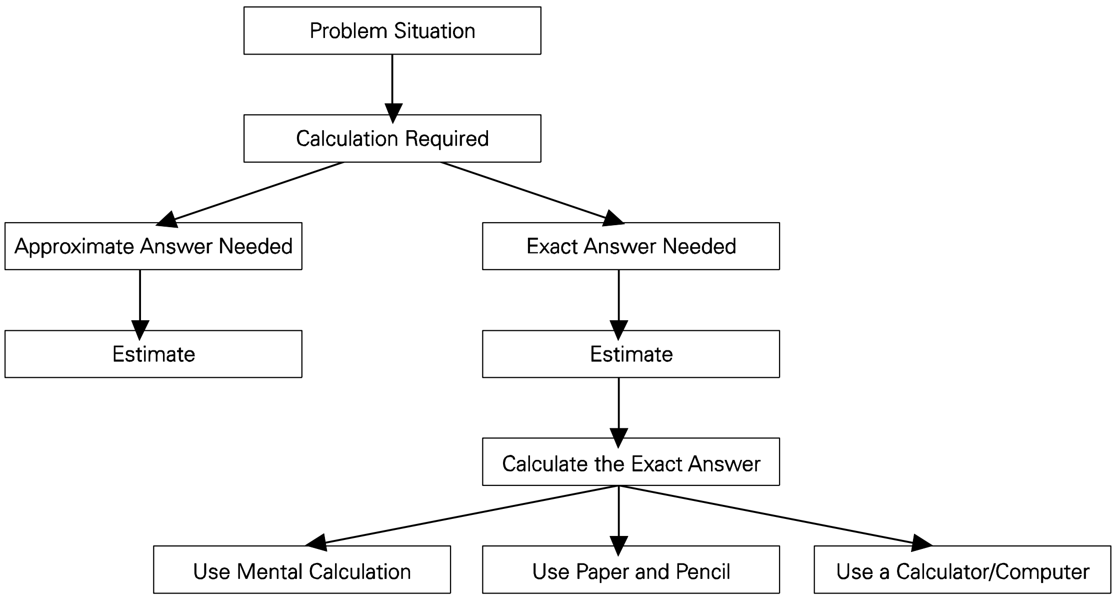
Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math.” (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers, and are more able to use multiple approaches to problem solving.” (Rubenstein 2001) Mental mathematics “provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers.” (Hope et al. 1988, v)

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when, and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process as illustrated below.



The skill of estimation requires a sound knowledge of mental mathematics. Both are necessary to many everyday experiences, and students should be provided with frequent opportunities to practise these skills.

Technology [T]

Technology can be effectively used to contribute to and support the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators, computers, and other technologies can be used to

- explore and represent mathematical relationships and patterns in a variety of ways
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of foundational concepts
- develop personal procedures for mathematical operations
- simulate situations
- develop number and spatial sense
- generate and test inductive conjectures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world." (Armstrong 1993, 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and to know several estimation strategies. (Shaw and Cliatt 1989, 150)

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, Why do you believe that's true/correct? or What would happen if ...

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true.

Mathematics reasoning involves informal thinking, conjecturing, and validating—these help students understand that mathematics makes sense. Students are encouraged to justify, in a variety of ways, their solutions, thinking processes, and hypotheses. In fact, good reasoning is as important as finding correct answers. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics, and these are woven throughout this document. These components include change, constancy, number sense, relationships, patterns, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen 1990, 184).

Students need to learn that new concepts of mathematics as well as changes to previously learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers, and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms **stability**, **conservation**, **equilibrium**, **steady state** and **symmetry** (AAAS–Benchmarks 1993, 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.
- Lines with constant slope.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy. (British Columbia Ministry of Education, 2000, 146) Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities, and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Curriculum Document Format

This guide presents the mathematics curriculum so that a teacher may readily view the scope of the outcomes that students are expected to achieve during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how students' learning at a particular grade level is part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes.

When a specific curriculum outcome is introduced, it is followed by the mathematical processes and performance indicators for that outcome. A scope and sequence is then provided, which relates the SCO to previous and next grade SCOs. Also for each SCO, there are background information, assessment strategies, suggested instructional strategies, and suggested models and manipulatives, mathematical vocabulary, and resource notes. For each section, the guiding questions should be used to help with unit and lesson preparation.

Assessment Strategies

Guiding Questions
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

FOLLOW-UP ON ASSESSMENT

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcome and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SCO

Mathematical Processes		
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections
[ME] Mental Mathematics and Estimation		
[T] Technology	**[V]** Visualization	**[R]** Reasoning

Performance Indicators

Describes observable indicators of whether students have achieved the specific outcome.

Scope and Sequence

Previous grade or course SCOs	Current grade SCO	Following grade or course SCOs

Background

Describes the "big ideas" to be learned and how they relate to work in previous grade and work in subsequent courses.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Sample tasks that can be used to determine students' prior knowledge.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Some suggestions for specific activities and questions that can be used for both instruction and assessment

FOLLOW-UP ON ASSESSMENT

Planning for Instruction

Choosing Instructional Strategies

Suggested strategies for planning daily lessons.

SUGGESTED LEARNING TASKS

Suggestions for general approaches and strategies suggested for teaching this outcome.

SUGGESTED MODELS AND MANIPULATIVES

MATHEMATICAL LANGUAGE

Teacher and student mathematical language associated with the respective outcome.

Resources/Notes

Contexts for Learning and Teaching

Beliefs about Students and Mathematics Learning

“Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.” (National Council of Teachers of Mathematics 2000, 20).

- The Nova Scotia mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning that have grown out of research and practice. These beliefs include the following:
- Mathematics learning is an active and constructive process.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.
- Learners are individuals who bring a wide range of prior knowledge and experiences and who learn via various styles and at different rates.
- Leaning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best constructed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals, and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics. The learning environment should value, respect, and address all students’ experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to understand that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals of Mathematics Education

The main goals of mathematics education are to prepare students to
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- become mathematically literate adults, using mathematics to contribute to society
- commit themselves to lifelong learning

Students who have met these goals will

- gain understanding and appreciation of the contributions of mathematics as a science, a philosophy, and an art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding. Students should be encouraged to

- take risks
- think and reflect independently
- share and communicate mathematical understanding
- solve problems in individual and group projects
- pursue greater understanding of mathematics
- appreciate the value of mathematics throughout history

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for student success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations, and engage in reflective practices.

To experience success, students must be taught to set achievable goals and assess their progress as they work toward these goals. Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Engaging All Learners

"No matter how engagement is defined or which dimension is considered, research confirms this truism of education: *The more engaged you are, the more you will learn."* (Hume 2011, 6)

Student engagement is at the core of learning. Engagement in learning occurs when students are provided with opportunities to become more invested in their learning. This is critical for teachers to take into account when planning and implementing instruction. Effective instruction engages, embraces, and supports all learners through a range of learning experiences that are both age and developmentally appropriate.

This curriculum is designed to provide learning opportunities that are equitable, accessible, and inclusive of the many facets of diversity represented in today's classrooms. When teachers know their students as individual learners and as individual people, their students are more likely to be motivated to learn, persist in challenging situations, and apply reflective practices.

Supportive Learning Environments

A supportive and positive learning environment has a profound effect on students' learning. Students need to feel physically, socially, emotionally, and culturally safe in order to take risks with their learning. In classrooms where students feel a sense of belonging, see their teachers' passion for learning and teaching, are encouraged to actively participate, and are challenged appropriately, they are more likely to be successful.

Teachers recognize that not all students progress at the same pace nor are they equally positioned in terms of their prior knowledge of particular concepts, skills, and learning outcomes. Teachers are able to create more equitable access to learning when

- instruction and assessment are flexible and offer multiple means of representation
- students have options to engage in learning through multiple ways
- students can express their knowledge, skills, and understanding in multiple ways

(Hall, Meyer, and Rose 2012)

In a supportive learning environment, teachers plan learning experiences that support *each* student's ability to achieve curriculum outcomes. Teachers use a variety of effective instructional approaches that help students to succeed, such as

- providing a range of learning opportunities that build on individual strengths and prior knowledge
- providing all students with equitable access to appropriate learning strategies, resources, and technology
- involving students in the creation of criteria for assessment and evaluation
- engaging and challenging students through inquiry-based practices
- verbalizing their own thinking to model comprehension strategies and new learning
- balancing individual, small-group, and whole-class learning experiences
- scaffolding instruction and assignments as needed and giving frequent and meaningful descriptive feedback throughout the learning process
- integrating "blended learning" opportunities by including an online environment that extends learning beyond the physical classroom
- encouraging students to take time and to persevere, when appropriate, in order to achieve a particular learning outcome

Multiple Ways of Learning

"Advances in neuroscience and education research over the past 40 years have reshaped our understanding of the learning brain. One of the clearest and most important revelations stemming from brain research is that there is no such thing as a 'regular student.'" (Hall, Meyer, and Rose 2012, 2) Teachers who know their students well are aware of students' individual learning differences and use this understanding to inform instruction and assessment decisions.

The ways in which students make sense of and demonstrate learning vary widely. Individual students tend to have a natural inclination toward one or a few learning styles. Teachers are often able to detect learning strengths and styles through observation and through conversation with students. Teachers can also get a sense of learning styles through an awareness of students' personal interests and talents.

Instruction and assessment practices that are designed to account for multiple learning styles create greater opportunities for all students to succeed.

While multiple learning styles are addressed in the classroom, the three most commonly identified are:
- auditory (such as listening to teacher-modelled think-aloud strategies or participating in peer discussion)
- kinesthetic (such as examining artifacts or problem-solving using tools or manipulatives)
- visual (such as reading print and visual texts or viewing video clips)

For additional information, refer to *Frames of Mind: The Theory of Multiple Intelligences* (Gardner 2007) and *How to Differentiate Instruction in Mixed-Ability Classrooms* (Tomlinson 2001).

A Gender-Inclusive Curriculum and Classroom

It is important that the curriculum and classroom climate respect the experiences and values of all students and that learning resources and instructional practices are not gender-biased. Teachers promote gender equity and inclusion in their classrooms when they
- articulate equally high expectations for all students
- provide equal opportunity for input and response from all students
- model gender-fair language, inclusive practices, and respectful listening in their interactions with students
- identify and openly address societal biases with respect to gender and sexual identity

Valuing Diversity: Teaching with Cultural Proficiency

"Instruction that is embedded in socially meaningful contexts, and tasks that are meaningful and relevant to the lives of students, will engage students in high-level problem-solving and reasoning and enhance students' engagement (Frankenstein 1995; Gutstein 2003; Ladson-Billings 1997; Tate 1995)." (Herzig 2005)

Teachers appreciate that students have diverse life and cultural experiences and that individual students bring different prior knowledge to their learning. Teachers can build upon their knowledge of their students as individuals, value their prior experiences, and respond by using a variety of culturally-proficient instruction and assessment practices in order to make learning more engaging, relevant, and accessible for all students. For additional information, refer to *Racial Equity Policy* (Nova Scotia Department of Education 2002) and *Racial Equity / Cultural Proficiency Framework* (Nova Scotia Department of Education 2011).

Students with Language, Communication, and Learning Challenges

Today's classrooms include students who have diverse language backgrounds, abilities, levels of development, and learning challenges. By observing and interacting with students and by conversing with students and/or their families, teachers gain deeper insights into the student as a learner. Teachers can use this awareness to identify and respond to areas where students may need additional support to achieve their learning goals. For students who are experiencing difficulties, it is important that teachers distinguish between those students for whom curriculum content is challenging and those for whom language-based factors are at the root of apparent academic difficulties. Students who are learning English as an additional language may require individual support, particularly in language-based subject areas, while they become more proficient in their English language skills. Teachers understand that many students who appear to be disengaged may be experiencing difficult life or family circumstances, mental health challenges, or low self-esteem, resulting in a loss of confidence that affects their engagement in learning. A caring, supportive teacher demonstrates belief in the students' abilities to learn and uses the students' strengths to create small successes that help nurture engagement in learning and provide a sense of hope.

Students who Demonstrate Exceptional Talents and Giftedness

Modern conceptions of giftedness recognize diversity, multiple forms of giftedness, and inclusivity. Some talents are easily observable in the classroom because they are already well developed and students have opportunities to express them in the curricular and extracurricular activities commonly offered in schools. Other talents only develop if students are exposed to many and various domains and hands-on experiences. Twenty-first century learning supports the thinking that most students are more engaged when learning activities are problem-centred, inquiry-based, and open-ended. Talented and gifted students usually thrive when such learning activities are present. Learning experiences may be enriched by offering a range of activities and resources that require increased cognitive demand and higher-level thinking with different degrees of complexity and abstraction. Teachers can provide further challenges and enhance learning by adjusting the pace of instruction and the breadth and depth of concepts being explored. For additional information, refer to *Gifted Education and Talent Development* (Nova Scotia Department of Education 2010).

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in business education, career education, literacy, music, physical education, science, social studies, technology education, and visual arts.

Trigonometry
35–40 hours

GCO: Students will be expected to develop trigonometric reasoning.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO T01 Students will be expected to demonstrate an understanding of angles in standard position, expressed in degrees and radians. [CN, ME, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

T01.01 Sketch, in standard position, an angle (positive or negative) when the measure is given in degrees.
T01.02 Describe the relationship among different systems of angle measurement, with emphasis on radians and degrees.
T01.03 Sketch, in standard position, an angle with a measure of one radian.
T01.04 Sketch, in standard position, an angle with a measure expressed in the form $k\pi$ radians, where $k \in$ Q.
T01.05 Express the measure of an angle in radians (exact value or decimal approximation), given its measure in degrees.
T01.06 Express the measure of an angle in degrees, given its measure in radians (exact value or decimal approximation).
T01.07 Determine the measures, in degrees or radians, of all angles in a given domain that are coterminal with a given angle in standard position.
T01.08 Determine the general form of the measures, in degrees or radians, of all angles that are coterminal with a given angle in standard position.
T01.09 Explain the relationship between the radian measure of an angle in standard position and the length of the arc cut on a circle of radius *r*, and solve problems based upon that relationship.

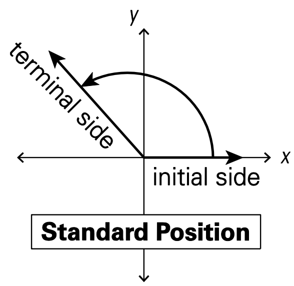
Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
T01 Students will be expected to demonstrate an understanding of angles in standard position (0° to 360°). (PC11)** **T02** Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. (PC 11)** **G03** Students will be expected to solve problems that involve the cosine law and sine law, including the ambiguous case. (M11)*	**T01** Students will be expected to demonstrate an understanding of angles in standard position, expressed in degrees and radians.

* M11—Mathematics 11
** PC11—Pre-calculus 11

Background

In Pre-calculus 11, students were introduced to angles in standard position between 0°and 360° (T01). This will be their first exposure to negative angles and to angles greater than 360°.



Up to this point, students have worked only with angles in degree measure. They will be introduced to radian measure and explore the relationship between radians and degrees. Students will also be introduced to coterminal angles and arc length.

To introduce students to radian measure, they could picture a 90° angle in standard position in the coordinate plane. This angle is subtended by an arc equal to one-fourth the circumference of any circle centred at the origin. Another way of saying this is that an angle of 90° intercepts an arc that is one-fourth of the circumference of the circle.

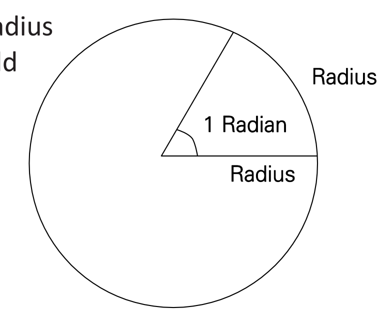
Students should observe that

- in a unit circle with radius 1, a 90° angle cuts an arc $\frac{2\pi}{4}$ or $\frac{\pi}{2}$ since the complete circle has a circumference of $2\pi(1)$ or 2π
- in a circle of radius 5, the circumference is 10π, so a 90° angle cuts an arc length of $\frac{10\pi}{4}$ or $\frac{5\pi}{2}$

Students should examine the ratio of the arc length cut by a 90° angle to the radius of the circle for various size circles, that is $\frac{\frac{1}{4}\text{ of circumference}}{\text{radius}} = \frac{\frac{1}{4}(2\pi r)}{r}$. They should conclude that these ratios are all equal to $\frac{\pi}{2}$. The fact that the ratio is constant is the basis for the radian measure for angles.

Radian measure $= \frac{\text{arc length}}{\text{radius}}$ or $\theta_{radian} = \frac{A}{r}$

Since the radian measure of an angle tells how many times the circle's radius is contained in the length of the subtended or cut-off arc, students should conclude that the radian measure of an angle will be 1 if the length of the subtended arc is equal to the radius.



Based on the definition of a radian, the relationship between a central angle θ and the length of the arc cut on a circle of radius r can be developed.

Radian measure $= \dfrac{\text{arc length}}{\text{radius}}$ or $\theta_{radian} = \dfrac{a}{r}$

$\therefore$ arc length $=$ (Radian measure)(radius) or $a = (\theta_{radian})(r)$

Students should be able to determine any variable in the relationship, given the measure of the other two.

Example:

Given an arc length of 20 cm cut on a circle of radius 5.4 cm, for example, students could be asked to determine the measure of the central angle in radians or degrees. They have the choice of rearranging the equation in terms of θ_{radian} first, or substituting the known values before solving for θ_{radian}.

$\text{arc length} = \theta_{radian} \times \text{radius}$

$20 = \theta_{radian} \times 5.4$

$\theta_{radian} \approx 3.7$ radians or $\theta_{\text{degree}} = 212^\circ$

After an introduction to radian measure using a circle of radius 1 unit, students should understand that 360° is equivalent to 2π radians. Therefore, 180° is equivalent to π radians. There are certain radian measures that occur frequently. Students should become familiar with $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$, and also with the multiples of each of these, such as $\frac{2\pi}{3}, \frac{3\pi}{4}$, and $\frac{5\pi}{6}$.

When sketching an angle with a measure of $k\pi$ radians, students should visualize angles as a fraction of π or 2π. For example, since π is equivalent to half a rotation and $\frac{\pi}{2}$ is $\frac{1}{2}$ of π, then $\frac{\pi}{2}$ is a quarter of a rotation.

Since the circumference of a unit circle is 2π, which is approximately 6.28, students are expected to understand there are approximately 6.28 radians in a circle and 3.14 radians in a half circle. Students should also be able to visualize the approximate size of angles, such as two radians, four radians, or five radians.

For students having trouble visualizing angles in radian measure, it could be suggested to convert to degree measure to check the accuracy of their sketch. The goal, however, is to work with radians without having to convert to degree measure first.

Once the relationship π radians $= 180^\circ$ has been developed, students can convert between degree and radian measure. To convert from radians to degrees, they can solve this equation in terms of radians.

$\pi \text{ radians} = 180^\circ$

$\therefore 1 \text{ radian} = \dfrac{180^\circ}{\pi}$

To rewrite $\dfrac{9\pi}{2}$ radians as degrees, for example, students multiply by one radian, or $\dfrac{180^\circ}{\pi}$.

Similarly, to convert to radians, they should solve the equation in terms of degrees: $180° = \pi$ radians, so $1° = \frac{\pi}{180°}$.

Alternatively students can use a ratio such as $\frac{\theta_{\text{degrees}}}{180°} = \frac{\theta_{\text{radians}}}{\pi}$ to convert between degrees and radians.

Students should become comfortable expressing radian measure in both exact and approximate values.

Students should note that any angle measure given without a degree symbol is assumed to be in radians.

The concept of coterminal angles is new to students. **Coterminal** angles are angles in standard position with the same terminal arms and can be measured in degrees or radians. Examples should include both positive and negative coterminal angles, found by adding or subtracting multiples of 360° or 2π. This leads to developing the general form, expressed as $\theta \pm (360°)k$ or $\theta \pm 2\pi k$ where $k \in$ W. The general solution could also be expressed as $\theta + (360°)k$ or $\theta + 2\pi k$, $k \in$ Z.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Draw the following angles in standard position.
  - (a) 120°
  - (b) 300°
  - (c) 200°
  - (d) 80°
- What is the circumference of a circle with a radius of 6 cm?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Draw a circle and show approximate angle measures of one radian, two radians, three radians, four radians, five radians, and six radians.
- Explain how to determine, with the aid of a diagram, whether 2.8 radians or 180° is a greater measure.
- Explain which is greater, 4.2 radians or 1.4π radians.
- Your classmate has missed the introduction to radian measure. Describe, with the aid of a diagram, how to sketch an angle with a measure of two radians.

- Sketch the following angles in standard position.

  (a) $\frac{\pi}{4}$

  (b) $\frac{2\pi}{3}$

  (c) $\frac{7\pi}{6}$

  (d) $-\frac{3\pi}{2}$

  (e) 3.14

  (f) 5

  (g) –1.57

- Express the measures of the following angles in radian measure
  - in terms of pi
  - as a decimal approximation

  (a) 60°

  (b) 150°

  (c) –225°

  (d) –144°

  (e) 214.5°

- Express the following radian measures in degrees.

  (a) $\frac{2\pi}{3}$

  (b) $-\frac{7\pi}{4}$

  (c) $\frac{11\pi}{12}$

  (d) –5

- Determine a positive and a negative angle measure that is coterminal to the following angle measures.

  (a) $\frac{\pi}{4}$

  (b) $\frac{2\pi}{3}$

  (c) $\frac{7\pi}{6}$

  (d) $-\frac{3\pi}{2}$

- Determine the measure of all angles that are coterminal with

  (a) 310°

  (b) $-\frac{5\pi}{4}$

- Determine the measures of the arc length subtended by the angles and radii below.

  (a) Central angle of $\frac{2\pi}{3}$ with radius 10 cm.

  (b) Central angle of 2.6 radians with radius 4.9 cm.

  (c) Central angle of 240° with radius of 5 cm.

- Determine the measure of the radius of a circle if an arc length of 42 ft. is subtended by an angle of $\frac{7\pi}{4}$.

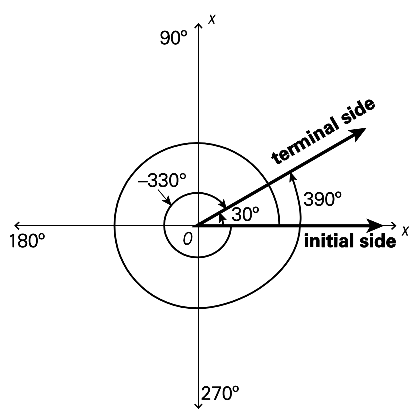
- During a family vacation, you go to dinner at the Niagra Falls Skylon Tower. There is a rotating restaurant at the top of the needle that is circular and has a radius of 54 feet. It makes one rotation per hour. At 6:42 p.m., you take a seat at a window table. You finish dinner at 8:28 p.m.
  (a) Through what angle did your position rotate during your stay?
  (b) How many feet did your position revolve?

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- It will be important to revisit sketching positive angles given in degree measure. A review of terminology such as **initial arm**, **terminal arm**, **vertex**, **sector**, and **standard position** may be required. Students should then be introduced to clockwise versus counter-clockwise as it pertains to angle rotations from standard position.
- When introducing coterminal angles, encourage students to sketch the angles. This highlights the fact that coterminal angles share a terminal arm.



- When an angle measure is given without units, remind students that it is assumed to be in radian measure. Many students may have developed the habit of writing sin 30 when they mean $\sin 30^\circ$. Remind students that this would be similar to writing $\sqrt{8}$ when they mean $\sqrt[3]{8}$ where the absence of an index implies an index of 2.
- Ensure that students always verify that their calculators are in the appropriate mode and that students know how to change between modes. **Note:** Some calculators will also have a GRAD as well as DEG and RAD functions. Students could be asked to look up GRADIENT as a type of angle measurement and determine when it is used.
- When calculating arc length using the formula $A = (\theta_{radian})(r)$, remind students that the angle must be measured in radians. Using subscripts for the angle, as shown above, will often avoid much confusion.
- Ask students to provide an example where it would be more convenient to work with an angle in radian measure rather than degree measure.
- Some students may have difficulty equating counter-clockwise with positive and clockwise with negative. It may be helpful to suggest that a positive rotation opens upward from standard position,

whereas a negative angle opens downward. The *x*-axis could also be considered the “west(−) to east(+)” line and the *y*-axis the “south(−) to north(+)” line. Counter-clockwise rotation from the *x*-axis can then be considered naturally positive.

- To introduce radian measure for angles,
  - ask students to think about the different ways they can measure things (Length can be measured in centimetres or in yards, temperature can be measured in degrees Celsius or Fahrenheit.)
  - ask students to think about the different ways to measure angles (They may refer to degree measure or use the concept of turns.)
  - introduce students to radian measure as an alternative way to express the size of an angle (This will be their first time being exposed to the concept of radian measure.)
  - ask students to visualize a circle where the radius equals one unit

  Teachers can use the following questions to help students connect the concepts of degrees and radians.)
  - What is the circumference of a circle?
  - What is a complete revolution of a circle?
  - Why must the two equations be equal to each other?

Suggested Models and Manipulatives

- chart paper
- masking tape
- measuring tape
- protractors
- rulers
- string

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- arc length
- coterminal angles
- general form
- radian
- unit circle

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 4.1 Angles and Angle Measure
    - > Student Book: pp. 166–179
    - > Teacher Resource: pp. 92–96

SCO T02 Students will be expected to develop and apply the equation of the unit circle. [CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

T02.01 Derive the equation of the unit circle from the Pythagorean theorem.
T02.02 Describe the six trigonometric ratios, using a point P (x, y) that is the intersection of the terminal arm of an angle and the unit circle.
T02.03 Generalize the equation of a circle with centre (0, 0) and radius r.

Scope and Sequence

Pre-calculus 11	Pre-calculus 12
T01 Students will be expected to demonstrate an understanding of angles in standard position (0° to 360°). **T02** Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. **RF11** Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).	**T02** Students will be expected to develop and apply the equation of the unit circle.

Background

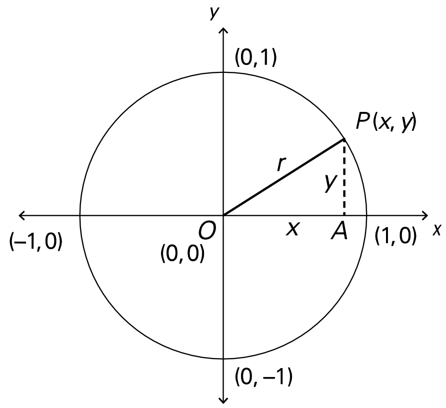
In Pre-calculus 11, students determined the exact values of the primary trigonometric ratios for angles between 0° and 360° using the relationship between the sides of special right triangles (T02). Students may have been introduced to the unit circle as a strategy for finding exact values; however, it was not a direct outcome.

In Pre-calculus 11, students also worked with special triangles and the exact values for the three primary trigonometric ratios for 30°, 45°, and 60°.

Now students will be formally introduced to the unit circle and use its equation to generalize the equation of any circle centred at the origin.

Unit circle: A circle of radius 1 unit with its centre at the origin on the Cartesian plane is known as *the* unit circle.

Students can find the equation of the unit circle using the Pythagorean theorem. Using a circle of radius 1 unit centred at the origin, students should mark a point *P* on the circle and draw a right triangle. Ask them to consider why the absolute value of the *y*-coordinate represents the distance from a point to the *x*-axis.

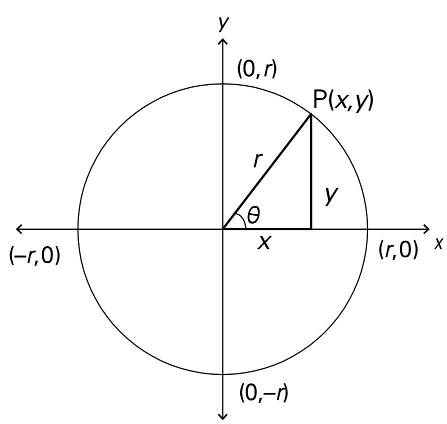


$x^2 + y^2 = r^2$

since $r = 1$

$x^2 + y^2 = 1$

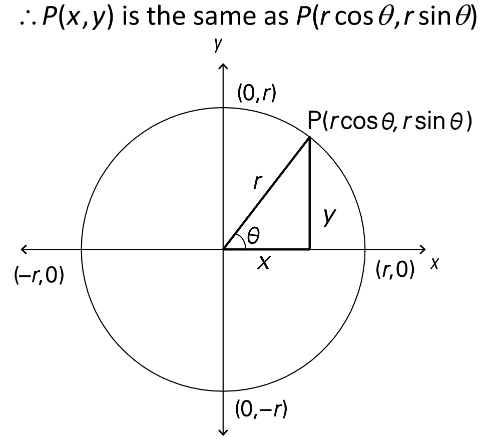
Ask students how the equation would differ if the radius was *r* instead of 1. From this, they should generalize the equation of a circle with centre (0, 0) and radius *r* to be $x^2 + y^2 = r^2$.



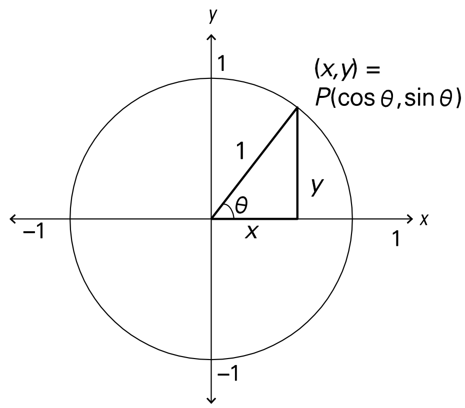
Since $\cos\theta = \dfrac{x}{r}$, we know that $r\cos\theta = x$.

Since $\sin\theta = \dfrac{y}{r}$, we know that $r\sin\theta = y$.

$\therefore P(x, y)$ is the same as $P(r\cos\theta, r\sin\theta)$.



Given an angle θ in standard position, expressed in degrees or radians, students should determine the coordinates of the corresponding point on the unit circle. Conversely, they should determine an angle in standard position that corresponds to a given point on the unit circle.



$r\sin\theta = y$ when $r = 1$; therefore, $\sin\theta = y$.
$r\cos\theta = x$ when $r = 1$; therefore, $\cos\theta = x$.

Therefore, the point of the circumference, $P(x, y)$, can be written as $P(r\cos\theta, r\sin\theta)$ or when the radius is one unit, $P(\cos\theta, \sin\theta)$.

In Mathematics 10 and 11, students worked with the three primary trigonometric ratios. This is their first exposure to the reciprocal ratios: cosecantθ ($\csc\theta$), secantθ $(\sec\ \theta)$, and cotangentθ $(\cot\ \theta)$.

Revisiting the unit circle above, students should observe that, in addition to $x = \cos\theta$ and $y = \sin\theta$, $\tan\theta = \frac{y}{x}$. Therefore, $\tan\theta$ represents the slope $\left(\frac{rise}{run} \text{ or } \frac{\Delta y}{\Delta x}\right)$ of the terminal arm of the angle θ.

Once students have been introduced to the reciprocal ratios:
$\sec\theta = \frac{1}{x}$ or $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{y}$ or $\csc\theta = \frac{1}{\sin\theta}$, and $\cot\theta = \frac{x}{y}$ or $\cot\theta = \frac{1}{\tan\theta}$,
it follows as a natural extension, based on work with reciprocals in Pre-calculus 11, to discuss the following relationships:

$(\csc\theta)(\sin\theta) = 1$
$(\sec\theta)(\cos\theta) = 1$
$(\cot\theta)(\tan\theta) = 1$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$\cot\theta = \frac{\cos\theta}{\sin\theta}$

Identity proofs will be addressed in detail in outcome T06.

Students are expected to be aware of the restrictions on θ for $\tan\theta$, $\csc\theta$, $\sec\theta$, and $\cot\theta$ as values of θ that create division by zero.

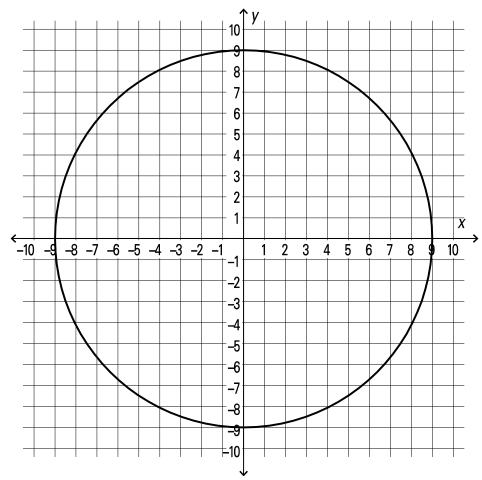
Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- For the circle shown to the right,
  - (a) What is the radius of this circle?
  - (b) What is the centre of this circle?
  - (c) Label the points where the circle intersects the x- and y-axes.
  - (d) The point $(6, 3\sqrt{5})$ is on the circumference of this circle, are there any other points on the circumference that have an x-coordinate of 6?
  - (e) Determine the y-coordinate of the points $(3, y)$ on the circumference of the circle.
  - (f) Determine the x-coordinate of the points $(x, 4\sqrt{2})$.



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the equation of a circle whose centre is at (0, 0) with a radius of
  - (a) 4
  - (b) $\sqrt{34}$
  - (c) $2\sqrt{5}$
- Determine whether the given point is on the circle whose equation is given.
  - (a) $P(-2,6)$, $x^2 + y^2 = 49$
  - (b) $P\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, on the unit circle
- Determine the x- or y-coordinate on the unit circle given the other coordinate.
  - (a) $P\left(\frac{1}{5}, y\right)$, Quadrant I
  - (b) $P\left(x, -\frac{3}{8}\right)$, Quadrant III

- Identify a measure for the central angle θ on the unit circle in the interval $0 \le \theta \le 2\pi$ such that $P(\theta)$ is the given point.

  (a) $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

  (b) $P(0, -1)$

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- When developing the unit circle, ensure that you refer to special triangles that have been developed in Pre-calculus 11.
- The unit circle is an interesting concept that ties together several important mathematical ideas, such as Euclidean geometry, coordinate geometry, and trigonometry. It is important to ensure that students understand the connection between the unit circle and the fundamental trigonometric ratios. For this reason, take time to develop the idea of the unit circle rather than just providing students with a copy of the unit circle.
- Have students create a unit circle using an activity, such as Serving Unit-Circle Trigonometry on a Paper Plate (Lyons and Vawdrey 2008) found at www.lcsd.logan.k12.ut.us/curriculum/mathematics/Paper_plate_unit_circle.pdf.

Suggested Models and Manipulatives

- can or other cylinder
- compass
- construction paper or card stock
- markers
- paper plates
- straight edge
- tape

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- reciprocal trigonometric functions
- unit circle

Resources/Notes

Digital

- *Serving Unit Circle Trigonometry on a Paper Plate* (presented at the 2008 NCTM National Conference) (Lyons and Vawdrey 2008): www.lcsd.logan.k12.ut.us/curriculum/mathematics/Paper_plate_unit_circle.pdf

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 4.2 The Unit Circle
    - Student Book: pp. 180–190
    - Teacher Resource: pp. 97–101

SCO T03 Students will be expected to solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

T03.01 Determine, with technology, the approximate value of a trigonometric ratio for any angle with a measure expressed in either degrees or radians.

T03.02 Determine, using a unit circle or reference triangle, the exact value of a trigonometric ratio for angles expressed in degrees that are multiples of 0°, 30°, 45°, 60°, or 90°, or for angles expressed in radians that are multiples of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, or $\frac{\pi}{2}$ and explain the strategy.

T03.03 Determine, with or without technology, the measures, in degrees or radians, of the angles in a specified domain, given the value of a trigonometric ratio.

T03.04 Explain how to determine the exact values of the six trigonometric ratios, given the coordinates of a point on the terminal arm of an angle in standard position.

T03.05 Determine the measures of the angles in a specified domain in degrees or radians, given a point on the terminal arm of an angle in standard position.

T03.06 Determine the exact values of the other trigonometric ratios, given the value of one trigonometric ratio in a specified domain.

T03.07 Sketch a diagram to represent a problem that involves trigonometric ratios.

T03.08 Solve a problem, using trigonometric ratios.

Scope and Sequence

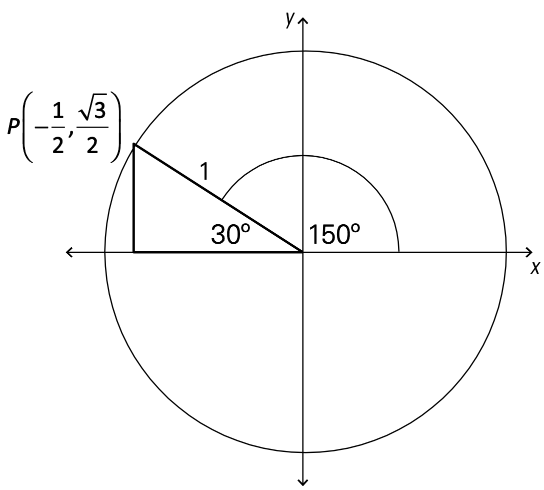
Pre-calculus 11	**Pre-calculus 12**
AN02 Students will be expected to solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. **AN04** Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). **T01** Students will be expected to demonstrate an understanding of angles in standard position (0° to 360°). **T02** Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.	**T03** Students will be expected to solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

Background

In Pre-calculus 11, students solved problems using the three primary trigonometric ratios (T02). Angles were expressed in degrees. This will now be extended to include the reciprocal ratios. Students will work with angles expressed in both degrees and radians.

Students should understand how to properly use a scientific or graphing calculator to evaluate all six of the trigonometric ratios. They should be able to efficiently use their calculators in both degree and radian mode, being careful to check for the appropriate mode in all calculations.

Students have had previous experience determining reference angles for positive angles. This can also be applied to negative angles. Students can also use their previous knowledge of reference triangles and/or the unit circle to determine the exact value of the six trigonometric ratios. From the sketch below, for example, they can determine the value of the trigonometric ratios for 150°.



$$\tan 150^\circ = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

$$\cot 150^\circ = \frac{x}{y} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \left(-\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right)$$

$$= \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}$$

Check:

Since $(\tan\theta)(\cot\theta) = 1$

$$\left(-\sqrt{3}\right)\left(-\frac{\sqrt{3}}{3}\right) = +\frac{\left(\sqrt{3}\right)^2}{3} = \frac{3}{3} = 1$$

Students should also simplify expressions, such as $\dfrac{\cos\left(\frac{\pi}{6}\right) + \sin(-\pi)}{\tan 30^\circ}$.

Expressions requiring rationalizing are limited to those with monomial denominators. From Pre-calculus 11, students are familiar with performing operations on rational expressions (AN05).

Students could use a calculator to verify their answers. The emphasis here, however, is on finding exact values using the unit circle, reference triangles, and mental mathematics strategies.

Students could be asked to determine the value of θ (e.g., when $\sec\theta = -\sqrt{2}$ for the domain $-2\pi \le \theta \le 2\pi$. To determine the reference angle, they could think about a triangle with hypotenuse $\sqrt{2}$ and adjacent side 1.

Alternatively, students could apply the reciprocal ratio $\cos\theta = -\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$.

Once the reference angle of $\frac{\pi}{4}$ is determined, students identify the quadrants where the secant ratio is negative.

Quadrant II

$$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Quadrant III

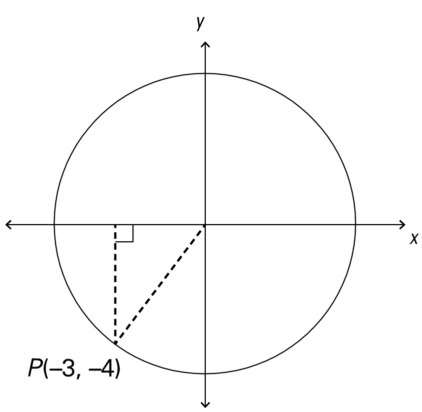
$$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

The final step focuses on identifying all possible values within the given domain.

$$0\left\{-\frac{5\pi}{4},\ -\frac{3\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4}\right\}$$

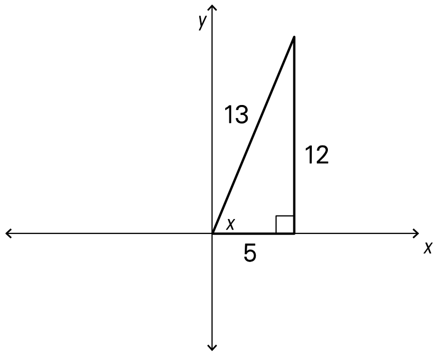
Given the coordinates of a point on the terminal arm of an angle in standard position, students will be expected to determine the six trigonometric ratios.

To determine the ratios, given that the point *P*(−3, −4) lies on the terminal arm of the angle, for example, students should first sketch a diagram.



Students should expect that $\sin\theta$, $\cos\theta$, and their reciprocals, $\csc\theta$ and $\sec\theta$, will be negative in Quadrant III since both the coordinates of the point are negative. They should expect that $\tan\theta$ and its reciprocal $\cot\theta$ will be positive in Quadrant III since the slope of the terminal arm is positive in this quadrant.

Students have had previous experience finding the other two primary trigonometric ratios when they have been given one trigonometric ratio. This is now extended to include the reciprocal trigonometric ratios. Given $\cos\theta = \frac{5}{13}$, for example, students could be asked to determine the value of $\csc\theta$ for $0° \le \theta \le 90°$. They should be encouraged to draw a sketch similar to the following to help visualize a solution.



Students should be exposed to cases with a variety of domains, including negative values.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- State the exact values for each of the following:
  - (a) $\sin 30°$
  - (b) $\sin 45°$
  - (c) $\sin 60°$
  - (d) $\cos 30°$
  - (e) $\cos 45°$
  - (f) $\cos 60°$
- State two angles for which $\sin\theta = \frac{1}{2}$.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Find the exact value of the following expressions:
  - (a) $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right)$
  - (b) $\dfrac{\tan\left(\frac{2\pi}{3}\right)}{\sin\left(-\frac{3\pi}{4}\right)}$
  - (c) $\sin\left(-\frac{2\pi}{3}\right)+\cos^2\left(\frac{11\pi}{6}\right)$
  - (d) $\csc\left(\frac{\pi}{3}\right)+\cot\left(\frac{11\pi}{4}\right)$
  - (e) $\cot^2\left(-\frac{7\pi}{6}\right)$
  - (f) $\dfrac{\cot(-60°)+\cos(300°)}{\csc(-240°)}$

- Solve the following trigonometric equations.

  (a) $\sin\theta = -\frac{\sqrt{3}}{2};\ 0 \le \theta \le 2\pi$

  (b) $\csc\theta = -2;\ 0° \le \theta \le 360°$

  (c) $\sec\theta = \frac{2\sqrt{3}}{3};\ -2\pi \le \theta \le 2\pi$

  (d) $\cot\theta = -\frac{\sqrt{3}}{3};\ -\pi \le \theta \le 2\pi$

- Given that each of the following points lie at the intersection of the unit circle and the terminal arm of an angle in standard position,
  - sketch the diagram
  - determine the values of the six trigonometric ratios
  - determine the angle of rotation from standard position

  (a) $P\left(\frac{5}{13}, -\frac{12}{13}\right)$

  (b) $P\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

  (c) $P(-5, 4)$
- Give an example of a trigonometric equation that does not have a solution. Explain why.
- Summarize what you have learned about the location of positive and negative trigonometric ratios in the four quadrants.
- Given $\sin\theta = -\frac{1}{2}$, $180° \le \theta \le 270°$, determine the value of $\cot\theta$.
- Given sec x = 5.5, where $\frac{3\pi}{2} \le x \le 2\pi$, determine $\tan x$.
- A clock is drawn on the unit circle. What are the exact coordinates of each of the hours? Explain your reasoning.

Planning for Instruction

SUGGESTED LEARNING TASKS

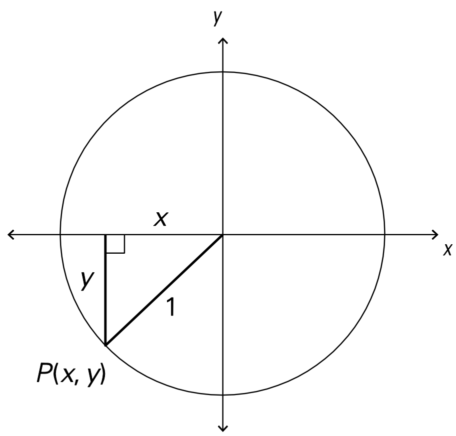
Consider the following sample instructional strategies when planning lessons.

- Review with students how to determine a reference angle for a given angle, θ.
- Students determined the measures of angles in degrees for the three primary trigonometric ratios. They have also used reference triangles and the unit circle to evaluate the six trigonometric ratios. Determining the measure of angles for all six trigonometric ratios in both radians and degrees should be a natural extension.
- It is recommended that students determine which trigonometric ratio is positive in which quadrant by thinking about the meaning of the ratio rather than by memorizing this information.
- Students should be able to determine, without using a chart, the exact values of all special and related angles. To help students do this, consider the following:

– On the unit circle, the value of $P(x, y)$ can be represented by $P(\cos\theta, \sin\theta)$; therefore, where x is positive, cosine and secant are positive. Similarly where y is positive, sine and cosecant are positive.
– Tangent of an angle is the slope of the line and, therefore, will be positive in those quadrants where the slope is positive and negative where the slope is negative.

Example:

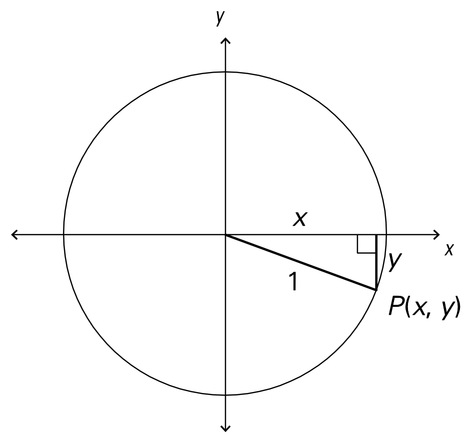
To determine the exact value of $\sin 225°$
A quick sketch



Since the y-coordinate in this point is a negative value, the $\sin 225°$ will be negative. The y-values and x-values are the same at this point. Therefore, the value will be $\frac{\sqrt{2}}{2}$.

$\therefore \sin 225° = -\frac{\sqrt{2}}{2}$.

To determine the exact value of $\sec 330°$
A quick sketch



Since the x-coordinate in this point is a positive value, the $\sec 330°$ will be positive. The x-value is larger than the y-value at this point.

Therefore, the value will be $\frac{\sqrt{3}}{2}$.

$\therefore \sec 330° = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$ or

$\sec 330° = \left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3}$

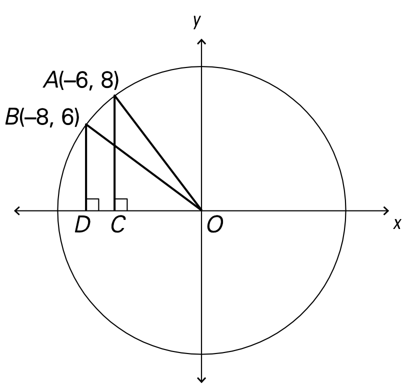
- For the Four Corners activity, each student is given a card containing a different trigonometric ratio. The four corners of the room correspond to the four quadrants. Students decide which quadrant the angle on the card lies in, and they move to the appropriate corner. The small group should then verify that the angles are appropriately placed in the quadrant, and evaluate each of the trigonometric ratios.

  Cards should include a mixture of angles in degree and radian measure, positive and negative angles, and those that have exact and approximate values.

Sample Ratios

$\cos\left(\frac{4\pi}{3}\right)$	$\tan\left(-210^\circ\right)$	$\csc\left(\frac{11\pi}{4}\right)$	$\sin\left(-310^\circ\right)$
$\cot\left(\frac{5\pi}{3}\right)$	$\sec\left(315^\circ\right)$	$\sin\left(3^\circ\right)$	$\sin\left(3.6\right)$

- In groups, students can play the Carousel game. Stations are set up around the room, with simplified expressions, containing errors, posted on the wall. Students identify the error(s) and write the correct simplification.
- Students can play a match game. One set of cards contains trigonometric expressions, and the second set contains the completed solutions. Teachers can vary the set-up, whereby they can have the cards face up or face down.
- Remind students that $\sin^{-1} x \neq (\sin x)^{-1}$. Specifically $\sin^{-1} x$ represents the inverse of $\sin x$, sometimes referred to as arcsin *x* to avoid this confusion. In contrast, $(\sin x)^{-1}$ represents the reciprocal of $\sin x$ or $\frac{1}{\sin x}$. This is particularly confusing for students since $\sin^2(x) = (\sin x)^2$.
- As an extension students could be asked to find the arc length between points *A*(−6, 8) and *B*(−8,6).



Method A

The radius of this circle is 10 units since
$(-6)^2 + (8)^2 = r^2$

Using *A*(–6, 8) $\tan\angle AOC = \frac{8}{6}$

$\therefore \angle AOC = 53^\circ$

Using *B*(–8, 6) $\tan\angle BOD = \frac{6}{8}$

$\therefore \angle BOD = 37^\circ$

This results in $\angle AOB = 16^\circ$, and thus a curved arc length between points *A* and *B* is
$\left(\frac{16^\circ}{360^\circ}\right)(2\pi)(10) = 2.8$ units.

Method B

The radius of this circle is 10 units since
$(-6)^2 + (8)^2 = r^2$

Using *A*(–6, 8) $\tan\angle AOC = \frac{8}{6}$

$\therefore \angle AOC = 0.927$

Using *B*(–8, 6) $\tan\angle BOD = \frac{6}{8}$

$\therefore \angle BOD = 0.6435$

$\angle AOB = 0.9273 - 0.6435 = 0.2838$

arc length = (θ_R)(radius)

arc length = (0.2838)(10) = 2.838 units

SUGGESTED MODELS AND MANIPULATIVES

- compass
- grid paper
- straight edge
- unit circle

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- cosecant ratio
- cotangent ratio
- secant ratio

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 4.3 Trigonometric Ratios
    - > Student Book: pp. 191–205
    - > Teacher Resource: pp. 102–107

Notes

Simplifying trigonometric expressions involving rationalizing requires supplementary practice questions.

SCO T04 Students will be expected to graph and analyze the trigonometric functions sine, cosine, and tangent to solve problems. [CN, PS, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

T04.01 Sketch, with or without technology, the graph of $y = \sin x$, $y = \cos x$, or $y = \tan x$.
T04.02 Determine the characteristics (amplitude, asymptotes, domain, period, range, and zeros) of the graph of $y = \sin x$, $y = \cos x$, or $y = \tan x$.
T04.03 Determine how varying the value of a affects the graphs of $y = a \sin x$ and $y = a \cos x$.
T04.04 Determine how varying the value of d affects the graphs of $y = \sin x + d$ and $y = \cos x + d$.
T04.05 Determine how varying the value of c affects the graphs of $y = \sin (x + c)$ and $y = \cos (x + c)$.
T04.06 Determine how varying the value of b affects the graphs of $y = \sin bx$ and $y = \cos bx$.
T04.07 Sketch, without technology, graphs of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$, using transformations, and explain the strategies.
T04.08 Determine the characteristics (amplitude, asymptotes, domain, period, phase shift, range and zeros) of the graph of a trigonometric function of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$.
T04.09 Determine the values of a, b, c, and d for functions of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$ that correspond to a given graph, and write the equation of the function.
T04.10 Determine a trigonometric function that models a situation to solve a problem.
T04.11 Explain how the characteristics of the graph of a trigonometric function relate to the conditions in a problem situation.
T04.12 Solve a problem by analyzing the graph of a trigonometric function.

Scope and Sequence

Pre-calculus 11	**Pre-calculus 12**
AN04 Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). **T02** Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. **RF11** Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).	**T04** Students will be expected to graph and analyze the trigonometric functions sine, cosine, and tangent to solve problems.

Background

In Mathematics 10 students first encountered and applied the three primary trigonometric ratios (M04). In Pre-calculus 11, students developed an understanding of angles as rotations, and solved simple trigonometric equations in degrees only. They also graphed reciprocal functions (limited to the reciprocal of linear and quadratic functions) and developed an understanding of vertical asymptotes. Earlier in this course, students were exposed to work in radian measure, the unit circle, and solved more complex trigonometric equations (T01, T02, T03, and T05).

In this unit, students develop the graphs for $y = \sin(x)$, $y = \cos(x)$, and later $y = \tan(x)$ and identify the characteristic features of each. Transformations of $y = \sin(x)$ and $y = \cos(x)$ are performed and are used to model and solve problems. Students are not expected to transform $y = \tan(x)$ or solve problems that involve the tangent function.

In Pre-calculus 11 students described how the three primary trigonometric functions changed as the angle changed from 0° to360°. Students now apply this understanding to plot points from the unit circle and then verify the graphs for $f(\theta) = \sin\theta$, $g(\theta) = \cos\theta$, and $h(\theta) = \tan\theta$. These results can be effectively verified using graphing technology.

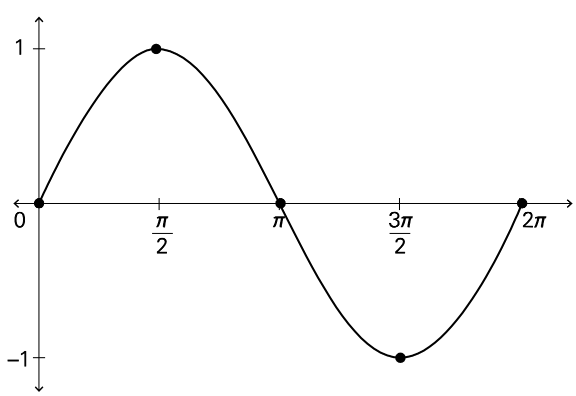
Students are expected to develop a proficiency in using patterning to continue the skeleton and to sketch the graph. Their graphs should show the periodic and continuous nature of the graphs of sinusoidal functions. This requires that students show more than one period of a sinusoidal graph. Students should identify local maximums and minimums, and the equation of the sinusoidal axis, also referred to as the horizontal central axis, and the period.

Students have developed an understanding of transformations for graphs of the form $y = a\,f\,[b(x - h)] + k$ earlier in this course (R02, R03, R04, R05), and will now apply that knowledge to sinusoidal graphs, $y = a\sin[b(x - c)] + d$ or $y = a\cos[b(x - c)] + d$, noting that parameters h and k are now referred to as c and d.

Students should determine the position of the five key points using transformations, and then sketch the graph appropriately. A mapping rule can assist them in describing the determining these key points.

Example:

Five key points marked
y* = sin *x



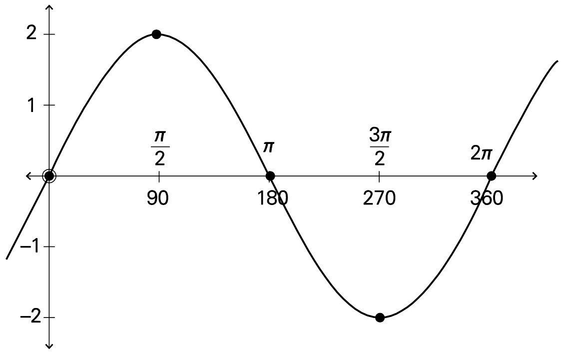
Five key Points on
y* = sin *x

x	*y*
0	0
90° or $\frac{\pi}{2}$	1
180° or π	0
270° or $\frac{3\pi}{2}$	–1
360° or 2π	0

y* = 2sin *x
(*x*, *y*) → (*x*, 2*y*)

x	*y*
0	0(2) = 0
90° or $\frac{\pi}{2}$	1(2) = 2
180° or π	0(2) = 0
270° or $\frac{3\pi}{2}$	–1(2) = –2
360° or 2π	0(2) = 0

Graphing $y = 2\sin x$



Students are expected to know what characteristics of the graph change with each parameter. To achieve this, they should investigate how each parameter change affects the resulting graph, one parameter at a time, and match these changes to the defining characteristics of sinusoidal graphs. Changing the value of *a*, for example, affects the amplitude of the graph, and negative values result in a reflection in the *x*-axis. These explorations can be effectively carried out using graphing technology and could be accomplished through a guided exploration of the parameters.

Ensure that students work with non-integer values for these parameters since many real-world applications do not use integer values.

Students should be able to determine the characteristics of a trigonometric function using the value of the parameters in the equation without necessarily having to graph the equation. Linking students' knowledge of transformations to their understanding of the language of sinusoidal functions should be emphasized here rather than memorizing formulas. Rather than only memorizing period as $\frac{2\pi}{b}$, for example, the concept should be developed from an understanding that *b* affects the horizontal stretch, and the period of the base graph for sine is 2π.

Students should know from their work with functions (RF02) that the *c* parameter relates to horizontal translation. With sinusoidal functions this is referred to as the phase shift, and will determine how the five key points will be translated horizontally.

Students should be encouraged to use appropriate mathematical language. They are expected to understand the terms **phase shift** and **vertical displacement**, recognizing that they refer to horizontal and vertical translations, respectively.

Students should also analyze sinusoidal functions that may require factoring. The function $y = \sin(4x - 6\pi)$, for example, should be factored to $y = \sin\left[4\left(x - \frac{3}{2}\pi\right)\right]$ in order to correctly identify the phase shift as $\frac{3}{2}\pi$.

To determine the equation of a sinusoidal function, students will have to determine the key characteristics of the graph, and then link them to the parameters in the equation $y = a\sin\left[b\left(x - c\right)\right] + d$ or $y = a\cos\left[b\left(x - c\right)\right] + d$.

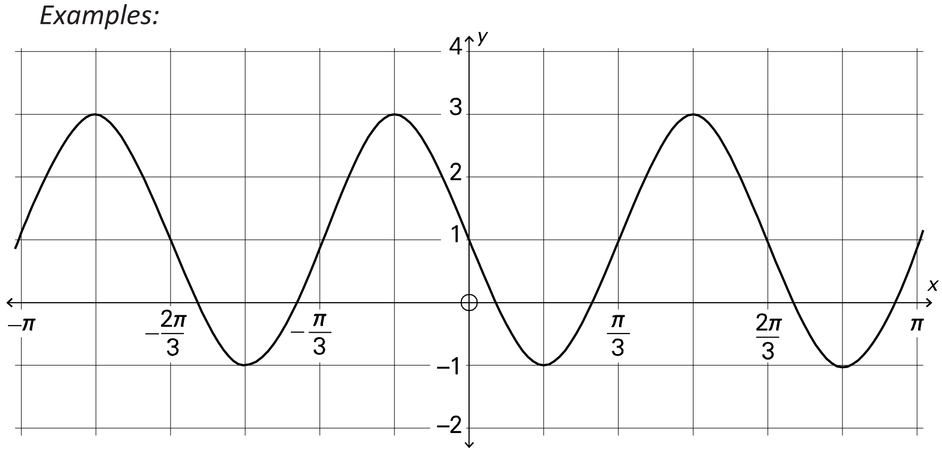
Parameter	Equation	Transformation	Characteristic	Mapping Rule
a = 3	$y = 3\sin x$	vertical stretch of 3	amplitude of 3	$(x, y) \to (x, 3y)$
b = 3	$y = \sin 3x$	horizontal stretch of $\frac{1}{3}$	wave is three times as frequent; therefore, period of $\frac{2\pi}{3}$	$(x, y) \to (\frac{1}{3}x, y)$
c = 3	$y = \sin(x - 3)$	horizontal translation of 3	phase shift of 3	$(x, y) \to (x + 3, y)$
d = 3	$y = \sin(x) + 3$	vertical translation of 3	sinusoidal axis *y* = 3	$(x, y) \to (x, y + 3)$
a < 0	$y = -\sin x$	vertical reflection		$(x, y) \to (x, -y)$

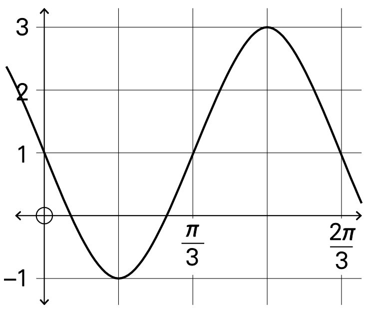
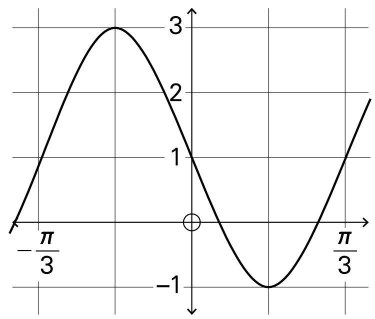
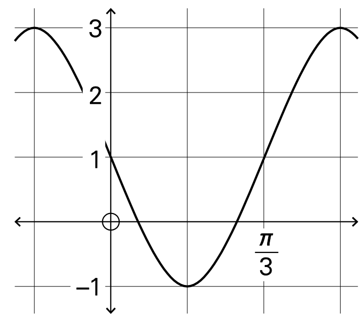
If students determine that the amplitude for the graph is 4, for example, they should understand that *a* = 4.

While there are an infinite number of correct choices for the phase shift, the convention is to use the smallest positive value. Teachers need to be careful to accept all correct answers, not just the conventional answer, when assessing student work. Students should also be made aware that more than one correct equation is possible, and they should be able to identify equations that produce the same graph.

Students should be cautioned when using negative values for *a* and *b*, since these would involve reflections. Consequently, when determining the value of the phase shift, different points on the graph need to be considered.

Examples:



<table>
<tr>
<td>Considering one period of the graph beginning at $x=0$ would give a sine function with a vertical reflection.</td>
<td>Considering one period of the graph beginning at $x=-\frac{\pi}{3}$ would give a sine function without a reflection.</td>
<td>Considering one period of the graph beginning at $x=-\frac{\pi}{6}$ would give a cosine function without a reflection.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Points considered:
$(0,1)$, $\left(\frac{\pi}{6},-1\right)$, $\left(\frac{\pi}{3},1\right)$, $\left(\frac{\pi}{2},3\right)$, $\left(\frac{2\pi}{3},1\right)$</td>
<td>Points considered:
$\left(-\frac{\pi}{3},1\right)$, $\left(-\frac{\pi}{6},3\right)$, $(0,1)$, $\left(\frac{\pi}{6},-1\right)$, $\left(\frac{\pi}{3},1\right)$</td>
<td>Points considered:
$\left(-\frac{\pi}{6},3\right)$, $(0,1)$, $\left(\frac{\pi}{6},-1\right)$, $\left(\frac{\pi}{3},1\right)$, $\left(\frac{\pi}{2},3\right)$</td>
</tr>
<tr>
<td>Mapping Rule:
$(x,y)\to\left(\frac{1}{3}x,-2y+1\right)$</td>
<td>Mapping Rule:
$(x,y)\to\left(\frac{1}{3}x-\frac{\pi}{3},2y+1\right)$</td>
<td>Mapping Rule:
$(x,y)\to\left(\frac{1}{3}x-\frac{\pi}{6},2y+1\right)$</td>
</tr>
<tr>
<td>Equation:
$y=-2\sin(3x)+1$</td>
<td>Equation:
$y=2\sin\left[3\left(x+\frac{\pi}{3}\right)\right]+1$</td>
<td>Equation:
$y=2\cos\left[3\left(x+\frac{\pi}{6}\right)\right]+1$</td>
</tr>
<tr>
<td colspan="3">In all the cases, the amplitude, period, and sinusoidal axis is the same; only the phase shift is changed, depending upon what points are considered.</td>
</tr>
</table>

Trigonometric functions are commonly used to model problems that are periodic in nature, including circular motion, pistons, tides, climate, daylight, populations of species, electricity, etc.

Students should be encouraged to draw well-labelled sketches of the graphs that represent the problems. This will help them more easily identify the characteristics of the trigonometric function.

While students should be able to write a sine and a cosine equation to model any application, cosine is more often used since it is easier to identify a correct phase shift. Students are free to choose any correct equation to solve the problem.

When graphing the function $h(\theta)=\tan\theta$ and discussing the behaviour around the asymptotes, it is important to remember that students have no formal experience with limits, limit notation, or infinity.

Note that transformations of $y = \tan(x)$ will not be explored in this course.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

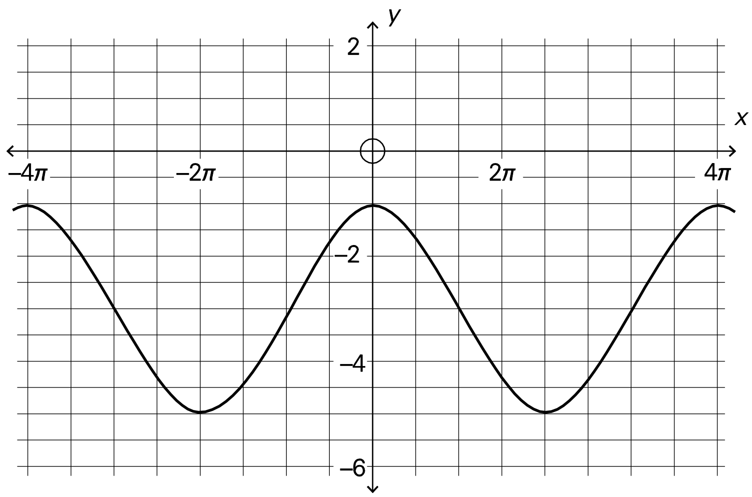
- Describe how $\sin\theta$ changes as θ changes from 0° to 90°.
- Describe how $\cos\theta$ changes as θ changes from 0° to 180°.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Create a Venn diagram to compare the characteristics of the functions $y = \sin(x)$ and $y = \cos(x)$.
- Determine if the point $P\left(-\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$, lies on the graph of $y = \sin(x)$, the graph of $y = \cos(x)$, or both of these graphs.
- Describe how the functions $y = 3\sin\lfloor 2(x - 30°)\rfloor + 6$ and $y = -4\cos\lfloor 2(x - 60°)\rfloor + 6$ are alike and how they differ.
- Use the function $y = 5\cos 3(x - 30°) + 2$ to answer the following:
  - (a) What is the period?
  - (b) What is the amplitude?
  - (c) What is the range?
  - (d) Suppose we wanted to write the equation in the form $y = a \sin b(x - c) + d$. What values could be used for a, b, c, and d?
  - (e) How could the equation be modified so that the resulting function will have no x-intercepts?

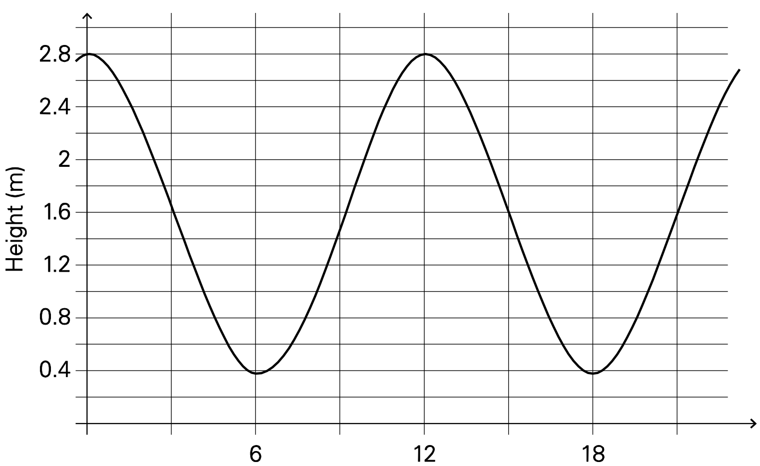
- Is the equation $y = 2\cos(2x) - 3$ correct for the graph shown here? If not, explain how the equation would need to be modified.



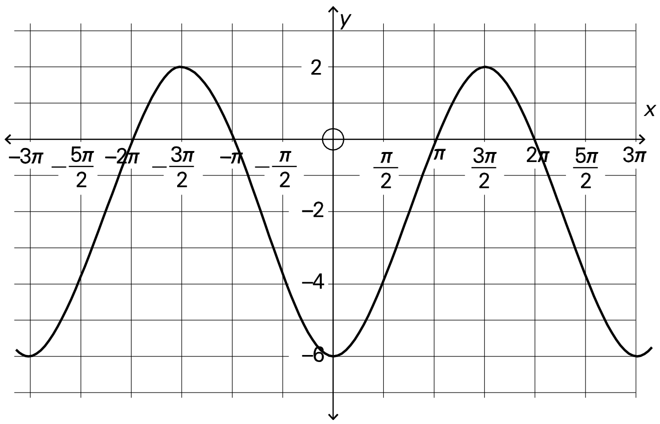
$y = 2\cos(2x) - 3$

- Identify questions that could be answered using the graph, below, that describes the height of tides since noon.

Height of tides since noon



- Determine the values of *a*, *b*, *c*, and *d* required to write an equation for the graph.



- Sketch a graph of *y* = sin(*x*) and *y* = tan(*x*) on the same grid. Explain how the *x*-intercepts of *y* = sin(*x*) relate to the *x*-intercepts of *y* = tan(*x*).
- Sketch the graph of *y* = cos(*x*) and *y* = tan(*x*) on the same grid. Explain how the *x*-intercepts of *y* = cos(*x*) relate to the asymptotes of *y* = tan(*x*).
- Is $x = 3\pi$ a vertical asymptote for the function *y* = tan(*x*)? Explain your reasoning.

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Have students create sinusoidal data sets by rolling and measuring the height of a point on the circumference of cylindrical objects with different diameters, as well as the distance that the cylinder has been rolled along a straight line.

  They could complete several charts, such as the one below, and plot the data they obtain. Recommend before students begin this exercise that
  - they record points that would represent the maximum and minimum heights of the point as well as at least two other points for each full rotation of the cylinder
  - they complete a minimum of two full rotations of the cylinder

Diameter of cylinder: 9 cm									
Distance rolled along straight line (cm)	0	7	14	21	28	35	42	49	56
Height of point on circumference of circle (cm)	0	4.5	9	4.5	0	4.5	9	4.5	0

  Teachers can use this activity to introduce sinusoidal graphs as well as the terms **period**, **sinusoidal axis**, and **amplitude**. It is important that these terms be linked to the specific characteristics of the cylindrical container.

- Create a model of the coordinate grid on the classroom floor. Use a metre stick, which can be rotated physically, to represent the radius of a circle. Ask students to record and graph the distance, in metres, from the tip of the metre stick to the *x*-axis over 360°. These values will produce the sine curve. Ask students to repeat the process measuring the distance, in metres, to the *y*-axis over 360°. These values produce the cosine curve.

- It is helpful to have students focus attention on the five key points associated with the graphs of $y = \sin x$ and $y = \cos x$, since these points help determine the characteristics of the graphs (amplitude, domain, period, range, and zeros) and will also be used to graph transformations of them.

$y = \sin(x)$:

0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
0 sinusoidal axis	1 maximum	0 sinusoidal axis	–1 minimum	0 sinusoidal axis

$y = \cos(x)$:

0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
1 maximum	0 sinusoidal axis	–1 minimum	0 sinusoidal axis	1 maximum

For each base graph, the five key points give a skeleton of the graph on the interval 0 to 2π, (one complete period) with points spaced a quarter period apart.

- When considering the graph of the trigonometric function $y = \tan x$, ask students to analyze the slope of the terminal arm on the unit circle as it rotates from 0° to 90°.
  - When is the slope 0?
  - What happens as the angle increases?
  - What happens to the slope as the angle approaches 90°?
  - What is the slope of the line at 90°?

  Then ask students the same questions concerning rotations from 0° to −90°.

  Have them compare these to slopes when the terminal arm is rotated past 90° from 90° to 180°, from 180° to 270°, and so on. They should see the periodic nature of the curve and the behaviour around the asymptotes.

  A similar discussion could be held to help students understand why the period of *y* = tan(*x*) is π.
- Students should develop the graph of *y* = tan(*x*), paying attention to the characteristics of this graph. As with sine and cosine, five key points (three points and two asymptotes) are conventionally used to graph one complete period of *y* = tan(*x*):

$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
asymptote	–1	0	1	asymptote

As with sine and cosine, ensure students understand that these key points help guide the construction of one period of the function, and they should be able to use these key points to produce reasonably accurate graphs. The graphs should be extended to include more than one period.

Discuss whether the graph of $y = \tan(x)$ has an amplitude. It may be helpful for students to see that "amplitude" is a characteristic of sine and cosine graphs and that it depends on a maximum and a minimum height.

- For a Jigsaw activity, students begin in a home group and each member is a given a number (1 to 4). The groups dissolve, and the students with the same numbers form expert groups. Each expert group explores a different type of transformation for $y = \sin x$ and $y = \cos x$. Once the exploration is complete, students return to their home groups. Each expert teaches the others about the transformation they explored.
- Review with the class the different types of transformations. Make connections between translations, and phase shifts and vertical displacements, and between stretches and amplitudes.
- Review with the class how to find vertical asymptotes and how they affect the shape of a graph.
- Make the connection between the tangent function and the slope of the terminal arm of θ.

Suggested Models and Manipulatives

- coloured pencils
- compass
- grid paper
- protractor
- ruler

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- amplitude
- period
- periodic function
- phase shift
- sinusoidal axis
- sinusoidal curve
- vertical asymptotes
- vertical displacement

Resources/Notes

Digital

- "Graphs from the Unit Circle," *Illuminations: Resources for Teaching Math* (National Council of Teachers of Mathematics 2015): http://illuminations.nctm.org/Lesson.aspx?id=2870

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 5.1 Graphing Sine and Cosine Functions
    - Student Book: pp. 222–237
    - Teacher Resource: pp. 118–123

- 5.2 Transformations of Sinusoidal Functions
  - > Student Book: pp. 238–255
  - > Teacher Resource: pp. 124–129
- 5.3 The Tangent Function
  - > Student Book: pp. 256–265
  - > Teacher Resource: pp. 130–133

Notes

- *Pre-Calculus 12* (McAskill et al. 2012)
  - Student Book: p. 263, #5–#7

  These questions develop the relationship between $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and the slope of the terminal arm, which connects nicely with Chapter 4: Trigonometry and the Unit Circle.

SCO T05 Students will be expected to solve, algebraically and graphically, first- and second-degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

T05.01 Verify, with or without technology, that a given value is a solution to a trigonometric equation.
T05.02 Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form, when possible.
T05.03 Determine, using technology, the approximate solution of a trigonometric equation in a restricted domain.
T05.04 Relate the general solution of a trigonometric equation to the zeros of the corresponding trigonometric function (restricted to sine and cosine functions).
T05.05 Determine, using technology, the general solution of a given trigonometric equation.
T05.06 Identify and correct errors in a solution for a trigonometric equation.

Scope and Sequence

Pre-calculus 11	Pre-calculus 12
T02 Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. **RF05** Students will be expected to solve problems that involve quadratic equations. **RF06** Students will be expected to solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.	**T05** Students will be expected to solve, algebraically and graphically, first- and second-degree trigonometric equations with the domain expressed in degrees and radians.

Background

In Pre-calculus 11, students solved simple trigonometric equations of the form $\sin\theta = a$ or $\cos\theta = a$, where $-1 \le a \le 1$, and $\tan\theta = a$, where a is a real number (T02). They worked with angles in degree measure.

This will now be extended to include trigonometric equations with all six trigonometric ratios. Students will solve first- and second-degree trigonometric equations with the domain expressed in degrees and radians.

When solving first-degree equations, rearrangement will sometimes be necessary to isolate the trigonometric ratio. When solving an equation such as $\sec\theta = 2$, students can consider which reference angle results when the hypotenuse is 2 and the adjacent side is 1. Alternatively, they can think about secant as the reciprocal of cosine and determine the reference angle for the equation $\cos\theta = \frac{1}{2}$.

Knowledge of exact values of the sine, cosine, or tangent of a 30°, 45°, or 60° angle, as well as their corresponding radian measures, and an understanding of the unit circle, continue to be important when solving trigonometric equations.

Students also solve second-degree equations through techniques such as factoring (e.g., $\sin^2\theta - 3\sin\theta + 2 = 0$, for all θ) or isolation and square-root principles (e.g., $\tan^2\theta = 3,\ 0 \le \theta < 2\pi$). Students sometimes mistakenly use only the principal square root when both negative and positive should be considered. In the equation $\tan^2\theta = 3,\ 0 \le \theta < 2\pi,$ isolating the trigonometric ratio results in $\tan\theta = \pm\sqrt{3}$.

Students should understand that using only the principal square root in this equation causes a loss of roots. Another common error occurs when students do not find all solutions for the given domain. In the example above, the reference angle is $\frac{\pi}{3}$, and since there are two cases to consider (tangent being negative and positive), there are solutions in all four quadrants.

Students should be encouraged to check all solutions with a calculator or using the unit circle where appropriate. When solving equations containing trigonometric ratios other than sine or cosine, students should also check that the solutions are defined for the domain of the specific ratios.

Students will also solve trigonometric equations for which the argument may include a horizontal stretch or horizontal translation, such as $\cos(2\theta - \pi) = -\frac{1}{2}$.

Students have had exposure to identifying and correcting errors throughout the mathematics curriculum. This approach is continued in the context of solving trigonometric equations.

Students should be encouraged to check the entire given solution for errors and not stop checking once they have encountered the first error.

The following solution, for example, contains two errors: the first error stems from improperly applying the zero-product principle and the second error occurs when all solutions for the given domain are not found.

$$2\cos^2\theta - \cos\theta - 1 = 0,\ 0 \le \theta < 360^\circ$$
$$(2\cos\theta + 1)(\cos\theta - 1) = 0$$
$$\cos\theta = \frac{1}{2},\ \cos\theta = -1$$
$$\theta = \cos^{-1}\left(\frac{1}{2}\right),\ \theta = \cos^{-1}(-1)$$
$$\therefore \theta = 60^\circ,\ 180^\circ$$

In Mathematics 11, students solved quadratic equations by graphing the related quadratic function and determining its *x*-intercepts (RF2). In Pre-calculus 11 they solved systems of equations (quadratic-quadratic and quadratic-linear) graphically and algebraically (RF06).

Earlier in this course, students solved first- and simple second-degree trigonometric equations algebraically. In this unit, they continue to solve equations algebraically. They also use the graphs of trigonometric functions to solve equations. For assessment purposes, students should analyze given graphs to determine solutions.

Students also solve trigonometric equations for which the argument may include a horizontal stretch or a horizontal translation.

To algebraically determine all solutions in radian measure for $\cos\left[4\left(x-\frac{\pi}{4}\right)\right]=-\frac{\sqrt{3}}{2}$ students could proceed as follows:

$$\cos\left[4\left(x-\frac{\pi}{4}\right)\right]=-\frac{\sqrt{3}}{2}$$

$$\text{let } \theta=\left[4\left(x-\frac{\pi}{4}\right)\right]$$

$$\cos\theta=-\frac{\sqrt{3}}{2}$$

Reference angle:

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$$

Cosine (*x*-coordinate of point on unit circle) is negative in quadrants II and III.

Using this reference angle and the fact that the cosine is negative in the quadrants where *x* is negative, the solutions found in Quadrants II and III can be determined.

Quadrant II : $\pi-\frac{\pi}{6}=\frac{5\pi}{6}$

$$\theta=\frac{5\pi}{6}+2\pi k,\ k\in Z$$

$$\text{since } \theta=\left[4\left(x-\frac{\pi}{4}\right)\right]$$

$$\left[4\left(x-\frac{\pi}{4}\right)\right]=\frac{5\pi}{6}+2\pi k,\ k\in Z$$

$$x-\frac{\pi}{4}=\frac{5\pi}{24}+\frac{\pi}{2}k,\ k\in Z$$

$$x=\left(\frac{5\pi}{24}+\frac{\pi}{4}\right)+\frac{\pi}{2}k,\ k\in Z$$

$$x=\left(\frac{11\pi}{24}\right)+\frac{\pi}{2}k,\ k\in Z$$

Quadrant III: $\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

$$\theta=\frac{7\pi}{6}+2\pi k,\ k\in Z$$

$$\text{since } \theta=\left[4\left(x-\frac{\pi}{4}\right)\right]$$

$$\left[4\left(x-\frac{\pi}{4}\right)\right]=\frac{7\pi}{6}+2\pi k,\ k\in Z$$

$$x-\frac{\pi}{4}=\frac{7\pi}{24}+\frac{\pi}{2}k,\ k\in Z$$

$$x=\left(\frac{7\pi}{24}+\frac{\pi}{4}\right)+\frac{\pi}{2}k,\ k\in Z$$

$$x=\left(\frac{13\pi}{24}\right)+\frac{\pi}{2}k,\ k\in Z$$

Students should be reminded to check solutions by evaluating each side of the original equation at all the solution values.

If the equation has a restricted domain, students should first solve for all solutions and then select the subset of solutions that fall within the restricted domain.

For example, solving $\cos\left[4\left(x-\frac{\pi}{4}\right)\right]=-\frac{\sqrt{3}}{2}$ for the domain $-\pi\le x\le\pi$, there will be multiple solutions since the period of this function is $\frac{\pi}{2}$ and the domain spans 2π; thus, there are four complete periods in the interval of interest. We would expect to have eight solutions generated since there are two solutions in each of the four periods. For example, for the solution $x=\left(\frac{11\pi}{24}\right)+\frac{\pi}{2}k,\ k\in \mathrm{Z}$,

$$k=0: x=\left(\frac{11\pi}{24}\right)$$

$$k=1: x=\left(\frac{11\pi}{24}\right)+\frac{\pi}{2}=\frac{23\pi}{24}$$

$$k=2: x=\left(\frac{11\pi}{24}\right)+\pi=\frac{35\pi}{24};\frac{35\pi}{24}>\pi$$

$$k=-1: x=\left(\frac{11\pi}{24}\right)-\frac{\pi}{2}=-\frac{\pi}{24}$$

$$k=-2: x=\left(\frac{11\pi}{24}\right)-\pi=-\frac{13\pi}{24}$$

$$k=-3: x=\left(\frac{11\pi}{24}\right)-\frac{3\pi}{2}=-\frac{25\pi}{4};-\frac{25\pi}{4}<-\pi$$

Therefore, the solutions generated by

$$x=\left(\frac{11\pi}{24}\right)+\frac{\pi}{2}k,\ k\in \mathrm{Z}$$

for the domain $-\pi\le x\le 2\pi$ are

$$-\frac{13\pi}{24};\ -\frac{\pi}{24};\ \frac{11\pi}{24};\ \frac{23\pi}{24}$$

A trigonometric function may have two, one, or zero *x*-intercepts, each period of the function.

Example:

$f(x)=2\sin(3x)-1$ has two *x*-intercepts per period.

$f(x)=0$ when $\sin(3x)=\frac{1}{2}$

if $\theta=3x$, then $\sin(\theta)=\frac{1}{2}$

$$3x=\theta=\frac{\pi}{6}+2\pi k;\ k\in \mathrm{Z}$$

$$3x=\theta=\frac{5\pi}{6}+2\pi k;\ k\in \mathrm{Z}$$

$$x=\frac{\pi}{18}+\frac{2\pi}{3}k;\ k\in \mathrm{Z}$$

$$x=\frac{5\pi}{18}+\frac{2\pi}{3}k;\ k\in \mathrm{Z}$$

$f(x)=\sin(3x)-1$ has only one *x*-intercept per period.

$f(x)=0$ when $\sin(3x)=1$

if $\theta=3x$, then $\sin(\theta)=1$

$$3x=\theta=\frac{\pi}{2}+2\pi k;\ k\in \mathrm{Z}$$

$$x=\frac{\pi}{6}+\frac{2\pi}{3}k;\ k\in \mathrm{Z}$$

$f(x)=\sin(3x)-2$ has no *x*-intercepts per period.

$f(x)=0$ when $\sin(3x)=2$

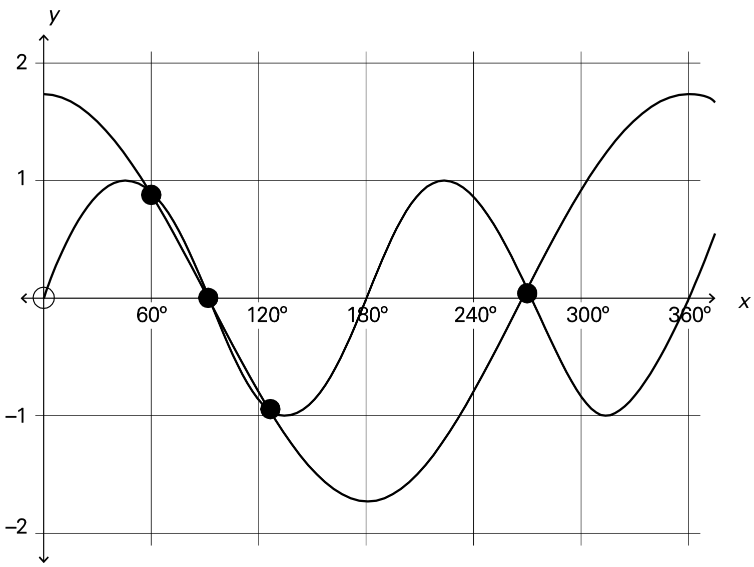
$f(x)=\sin\theta$ has a maximum of +1 and a minimum of –2 and, therefore, will never have a value of 2.

The identities encountered in (T06) can be applied to solve trigonometric equations. It is expected that teachers will have students solving trigonometric equations without the use of identities and then will revisit solving trigonometric equations where the use of identities is necessary after students have worked with the identities.

It is important to continue to emphasize the connection between graphical solutions and algebraic solutions for trigonometric equations. Students could, for example, be asked to find the solutions of $\sin 2x = 3\cos x$ for the domain $0° \le x < 360°$.

Graphically:

The graphs of $y = \sin 2x$ and $y = 3\cos x$ with domain $0° \le x < 360°$ show the intersection at 60°, 90°, 120°, and 270° are as shown below.



Once students have worked with the double angle identities, for example, a double-angle identity is used to solve this same equation algebraically.

$$\sin 2x = \sqrt{3}\cos x$$

$$2\sin x\cos x - \sqrt{3}\cos x = 0$$

$$\cos x\left(2\sin x - \sqrt{3}\right) = 0$$

$$\cos x = 0 \qquad 2\sin x - \sqrt{3} = 0$$

$$x = 90°,\ 270° \qquad \sin x = \frac{\sqrt{3}}{2}$$

$$x = 60°,\ 120°$$

Students are expected to see the relationship between the algebraic and graphical solutions.

Students are also expected to solve trigonometric equations with unrestricted domains resulting in a general solution, in degrees and radian measure.

Using the previous example with an unrestricted domain, the solutions in radians are $x = \frac{\pi}{2} + \pi k,\ k \in Z$; $x = \frac{\pi}{3} + 2\pi k,\ k \in Z$; and $x = \frac{2\pi}{3} + 2\pi k,\ k \in Z$. Here, students see that when the domain has no restriction, there are an infinite number of solutions. Extending the graph so that students can see more intersection points may help consolidate understanding of general solutions.

Students should verify that a particular value is a solution to a given trigonometric equation by simply substituting it into the equation and determining if it satisfies the equation.

Students should solve trigonometric equations for non-special angles as well. This can be done with the use of a scientific calculator where the degree or radian measure is found by finding the inverse trigonometric function of a ratio.

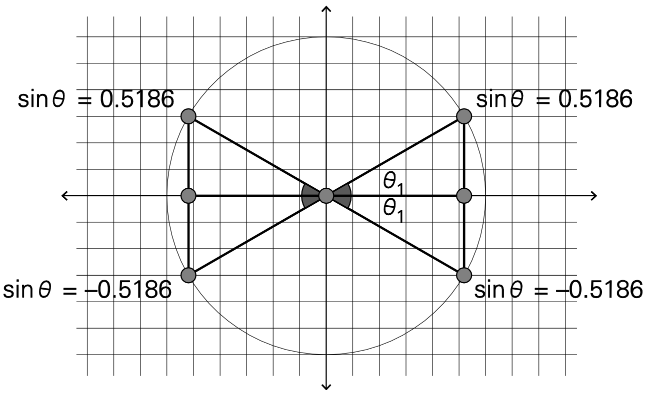
Students could solve $\cos 2\theta + \sin^2\theta = 0.7311$, for example, for the domain 0° ≤ θ < 360°.

$$\cos 2\theta + \sin^2\theta = 0.7311$$
$$1 - 2\sin^2\theta + \sin^2\theta = 0.7311$$
$$1 - \sin^2\theta = 0.7311$$
$$0.2689 = \sin^2\theta$$
$$\pm 0.5186 = \sin\theta$$

$$+0.5186 = \sin\theta$$
$$\theta = 31.24°$$

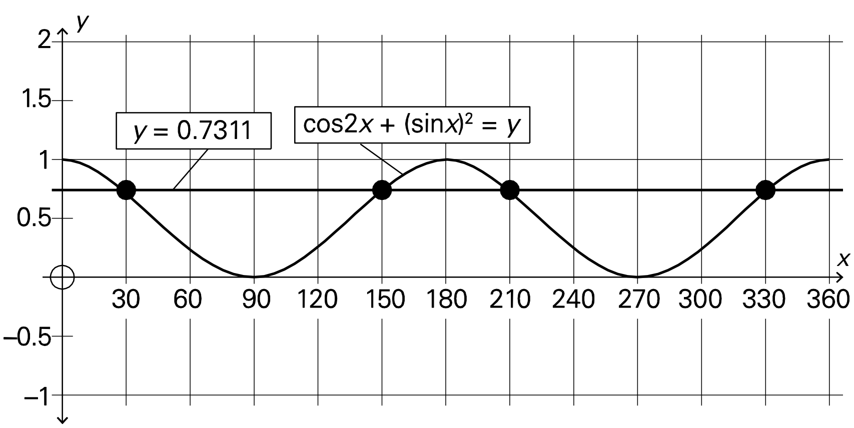
$$-0.5186 = \sin\theta$$
$$\theta = -31.24°$$

Graphing these angles, the other solutions become evident.



For $\theta_1 = 31.24°$, we know $\theta_2 = 180° - 31.24° = 148.76°$.
For $\theta_1 = -31.24°$ or $328.76°$, we know $\theta_2 = 180° + 31.24° = 211.24°$.

The solutions for the equation $\cos 2\theta + \sin^2\theta = 0.7311$ for the domain $0° \le \theta < 360°$ are $\theta = \{31.24°,\ 148.76°,\ 211.24°,\ 328.76°\}$.



Considering the intersection of the two graphs, $y_1 = \cos(2x) + \sin^2 x$ and $y_2 = 0.7311$, allows students to check the reasonableness of the answers obtained algebraically.

Students are also required to provide the general solution of a trigonometric equation. Remind them that the general solution of the equation above includes all angles that are coterminal with the solutions already found.

The general solution for $\cos 2\theta + \sin^2\theta = 0.7311$ would be as follows:

For $+0.5186 = \sin\theta$, the calculator gives $\theta_1 = 31.24°$.

$$\begin{cases} \theta_1 = 31.24° + 360°k;\ k \in Z \\ \theta_2 = (180° - 31.24°) + 360°k;\ k \in Z \end{cases}$$

For $-0.5186 = \sin\theta$, the calculator gives $\theta_1 = -31.24°$.

$$\begin{cases} \theta_1 = -31.24° + 360°k;\ k \in Z \\ \theta_2 = (180° + 31.24°) + 360°k;\ k \in Z \end{cases}$$

Identifying errors and providing the correct solution is a good technique for developing analytical skills. Students could be given a particular example with an error and asked to identify the mistake.

$$\sin^2 x - \sin x = 0$$
$$\sin^2 x = \sin x$$
$$\frac{\sin^2 x}{\sin x} = \frac{\sin x}{\sin x}$$
$$\sin x = 1$$
$$x = 90°$$

In this case, a solution has been lost as a result of dividing both sides of the equation by sin *x*.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

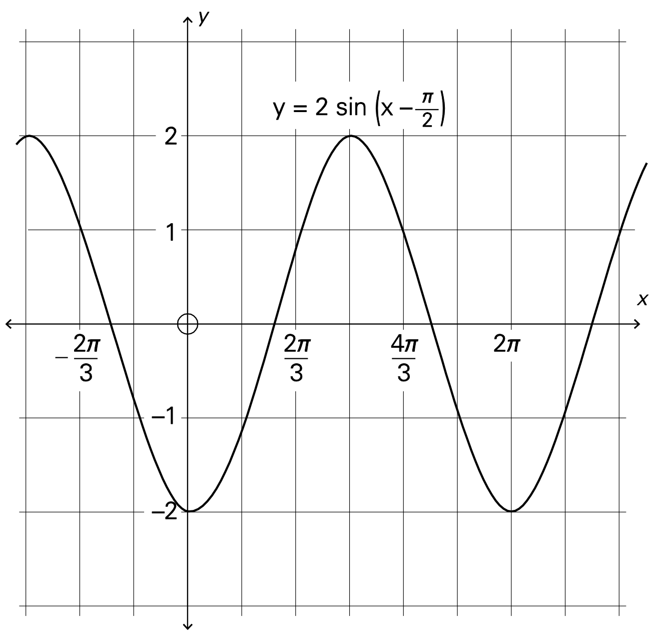
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve the following for $0° \le \theta \le 90°$.
  - (a) $\sin\theta = 0.5$
  - (b) $\cos\theta = 0.8$
  - (c) $\tan\theta = 1$
- Factor and solve each of the following equations.
  - (a) $2x^2 + x - 1 = 0$
  - (b) $4x^2 - 1 = 0$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Solve the following trigonometric equations.
  - (a) $\sqrt{2}\cos x - 1 = 0;\ [-2\pi, 2\pi]$
  - (b) $4\cot\theta + 3 = -2\cot\theta - 8;\ (0, 360°)$
  - (c) $2\csc^2\theta - 8 = 0$; for all θ in radians
  - (d) $2\sec^2 x = 1 - \sec x$; for all x in radians
  - (e) $2\sin^2 x + 5\sin x - 3 = 0;\ (-\pi, 2\pi]$
- Given $g(\theta) = \cos^2\theta - 3$ and $p(\theta) = 2\cos\theta$, ask students to determine the values of θ, such that $g(\theta) = p(\theta)$, where $[0, 4\pi]$.
- Use the graph shown to determine the general solution for the equation $2\sin\left(x - \frac{\pi}{2}\right) - 1 = 0$.



- Determine the number of solutions for each of the following:
  - (a) $2\sin\left[3\left(x - \frac{\pi}{2}\right)\right] = 1;\ [-\pi, 2\pi]$
  - (b) $4\cos\left[\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right] = 4;\ [-2\pi, 4\pi]$
  - (c) $\sin\left[2\left(x - \frac{\pi}{2}\right)\right] + 1 = 1;\ [-3\pi, \pi]$
  - (d) $4\cos\left[\frac{1}{2}\left(x - \frac{\pi}{3}\right)\right] = 8;\ [-\pi, 3\pi]$
- Find the zeros of the following functions:
  - (a) $g(x) = 2\sin x - 1$
  - (b) $f(x) = \cos(3x) - 1$
  - (c) $h(x) = 2\sin(x) + 4$

- Solve the following equations.
  - (a) $2\sin(3x)+1=2;\ -\pi\le x\le\pi$
  - (b) $6\sin(x-\pi)-1=2;\ -2\pi\le x\le 2\pi$
  - (c) $4\cos\left(3x+\frac{\pi}{2}\right)+2=2$; all solutions
  - (d) $4\sin^2(3x)-1=2;\ 0\le x\le 3\pi$
  - (e) $6\cos\left[2\left(x-\frac{\pi}{4}\right)\right]-1=2$; all solutions
  - (f) $2\cos^2(4x)-3\cos(4x)+3=2;\ 0\le x\le\pi$
- Solve each of the following equations for $0^\circ\le x\le 360^\circ$, giving exact solutions, where possible.
  - (a) $\sin 2x=\sqrt{2}\cos x$
  - (b) $\tan 2x=\dfrac{2}{\sec^2 x}$
- Write the general solution for the following equations in degrees and radian measure.
  - (a) $\sin 2x=\sqrt{2}\cos x$
  - (b) $\tan 2x=\dfrac{2}{\sec^2 x}$
- Solve $\cos 2x=\sin x$ graphically, and verify the answers numerically.
- A student's solution for $\tan^2 x=\sec x\tan^2 x$ for $0\le x\le\pi$ is shown below. Identify and explain the error(s) in this solution.

  $\dfrac{\tan^2 x}{\tan^2 x}=\dfrac{\sec x\tan^2 x}{\tan^2 x}$

  $1=\sec x$

  $x=0,\ 2\pi$
- Solve cos $2x$ = 0.8179 for 0° ≤ x ≤ 360°. Write the general solution in both degrees and radians.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Remind students that the strategies used when solving linear and quadratic equations also apply when solving trigonometric equations.
- Review with students how to determine a reference angle for a given angle, θ.

- Set up centres containing examples of trigonometric equations that have been solved incorrectly. Students should move around the centres to identify and correct the errors. Samples are shown below.

(1) $4\sin^2 x - 2 = 0$, for all values of x in radians

$\sin^2 x = \frac{1}{2}$

$\sin x = \sqrt{\frac{1}{2}}$

$\sin x = \frac{\sqrt{2}}{2}$

$\text{ref}\angle = \frac{\pi}{4}$

Quadrant I: $\frac{\pi}{4}$

Quadrant II: $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$x = \begin{cases} \frac{\pi}{4} + 2\pi n, n \in \mathrm{Z} \\ \frac{3\pi}{4} + 2\pi n, n \in \mathrm{Z} \end{cases}$$

(2) $2\tan^2 x - 5\tan x - 3 = 0, \; -360° \le x \le 360°$

$(\tan x - 3)(2\tan x + 1) = 0$

$\tan x = 3$	$\tan x = -\frac{1}{2}$
$\text{ref}\angle \doteq 71.6°$	$\text{ref}\angle \doteq 26.6°$
Quadrant I: 71.6°	Quadrant I: 153.4°
Quadrant III: 251.6°	Quadrant III: 333.4°

$x = 71.6°, 153.4°, 251.6°, 333.4°$

- Distribute half sheets of paper or index cards and ask students to describe the "muddiest point" of solving trigonometric equations. They should jot down any ideas or parts of the lesson that were difficult to understand. This is a quick monitoring technique that allows any difficulties to be addressed.

Suggested Models and Manipulatives

- compass
- grid paper
- ruler
- stop watch

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- trigonometric equation

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 4.4 Introduction to Trigonometric Equations
    - Student Book: pp. 206–214
    - Teacher Resource: pp. 108–111
  - 5.4 Equations and Graphs of Trigonometric Functions
    - Student Book: pp. 266–281
    - Teacher Resource: pp. 134–138
  - 6.4 Solving Trigonometric Equations Using Identities
    - Student Book: pp. 316–321
    - Teacher Resource: pp. 158–161

SCO T06 Students will be expected to prove trigonometric identities, using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identities (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent) [R, T, V]

[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

T06.01 Explain the difference between a trigonometric identity and a trigonometric equation.
T06.02 Verify a trigonometric identity numerically for a given value in either degrees or radians.
T06.03 Explain why verifying that the two sides of a trigonometric identity are equal for given values is insufficient to conclude that the identity is valid.
T06.04 Determine, graphically, the potential validity of a trigonometric identity, using technology.
T06.05 Determine the non-permissible values of a trigonometric identity.
T06.06 Prove, algebraically, that a trigonometric identity is valid.
T06.07 Determine, using the sum, difference, and double-angle identities, the exact value of a trigonometric ratio.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
LR01 Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. (M11)* **AN05** Students will be expected to perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). (PC11)** **RF01** Students will be expected to factor polynomial expressions of the following form where *a*, *b,* and *c* are rational numbers. ▪ $ax^2 + bx + c,\ a \neq 0$ ▪ $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$ ▪ $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$ ▪ $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$ (PC11)**	**T06** Students will be expected to prove trigonometric identities, using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identities (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent)

* M11—Mathematics 11
** PC11—Pre-calculus 11

Background

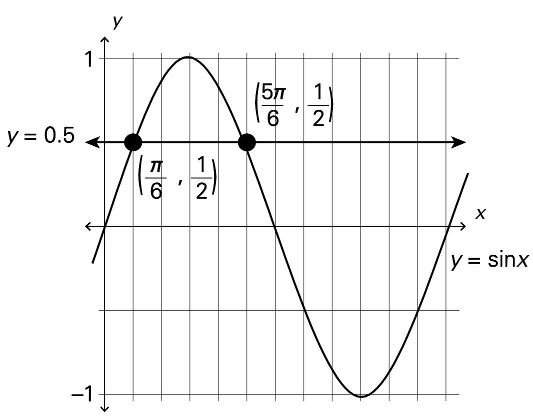
Earlier in this course, students had exposure to graphing trigonometric functions (T04) and solving trigonometric equations where identities were not needed (T05). They were also introduced to the reciprocal trigonometric ratios (T03). Students are now introduced to trigonometric identities as trigonometric equations that are true for all permissible values of the variable in the expressions on both sides of the equation.

Students are expected to verify identities both graphically and numerically, and prove identities using the Pythagorean identities, the quotient identities, the reciprocal identities, the sum/difference identities, and the double-angle identities.

Students are expected to explain the difference between a trigonometric equation and a trigonometric identity. An identity is true for all permissible values, whereas an equation is only true for a smaller subset of the permissible values. For example, the equation $\cos x = \cos(-x)$ is an identity since it is true for all values of *x*, and the graphs of $y = \cos x$ and $y = \cos(-x)$ are identical. The equation $\sin x = \frac{1}{2}$ is not an identity since it is only true for $x = \frac{\pi}{6} + 2\pi k$, $k \in$ Z and $x = \frac{5\pi}{6} + 2\pi k$, $k \in$ Z; and while the graphs $y = \sin x$ and $y = 1$ intersect, they are not the same graph.

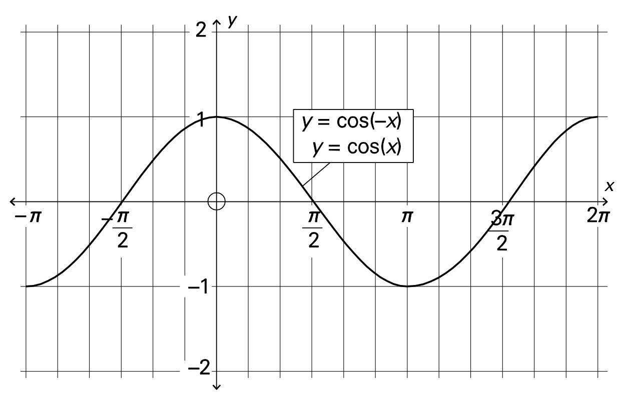
This difference can be demonstrated with the aid of graphing technology.

Not an identity; only true for some values of *x*.



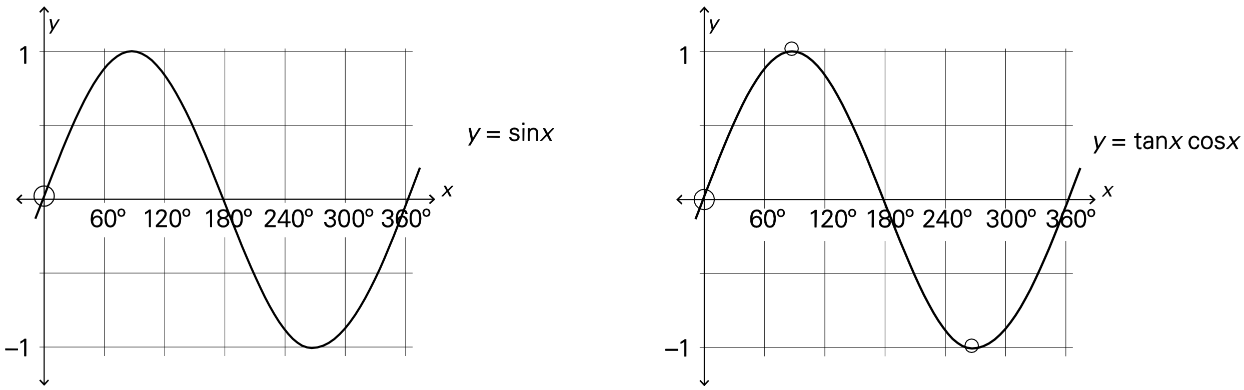
$$\sin x = \frac{1}{2}$$

An identity; true for all values of *x*.



$$\cos x = \cos(-x)$$

Working with the equation $\sin x = \tan x \cos x$, students are expected to observe that, when graphed, $y = \sin x$ and $y = \tan x \cos x$ are almost identical.



The only differences in the graphs occur at the points (90°, 1) and (270°, −1), which are non-permissible values of *x* and are called points of discontinuity for the graph. Therefore, $\sin x = \tan x \cos x$ is an identity, since the expressions are equivalent for all permissible values.

Students should discuss why there are points for which identities are not equivalent. Non-permissible values for identities occur where one of the expressions is undefined. In the above example, $y = \tan x \cos x$ is not defined when $x = 90^\circ + 180^\circ k,\ k \in \mathrm{Z}$ since $y = \tan x$ is undefined at these values.

Students should note that non-permissible values often occur when a trigonometric expression contains

- a rational expression, resulting in values that give a denominator of zero
- tangent, cotangent, secant, and cosecant, since these expressions all have non-permissible values in their domains.

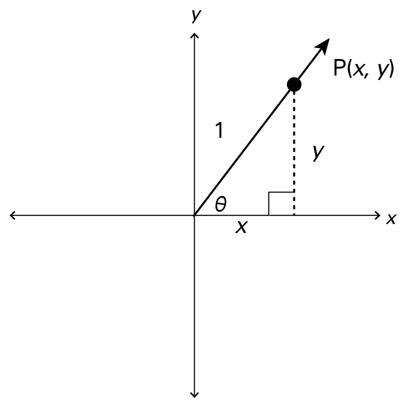
Students are expected to determine non-permissible values, both graphically and algebraically.

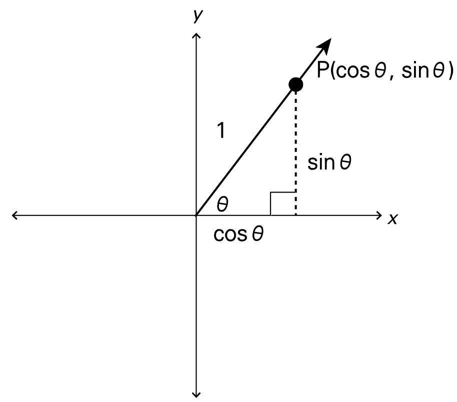
Students are also expected to verify numerically that an identity is valid by substituting numerical values into both sides of the equation. Angles in both degree and radian measures should be used.

In Mathematics 11, students were expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems (LR01).

To introduce the Pythagorean identities, students could be given the expression $\sin^2\theta + \cos^2\theta$ and asked to substitute in different values for θ. They should conclude inductively that $\sin^2\theta + \cos^2\theta = 1$ for all values θ.

Students are expected to understand that observing that a statement is true for different values is insufficient to conclude that the equation is an identity because only a limited number of values were substituted for θ, and there may be a certain group of numbers for which this identity does not hold. This discussion should lead to the idea of proof, a deductive argument that is used to show the validity of a mathematical statement. To prove $\sin^2\theta + \cos^2\theta = 1$, the unit circle (T02), the definitions of $\sin\theta$ and $\cos\theta$ (T02) and the Pythagorean theorem can be used.





To prove that $\sin^2\theta + \cos^2\theta = 1$, using the unit circle:

Since in a right triangle the sum of the squares of the two legs is equal to the square of the hypotenuse, we know that

$\sin^2\theta + \cos^2\theta = (1)^2$

$\therefore \sin^2\theta + \cos^2\theta = 1$

Using the above identity and the reciprocal and quotient identities, students should derive the other two Pythagorean identities, $1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$, as well as identify the non-permissible values. They should verify the identities numerically and validate them with proofs.

Students should also simplify expressions using the Pythagorean identities, the reciprocal identities, and the quotient identities.

Remind students to determine any non-permissible values of the variable in an expression. Students could be asked, for example, to identify the non-permissible values of θ in $\sin\theta\cos\theta\cot\theta$, and then simplify the expression. It is critical that students understand that $\sin\theta\cos\theta\cot\theta \neq \cos^2\theta$, unless the restriction $\sin\theta \neq 0$ is included.

Students often find simplifying trigonometric expressions more challenging than proving trigonometric identities because they may be uncertain of when an expression is simplified as much as possible. Developing a good foundation with simplifying expressions makes the transition to proving trigonometric identities easier.

Frequently, trigonometric relationships involve angle measures that are related as either the sum or difference of other angles or the double of other angles. In such cases, students can use formulas to evaluate trigonometric functions. Once introduced to these, they should understand that the advantage in using the sum and difference formulas or the double-angle formulas is that resulting evaluations can be expressed as exact values rather than as approximate decimal values. The formulas are also used to simplify trigonometric expressions and verify identities. Students should be exposed to the sum and difference identities for the primary trigonometric ratios.

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A+B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

After working with the sum and difference identities, students should be exposed to the double-angle identities for the primary trigonometric ratios:

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

These identities should be verified numerically. Teachers could expose students to the derivation of these formulas, but the proofs are not required for assessment.

Students should see that the three formulas for $\cos 2a$ are all equivalent and that they can use whichever one is most convenient in a problem.

The sum, difference, and double-angle identities are used to determine exact values of trigonometric expressions. Using the double-angle identities, students should determine the exact trigonometric ratios of angles that are not multiples of 30° or 45°, but are multiples of 15°. They could algebraically determine, for example, the exact values of $\cos\left(\dfrac{7\pi}{12}\right)$ and tan145°.

Students should also use these formulas to find the exact value of expressions, such as $\dfrac{\tan 80^\circ + \tan 55^\circ}{1 - \tan 80^\circ \tan 55^\circ}$. They should understand that the sum and difference identities can be applied in either direction.

The sum, difference, and double-angle identities are also needed to simplify certain trigonometric expressions. The expression $\dfrac{2\sin x}{1 - \cos 2x}$, for example, can be simplified to cot x, using the appropriate double-angle formula for cos $2x$.

Ask students to identify the non-permissible roots algebraically, and verify the solution numerically and/or graphically. To find the non-permissible roots algebraically, it is necessary to solve $1 - \cos 2x \neq 0$.

Students could use either of the following methods:

Method A
Substitute the appropriate double-angle formula and solve:
$1-\cos 2\theta \neq 0$
$1-\left(1-2\sin^2\theta\right)\neq 0$
$2\sin^2\theta \neq 0$
$\sin^2\theta \neq 0$
$\sin\theta \neq 0$
$\theta \neq \pi k,\ k \in \mathrm{Z}$

Method B
Solve the equation:
$1-\cos 2\theta \neq 0$
$1 \neq \cos 2\theta$
let $\beta = 2\theta$
$\cos\beta \neq 1$
$\beta \neq 0+2\pi k,\ k \in \mathrm{Z}$
$2\theta \neq 0+2\pi k,\ k \in \mathrm{Z}$
$\theta \neq \pi k,\ k \in \mathrm{Z}$

Students should use the reciprocal identities, the Pythagorean identities, the sum and difference identities, and the double-angle identities to prove other trigonometric identities. To prove identities, they must understand that one side of the identity is rewritten in terms of the functions found on the other side. Strategies for validating an identity include the following:

- writing expressions in terms of sine and cosine
- expressing the given trigonometric functions in terms of a single trigonometric function
- factoring expressions, including expressions with common factors, difference of squares, and trinomials
- writing expressions with a common denominator
- expanding an expression, such as multiplying two binomials together
- writing one fraction as two or more fractions
- multiplying by the conjugate
- multiplying an expression by a fraction equivalent to 1

Students must identify any non-permissible roots when proving identities.

For the identity $\left(1+\cot^2 x\right)\left(1-\cos 2x\right)=2$, the non-permissible roots are given by $\sin x \neq 0$ or $x \neq \pi k,\ k \in \mathrm{Z},$ since $\cot^2 x = \dfrac{\cos^2 x}{\sin^2 x}$. To prove this identity, encourage them to start with the side that appears more complicated. This proof uses both a Pythagorean identity and a double-angle identity. As they work through this proof, ask students how they decide which form of the double-angle formulas for cosine is most appropriate. They should see the importance of being able to express the trigonometric functions in terms of a single function.

Students can verify identities numerically, but remind them that showing an identity is true for certain values of *x* is not a proof, since there may be other values of *x* that do not work in the identity. It does give evidence, however, that the identity is valid. They can also verify the result graphically.

Students should be exposed to proofs that require more than one strategy.

In some cases, both sides of an identity may be independently simplified to a common expression. Subsequently, the proof is validated by the transitive property. For example, to prove that $\sin(2\theta)+\cos\theta=2\cos^2\theta\tan\theta+\cos\theta,\ \cos\theta\neq 0,$

Left side:

$\sin(2\theta)+\cos\theta$

$2\sin\theta\cos\theta+\cos\theta$

$\cos\theta(2\sin\theta+1)$

Right side:

$2\cos^2\theta\tan\theta+\cos\theta$

$2\cos^2\theta\left(\frac{\sin\theta}{\cos\theta}\right)+\cos\theta$

$2\sin\theta\cos\theta+\cos\theta$

$\cos\theta(2\sin\theta+1)$

$\therefore$ Left side = right side by transitive property of equality.

It is expected that students will be provided with a list of the identities for a formal assessment.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Norman noted that

  $(3)^2-1=8$

  $(5)^2-1=24$

  $(7)^2-1=48$

  $(9)^2-1=80$

  Norman concluded that when you subtract 1 from the square of any odd number, you get an even number that is a multiple of 8. Is he correct?

- Is it true that $\frac{x^2-4}{x-2}=x+2$ for all values of x? Explain.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Explain to a friend who missed today's class the difference between an identity and an equation.
- Determine graphically if the following are identities. Identify the non-permissible values for each identity.

  (a) $\sin\theta+\cos\theta\tan\theta=2\sin\theta$

(b) $\tan^2\theta + 1 = \sec^2\theta$

(c) $\dfrac{\cos\theta}{\sin\theta} = \sec\theta$

- Justify whether $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ is an identity, using numerical or graphical evidence.
- Explain whether or not $\sin\theta + \cos\theta = 1$, given that $\sin^2\theta + \cos^2\theta = 1$.
- Simplify $\dfrac{1-\sin^2\theta}{\cos\theta}$, identifying the non-permissible values.
- For one of your homework problems, the answer is sec *x* csc *x*. You and your friend get different answers. Which of them is correct? Explain.

(a) $\dfrac{\sec^2 x}{\tan x}$

(b) $\cot x + \tan x$

- Determine the exact value of the following:

(a) $\tan\left(\dfrac{23\pi}{12}\right)$

(b) sin(255°)

- Write each of the following expressions as a single trigonometric expression. Then give the exact value of the expression.

(a) $\cos\dfrac{\pi}{3}\cos\dfrac{\pi}{6} - \sin\dfrac{\pi}{3}\sin\dfrac{\pi}{6}$

(b) sin60°cos15° – cos60°sin15°

- If $\cos\theta = \dfrac{7}{25}$ where 270° ≤ θ < 360°, determine the exact values of $\tan(2\theta)$ and $\sin\left(\theta + \dfrac{3\pi}{2}\right)$.
- Explain whether or not sin2*x* = 2sin*x* is an identity.
- Use numerical examples to show that $\tan(x+y) \neq \tan(x) + \tan(y)$.
- Simplify.

(a) $\dfrac{\cos 2x + \sin^2 x}{\sin 2x}$

(b) $\sin\left(\theta + \dfrac{\pi}{2}\right) - \sin\left(\theta - \dfrac{\pi}{2}\right)$

- Prove that the following are identities.

(a) $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

(b) $\dfrac{1-\cos 2\theta + \sin 2\theta}{1+\cos 2\theta + \sin 2\theta} = \tan\theta$

(c) $1 + \sin 2x = (\sin x + \cos x)^2$

- Graphically verify that $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$. Determine the non-permissible roots and algebraically prove the identity.
- Which identity should be used as a substitution for cos2*x* when solving 1− cos2*x* = cos *x?* Justify your answer and solve the equation with domain $0 \le \theta \le 2\pi$.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Discuss specific strategies with students that they might use to begin the simplifications.
  - Replace a “squared” term with a Pythagorean identity.
  - Write the expression in terms of sine or cosine.
  - For expressions involving addition or subtraction, it may be necessary to use a common denominator to simplify a fraction.
- A common student error is to express tan 2*x* as 2 tan *x* or cos 2*x* as 2 cos *x*. Ensure students understand that these expressions are not equivalent.
- Students often ignore the fact that the sum and difference formulas are equally true when read from right to left. Make sure to reinforce this by asking for the exact value of expressions such as $\sin(15°)\cos(30°)+\cos(15°)\sin(30°)$ and tan(75°).
- Students can work in pairs to prove that $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$ in three ways, using a different identity for cos2*x* in each proof. Ask them to discuss which identity they found the easiest to work with, and identify any strategies that they learned. The teacher then interviews the student pairs to gain insight into the level of understanding and ability to put mathematical ideas into words.

  Ask questions, such as
  - Which identity for cos2*x* did you use first? Why did you choose this one?
  - Which identity did you find easiest to work with?
  - What strategies have you learned that might help you choose the best identity for cos2*x* for future proofs?

  Rather than interview each student for each topic, teachers may decide to select a sample of students to interview, ensuring all students are included over time.

- Prepare cards containing the steps in the proof of a trigonometric identity. Students work in small groups to decide on a logical sequence in which to place the cards. As students examine the cards, they should discuss their ideas about a possible sequence.
- Some students will have difficulty proving identities because the strategies used depend on the problem. Ensure that a number of examples using a variety of strategies is presented to students. Remind students each side of the identity must be transformed separately.

Correct proof of $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$	**Correct proof of** $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$	**Incorrect proof of** $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$
$\dfrac{\sin x + \tan x}{\cos x + 1}$ $= \dfrac{\sin x + \dfrac{\sin x}{\cos x}}{\cos x + 1}$ $= \dfrac{\dfrac{\sin x \cos x}{\cos x} + \dfrac{\sin x}{\cos x}}{\cos x + 1}$ $= \dfrac{\dfrac{\sin x(\cos x + 1)}{\cos x}}{\cos x + 1}$ $= \dfrac{\sin x(\cos x + 1)}{\cos x}\left(\dfrac{1}{\cos x + 1}\right)$ $= \dfrac{\sin x}{\cos x} = \tan x$	*Left-hand side:* $\dfrac{\sin x + \tan x}{\cos x + 1}$ $= \dfrac{\sin x + \dfrac{\sin x}{\cos x}}{\cos x + 1}$ $= \dfrac{\dfrac{\sin x \cos x}{\cos x} + \dfrac{\sin x}{\cos x}}{\cos x + 1}$ $= \dfrac{\dfrac{\sin x(\cos x + 1)}{\cos x}}{\cos x + 1}$ $= \dfrac{\sin x(\cos x + 1)}{\cos x}\left(\dfrac{1}{\cos x + 1}\right)$ $= \dfrac{\sin x}{\cos x}$ *Right-hand side:* $\tan x = \dfrac{\sin x}{\cos x}$ $\therefore \dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$ by transitive property of equality	$\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$ $\sin x + \tan x = \tan x(\cos x + 1)$ $\sin x + \tan x = \tan x \cos x + \tan x$ $\sin x + \tan x = \left(\dfrac{\sin x}{\cos x}\right)\cos x + \tan x$ $\sin x + \tan x = \sin x + \tan x$

- Often students will experience more success if they work with one side of the identity to obtain the other side of the identity when asked to simplify an expression.

For example, to prove that $\dfrac{1+\cos 2x}{\sin 2x} = \cot x$, ask students to simplify $\dfrac{1+\cos 2x}{\sin 2x}$ to obtain cot x.

Suggested Models and Manipulatives

- protractor
- ruler

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- double-angle identities
- equation
- identity
- points of discontinuity
- Pythagorean identities
- quotient identities
- reciprocal identities
- sum and difference identities
- trigonometric identity

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 6.1 Reciprocal, Quotient, and Pythagorean Identities
    - > Student Book: pp. 290–298
    - > Teacher Resource: pp. 146–149
  - 6.2 Sum, Difference, and Double-Angle Identities
    - > Student Book: pp. 299–308
    - > Teacher Resource: pp. 150–153
  - 6.3 Proving Identities
    - > Student Book: pp. 309–315
    - > Teacher Resource: pp. 154–157

Relations and Functions
60–65 hours

GCO: Students will be expected to develop algebraic and graphical reasoning through the study of relations.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

GUIDING QUESTIONS

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

GUIDING QUESTIONS

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

GUIDING QUESTIONS

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO RF01 Students will be expected to demonstrate an understanding of operations on, and compositions of, functions. [CN, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF01.01 Sketch the graph of a function that is the sum, difference, product, or quotient of two functions, given their graphs.
RF01.02 Write the equation of a function that is the sum, difference, product, or quotient of two or more functions, given their equations.
RF01.03 Determine the domain and range of a function that is the sum, difference, product, or quotient of two functions.
RF01.04 Write a function $h(x)$ as the sum, difference, product, or quotient of two or more functions.
RF01.05 Determine the value of the composition of functions when evaluated at a point, including $f[f(a)]$, $f[g(a)]$, and $g[f(a)]$.
RF01.06 Determine, given the equations of two functions $f(x)$ and $g(x)$, the equation of the composite function $f[f(x)]$, $f[g(x)]$, and $g[f(x)]$, and explain any restrictions.
RF01.07 Sketch, given the equations of two functions $f(x)$ and $g(x)$, the graph of the composite function $f[f(x)]$, $f[g(x)]$, and $g[f(x)]$.
RF01.08 Write a function $h(x)$ as the composition of two or more functions.
RF01.09 Write a function $h(x)$ by combining two or more functions through operations on, and compositions of, functions.

Scope and Sequence

Pre-calculus 11	**Pre-calculus 12**
RF01 Students will be expected to factor polynomial expressions of the following form where a, b, and c are rational numbers. ▪ $ax^2 + bx + c,\ a \neq 0$ ▪ $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$ ▪ $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$ ▪ $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$ **RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. **RF11** Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).	**RF01** Students will be expected to demonstrate an understanding of operations on, and compositions of, functions.

Pre-calculus 11 (continued)	Pre-calculus 12 (continued)
AN02 Students will be expected to solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. **AN04** Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials).	

Background

Students have experience with graphical representations of linear, absolute value, and quadratic functions, as well as practice determining the domain and range of each function. They have also been exposed to non-permissible values (restrictions) in terms of rational expressions that will serve as a foundation for the introduction of asymptotes in sketching graphs of combinations of functions (quotients).

Students' facility with functions now becomes increasingly more sophisticated as they learn to reason about new functions derived from familiar ones via the combination or composition of functions. For example, a profit relationship can be viewed as a combination of functions by subtracting the functions that represent revenue and expenses. Students should explore visually the effects upon the domains and ranges of combination situations, such as linear $\pm$ linear, linear $\pm$ quadratic, quadratic $\pm$ quadratic, and linear × linear. Students will have already worked with the product of two linear functions that form a quadratic function.

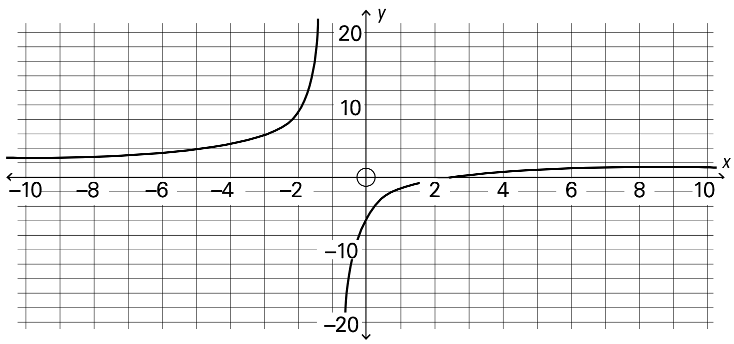
As students write equations of functions that are sums, differences, products, or quotients of two functions, they will be required to simplify their answers. For quotient functions, students will need to identify non-permissible values.

Vertical asymptotes and **points of discontinuity** (where the function does not exist) will be explored by stating non-permissible values with the quotients of functions. Horizontal asymptotes do not need to be addressed at this time.

When determining the domain and range of a function representing the sum, difference, product, or quotient of two functions, the domain will be the domain common to both of the original functions. The domain of the quotient function is further restricted by excluding values where the denominator would equal zero. The range will be determined by the graph.

For example the function $g(x)=\dfrac{f(x)}{h(x)}=\dfrac{2x-5}{x+1}$ would have a vertical asymptote at $x = -1$ since that value creates division by zero.

Its graph would confirm that the domain of $g(x)=\dfrac{f(x)}{h(x)}=\dfrac{2x-5}{x+1}$ is $\{x|x\neq -1;\ x\in \mathrm{R}\}$; by observation it appears that the range of $g(x)$ is $\{y|y\neq 2;\ y\in \mathrm{R}\}$.



Sometimes students will encounter a function within another function, where both functions are needed to answer a question or analyze a problem. The domain of the second function has to connect to the first function. The symbol $f\lfloor g(x)\rfloor$ or $(f\circ g)(x)$ is a composition of the two functions, *f* and *g*. Students should understand that the composition $f\lfloor g(x)\rfloor$ gives the final outcome when the independent value is substituted into the inner function *g*, and its value $g(x)$ is then substituted into the outer function f. The range of $g(x)$ becomes the domain of $f(x)$.

Example:

$f(x)=4(x-3)^2-1$ and $g(x)=-\frac{1}{2}(x-5)^2+9;$ evaluate $g\left[f(2)\right]$.

$f(2)=4(2-3)^2-1=4(-1)^2-1=3$

$g(3)=-\frac{1}{2}(3-5)^2+9$

$g(3)=-\frac{1}{2}(-2)^2+9$

$g(3)=-2+9$

$g(3)=7$

$\therefore g\left[f(2)\right]=7$

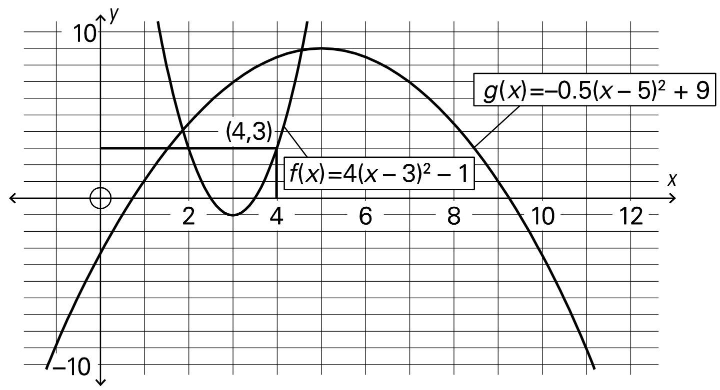
Students are expected to visualize the composition of functions using graphs. Students should be able to explain each step of the process.

For example, to determine $g\lfloor f(4)\rfloor$, students are expected to understand a visual approach as illustrated below.

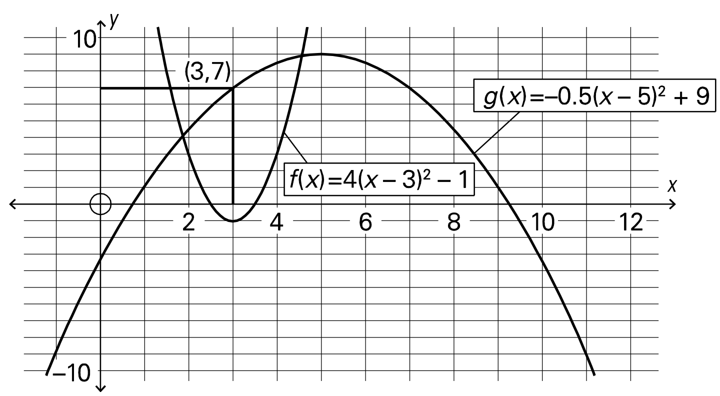
<table>
<tr>
<td>To determine $g\lfloor f(4)\rfloor$, begin by evaluating $f(4)$.

To do this trace a vertical line from x = 2 to the graph of $f(x)$.

Find the y-value of this point by tracing a horizontal line to the y-axis. You obtain $f(4)=3$.

This y-value of $f(4)$ becomes our value to substitute into the function $g(x)$.</td>
<td>
</td>
</tr>
<tr>
<td>Trace a vertical line from this x-value, $x=3$, to intersect the graph $g(x)$.

Find the y-value of this point by tracing a horizontal line to the y-axis. You obtain $g(3)=7$.

Therefore, $g\lfloor f(4)\rfloor=7$.</td>
<td>
</td>
</tr>
</table>

Students will be expected to sketch graphs of composition of functions using graphs of linear, quadratic, etc. In order to sketch the new graphs, students are expected to anticipate the type of graph that will be created by the composition. For example, they should expect that the composition of a linear and a quadratic function should yield a quadratic function.

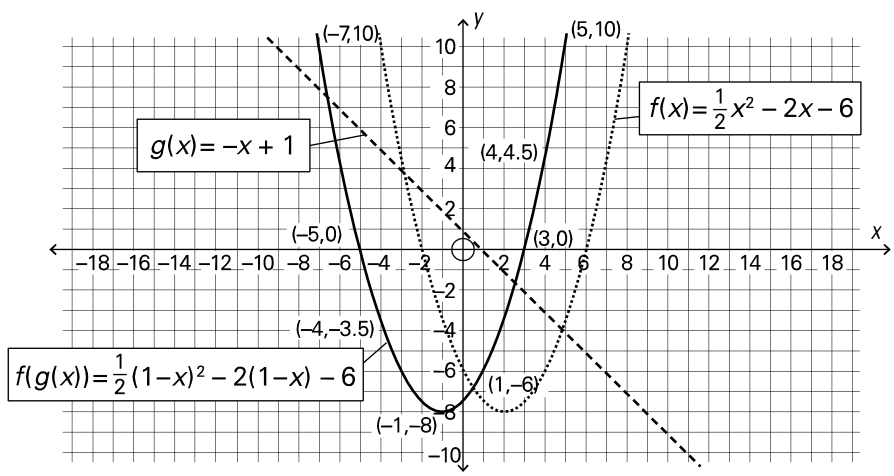
When given the two functions, $f(x)$ and $g(x)$, students would be expected to create a table of values for $f\lfloor g(x)\rfloor$. For example, given that $f(x)=\frac{1}{2}x^2-2x-6$ and $g(x)=-x+1$, to evaluate $f\lfloor g(3)\rfloor$, find $g(3)=-3+1=-2$ and then evaluate $f(-2)=\frac{1}{2}(-2)^2-2(-2)-6=2+4-6=0$.

Repeating this with more values of *x* obtains:

x	$g(x)=-1+x$	$f\left[g(x)\right]=\frac{1}{2}\left[g(x)\right]^2-2\left[g(x)\right]-6$	$f\left[g(x)\right]$
–7	$g(-7)=7+1=8$	$f\left[g(-7)\right]=f(8)=\frac{1}{2}(8)^2-2(8)-6$	32 – 16 – 6 = 10
–5	$g(-5)=5+1=6$	$f\left[g(-5)\right]=f(6)=\frac{1}{2}(6)^2-2(6)-6$	18 – 12 – 6 = 0
–4	$g(-4)=4+1=5$	$f\left[g(-4)\right]=f(5)=\frac{1}{2}(5)^2-2(5)-6$	12.5 – 10 – 6 = –3.5
–1	$g(-1)=1+1=2$	$f\left[g(-1)\right]=f(2)=\frac{1}{2}(2)^2-2(2)-6$	2 – 4 – 6 = –8
1	$g(1)=-1+1=0$	$f\left[g(1)\right]=f(0)=\frac{1}{2}(0)^2-2(0)-6$	0 – 0 – 6 = –6
4	$g(4)=-4+1=-3$	$f\left[g(4)\right]=f(-3)=\frac{1}{2}(-3)^2-2(-3)-6$	4.5 + 6 – 6 = 4.5
5	$g(5)=-5+1=-4$	$f\left[g(5)\right]=f(-4)=\frac{1}{2}(-4)^2-2(-4)-6$	8 + 8 – 6 = 10

Sketching the functions $f(x)$, $g(x)$, and $(f\circ g)(x)$.

x	$f\left[g(x)\right]$
–7	10
–5	0
–4	–3.5
–1	–8
1	–6
3	0
4	4.5
5	10



Because of symmetry, we know that $f\lfloor g(x)\rfloor$ has its vertex midway between *x* = –7 and *x* = 5 (both have *y*-value of 10) or between *x* = –5 and *x* = 3 (both have *y*-value of 0).

Therefore, (–1, –8) is its vertex. Its equation is $f\left[g(x)\right]=\frac{1}{2}(x+1)^2-8$.

Similarly the two *x*-intercepts of $f\lfloor g(x)\rfloor$ are (–5, 0) and (3, 0); therefore, its equation is

$f\left[g(x)\right]=\frac{1}{2}(x-5)(x-3)$.

Students will be required to write a function as a composition of two or more functions. This will require them to identify patterns in the function where they can insert an algebraic expression inside the given function. There will be many potential solutions to these types of questions.

For example, the following function, $h(x)$, can be written as a composition of two functions, $f(x)=2x+3$ and $g(x)=x^2+2x-5$.

If $h(x)=f[g(x)]$ or $h(x)=(f\circ g)(x)$	If $h(x)=g[f(x)]$ or $h(x)=(g\circ f)(x)$
$h(x)=f[g(x)]$ $h(x)=2[g(x)]+3$ $h(x)=2(x^2+2x-5)+3$ $h(x)=2x^2+4x-7$	$h(x)=g[f(x)]$ $h(x)=(2x+3)^2+2(2x+3)-5$ $h(x)=4x^2+12x+9+4x+6-5$ $h(x)=4x^2+16x+10$

Students are expected to note that $f[g(x)]\neq g[f(x)]$, unless the functions are inverse functions, such as if $f(x)=\frac{1}{2}x$ and $g(x)=2x$.

Assessment, Teaching, and Learning

Assessment Strategies

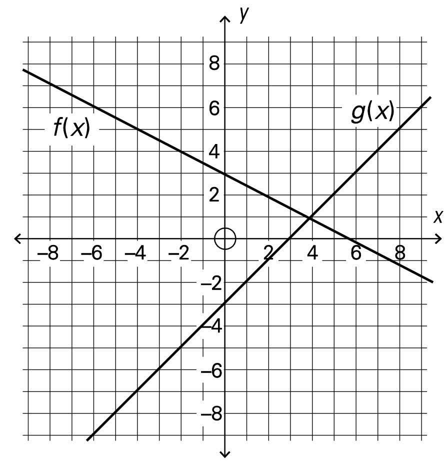
ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Expand and simplify the following:
  (a) $5(x^2+2x-3)+6$
  (b) $(5x+6)^2+2(5x+6)-3$
- State the domain for each of the following:
  (a) $y=2x-30$
  (b) $y=\frac{1}{2x-30}$
  (c) $y=\sqrt{2x-3}$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.



- Using the graph of two linear functions $f(x) = -\frac{1}{2}x + 3$ and $g(x) = x - 3$, shown below, use a table of values to sketch the graphs of the sum, difference, product, and quotient of the two functions.

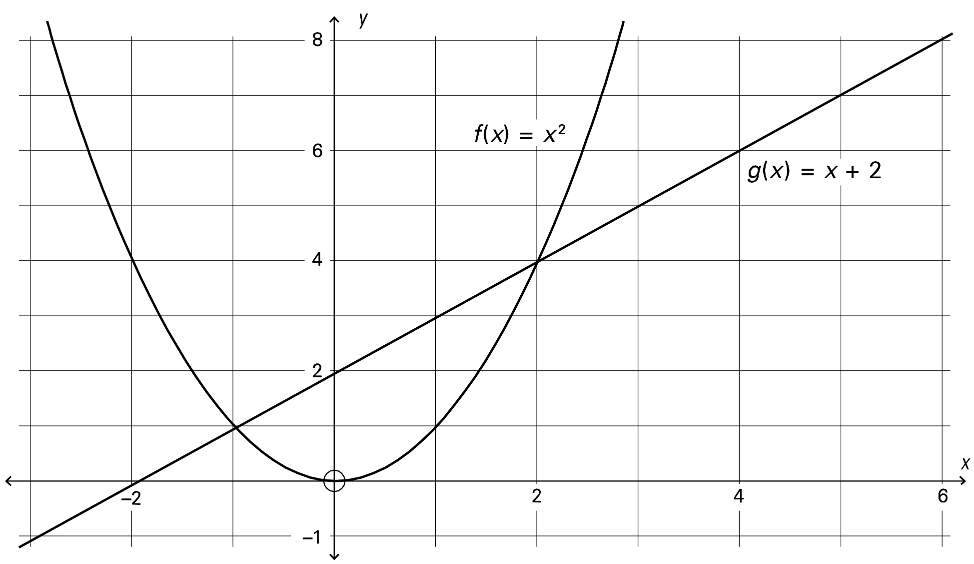
  (a) Sketch the function $h(x) = f(x) + g(x)$.

  (b) Sketch the function $h(x) = f(x) - g(x)$.

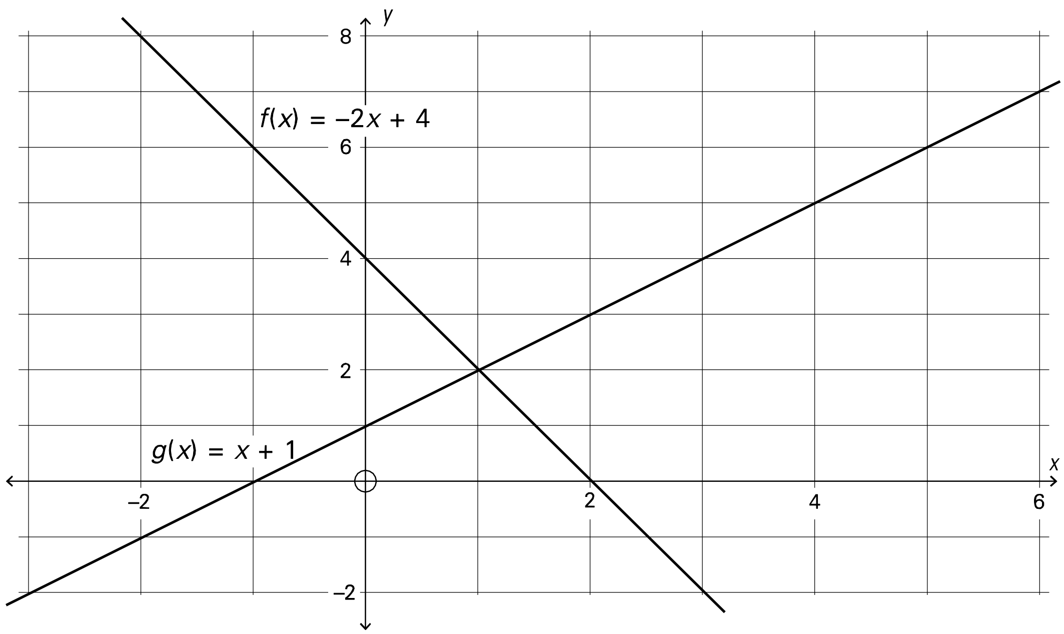
  (c) Sketch the function $h(x) = f(x) \times g(x)$.

  (d) Sketch the function $h(x) = f(x) \div g(x)$.

  Determine the domain and range for each of the functions (a) to (d).
- If $f(3) = 5$ and $g(3) = 7$ and $h(x) = f(x) + g(x)$, then $h(3) = ?$
- The graph below shows two functions $f(x) = x^2$ and $g(x) = x + 2$. Sketch the graphs of the sum, difference, product, and quotient of these two functions. Determine the domain and range for each of these combined functions.



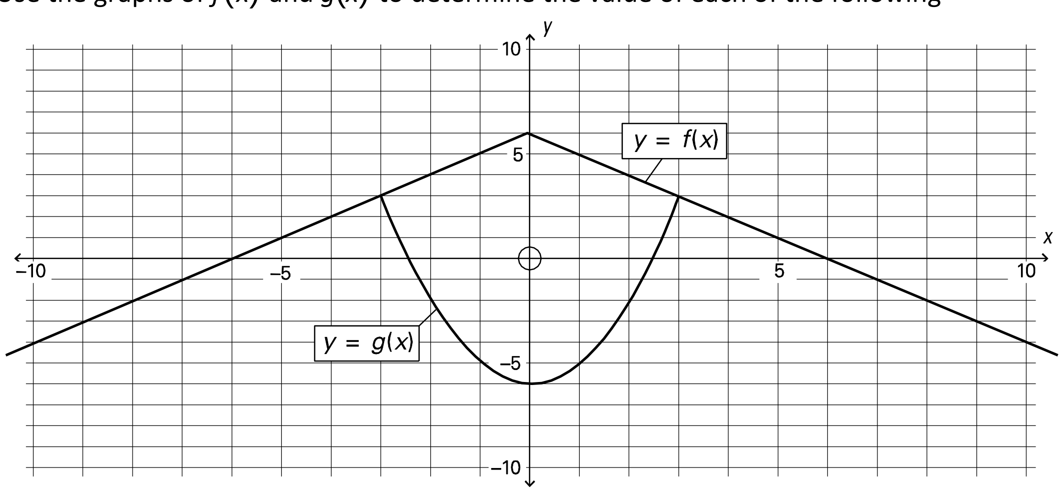
- The graph below shows two functions $f(x)=-2x+4$ and $g(x)=x+1$. Sketch the graphs of the sum, difference, product, and quotient of these two functions. Determine the domain and range for each of these combined functions.



- Given $f(x)=3x+6$ and $g(x)=x^2+x-2$, determine $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\times g(x)$, and $\dfrac{f(x)}{g(x)}$.

  Determine the domain and the range for the combined functions. (Use technology to sketch the graphs.)

- Use the graphs of $f(x)$ and $g(x)$ to determine the value of each of the following



(a) $(f\circ g)(2)$

(b) $\dfrac{f}{g}(1)$

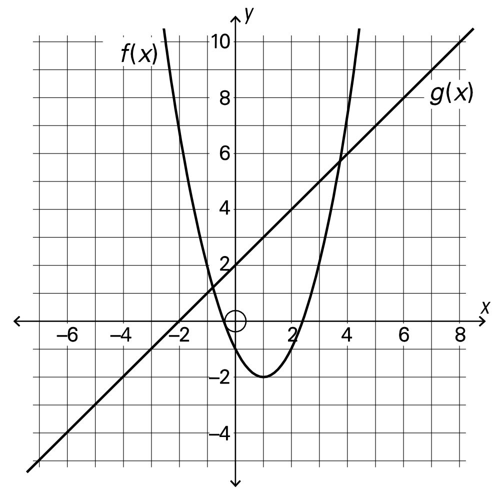
(c) $(g\circ f)(-3)$

(d) $\left(\dfrac{g}{f}\right)(0)$

- Given $f(x)=x-1$ and $h(x)=x^2-1$, determine $g(x)$ such that $(f\circ g)(x)=h(x)$.

- Given $f(x)=\sqrt{x}$ and $h(x)=2\sqrt{x}$, determine $g(x)$ such that $\left(\frac{f}{g}\right)(x)=h(x)$.
- If $f(x)=x$ and $g(x)=2-x$, sketch the graph of $h(x)=(f\circ g)(x)$.
- You need to save \$1500 to contribute to the cost of tuition in the fall and are considering mowing lawns for the summer. You would like to buy a battery-powered lawn mower with rechargeable batteries, and your parents are asking for a flat rate of \$100 for the summer to pay for the electricity for recharging the batteries each night. The cost of the mower is \$499 plus sales tax, and you plan to charge \$25 to cut a lawn.
  (a) Set up a chart with four columns: number of lawns cut, cost, income, profit.
  (b) Determine the equation for each column in relation to the number of lawns cut.
  (c) Graph the three equations on the same graph.
    (i) What does the *x* represent?
    (ii) What does the *y* represent?
    (iii) What do the slopes represent?
    (iv) Which lines appear parallel? Are they parallel? Explain.
    (v) How many lawns would you need to mow to break even?
    (vi) How many lawns would you need to mow to cover your costs and clear \$1500?
    (vii) On average, how many lawns would you need to cut per week? Is this a reasonable expectation?
    (viii) Do you think this would be a good business venture?
    (ix) Is there anything you would do differently?
- If $f(x)=2x+1$ and $g(x)=\frac{1}{x}$, determine the function composition for $\lfloor g(x)\rfloor$, and $g\lfloor f(x)\rfloor$.
- If $f(x)=5x-1$ and $f\lfloor g(x)\rfloor=5x^2+10x+4$, find $g(x)$.
- For $f(x)=\sqrt{x+2}$ and $g(x)=x^2-6x+7$, evaluate each of the compositions of functions below.
  (a) $f\lfloor g(0)\rfloor$
  (b) $(f\circ g)(3x)$
  (c) $g\lfloor f(-2)\rfloor$
  (d) $g\lfloor f(7)\rfloor$
  (e) $g\lfloor g(5)\rfloor$
  (f) $(g\circ g)(1)$

- For $f(x)=x^2+2$ and $g(x)=x-3$, sketch the graphs of each of the following:
  - (a) $f\lfloor g(x)\rfloor$
  - (b) $(f\circ g)(x)$
  - (c) $g\lfloor g(x)\rfloor$



- From the graph to the right, determine the value of each of the composition of functions that follow.
  - (a) $f\lfloor g(1)\rfloor$
  - (b) $(f\circ g)(-1)$
  - (c) $(g\circ f)(3)$
  - (d) $g\lfloor g(2)\rfloor$
- Use the table below to evaluate each of the following.

x	**–2**	**–1**	**0**	**1**	**2**
$\boldsymbol{f(x)}$	2	1	0	–1	–2
$\boldsymbol{g(x)}$	0	2	4	6	8

  - (a) $f\lfloor g(-2)\rfloor$
  - (b) $g\lfloor f(2)\rfloor$
  - (c) $f\lfloor f(1)\rfloor$
  - (d) $(g\circ g)(0)$
- For each pair of functions, find $(f\circ g)(x)$ and $(g\circ f)(x)$ and state the domain of each function.
  - (a) $f(x)=5x;\ g(x)=x^2+1$
  - (b) $f(x)=\sqrt{x};\ g(x)=x+1$
  - (c) $f(x)=\dfrac{x}{x-2};\ g(x)=2x$
- If $f(x)=|x+1|$ and $g(x)=4-x^2$, sketch the graphs of $(f\circ g)(x)$ and $(g\circ f)(x)$.
- Given $f(x)=2x+1$ and $h(x)=3-2x$, determine $g(x)$ such that $(f\circ g)(x)=\mathrm{h}(x)$.
- If $f(x)=\dfrac{x+1}{x}$ find $(f\circ f)(x)$ and state its domain.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Give the students a graph of two linear functions $f(x)$ and $g(x)$. Then have them use a table of values to sketch the graphs of the sum, difference, product, and quotient of the two functions. Students would be expected to state their domain and range. It is important that understanding be emphasized.
- This illustration of a function machine can clarify the idea of function composition.

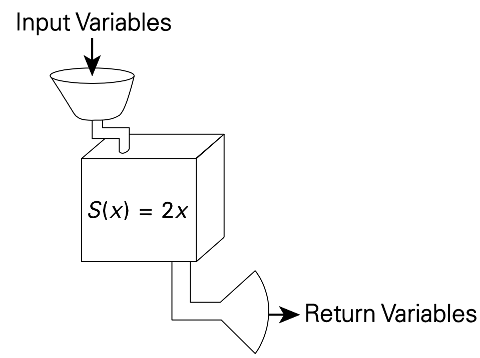
  Label three different containers, with lids, as function machines.

  For example,

  – a shoe box could be called $S(x)=2x$

  – a cookie tin could be called $C(x)=x^2$

  – a Mason jar could be called $M(x)=x-1$



  Then describe the function of each of these boxes.

  – If a number such as 10 is put in the shoe box, that number ill be doubled, and you will obtain 20.

  – If a number such as 10 is put in the cookie tin, that number will be squared, and you will obtain 100.

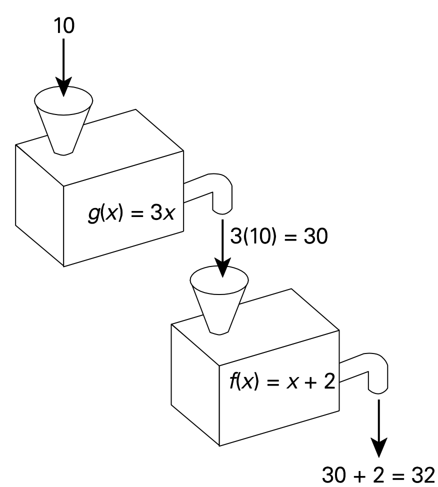
  – If a number such as 10 is put in the Mason jar, that number will have one subtracted from it, and you will obtain 9.

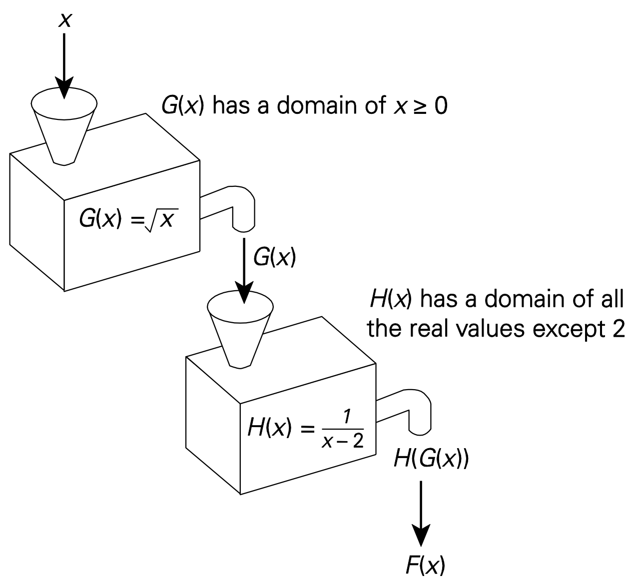
  Then run through some possibilities, such as

  – If $x=10$ is put into the shoe box and then the result is poured directly into the Mason jar, what will be the result from the Mason jar? Students will see this as $S(10)\rightarrow 20$; $M(20)\rightarrow 19$ or $M\lfloor S(10)\rfloor=19$ or $(M\circ S)(10)=19$.

  – If we put $x = 3$ into the cookie tin, then pour the result directly into the Mason jar, and then directly into the shoe box, what will be the result from the Mason jar? Students will see this as $C(3)\rightarrow 9$; $M(9)\rightarrow 8$; $S(8)\rightarrow 16$ or $S\{M[C(3)]\}=16$ or $(S\circ M\circ C)(3)=16$.

- Using function machines, such as the one shown to the right, to illustrate composition can be very helpful for many students when visualizing $\{f[g(10)]\}$ or $(f\circ g)(10)$.
- Ensure that students do not confuse the composition of two functions with the product of two functions.
- Ensure that students know how to find the domain of a composition of functions. This can be emphasized by using a function machines model, such as the one shown below, where each of the functions have a restricted domain.





Students should understand that the domain of the $G(x)$ is $x \geq 0$, and that of $H(x)$ is $x \neq 2.$ From the diagram, they can see that the range of $G(x)$ is the domain of $H(x)$. From this they can conclude that the domain of $F(x)=H\lfloor G(x)\rfloor$ or $F(x)=(H \circ G)(x)$ would also have its domain restricted to

– values of *x* for which $G(x)$ is defined (in this case all $x \geq 0$) *and*

– those values of *x* that would create range values of $G(x)$ for which $H(x)$ exists (in this case values of *x* for which $G(x) \neq 2$ or $\sqrt{x} \neq 2$ or $x \neq 4$.

– Therefore, for this example, the domain of $F(x)=H\lfloor G(x)\rfloor$ or $F(x)=(H \circ G)(x)$ is $x \geq 0$; $x \neq 4$.

- Remind students that the composition of functions does not follow the commutative property, that is $f\lfloor g(x)\rfloor \neq g\lfloor f(x)\rfloor$.
- Ensure that both common notations for composition $h\{g[f(x)]\}$ and $(h \circ g \circ f)(x)$ are used.

SUGGESTED MODELS AND MANIPULATIVES

- several empty containers with lids (e.g., shoe box, cookie tin, or Mason jar)

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- composite function

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 10.3 Composite Functions
    - Student Book: pp. 499–509

SCO RF02 Students will be expected to demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. [C, CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF02.01 Compare the graphs of a set of functions of the form $y-k=f(x)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of *k*.

RF02.02 Compare the graphs of a set of functions of the form $y=f(x-h)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effect of *h*.

RF02.03 Compare the graphs of a set of functions of the form $y-k=f(x-h)$ to the graph of $y=f(x)$, and generalize, using inductive reasoning, a rule about the effects of *h* and *k*.

RF02.04 Sketch the graph of $y-k=f(x)$, $y=f(x-h)$, or $y-k=f(x-h)$ for given values of *h* and *k*, given a sketch of the function $y=f(x)$, where the equation of $y=f(x)$ is not given.

RF02.05 Write the equation of a function whose graph is a vertical and/or horizontal translation of the graph of the function $y=f(x)$.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. (PC11)**	**RF02** Students will be expected to demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

* M11—Mathematics 11
** PC11—Pre-calculus 11

Background

In Mathematics 11, students were introduced to quadratic functions in vertex form $y = a(x - h)^2 + k$ (RF03). They discovered that *h* and *k* translated the graph horizontally and vertically. These transformations allow students to identify the vertex directly from the quadratic equation.

Students will now look at these translations for general functions and compare how the graph and table of values of $y = f(x)$ compares to $y - k = f(x - h)$.

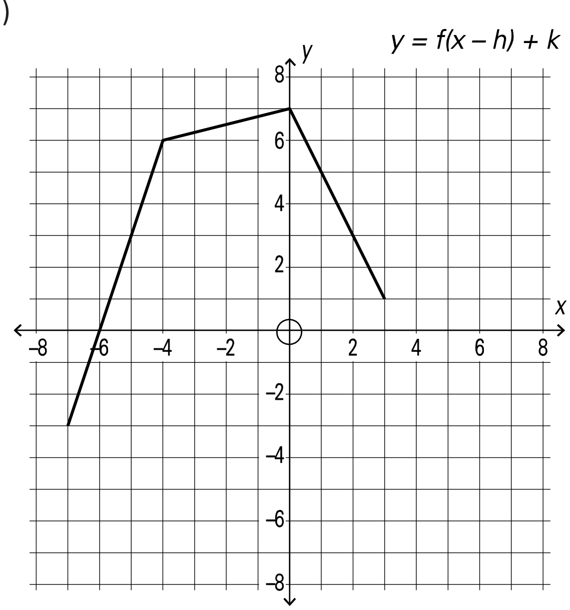
Students should investigate the effect of changing the value of k by comparing functions of the form $y - k = f(x)$ or $y = f(x) + k$ to the graph of $y = f(x)$ and then the effect of changing the value of h by comparing functions of the form $y = f(x - h)$ to the graph of $y = f(x)$.

Students should then use mapping notation $(x, y) \rightarrow (x, y + k)$ to describe the applicable vertical translation for each graph and the mapping notation $(x, y) \rightarrow (x + h, y)$ to describe the applicable horizontal translation for each graph.

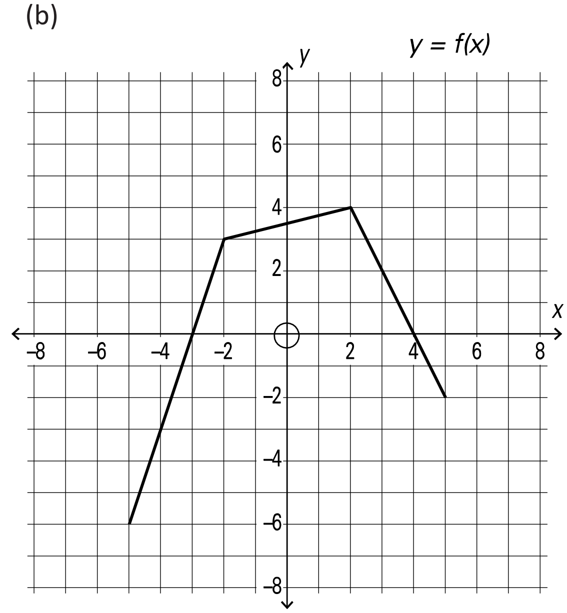
Students are also expected to work with functions that have been translated both horizontally and vertically when compared to a base function and to generalize those transformations using the mapping rule $(x, y) \rightarrow (x + h, y + k)$. They should be exposed to both forms of the transformed function, $y-k=f\left(x-h\right)$ and $y=f\left(x-h\right)+k$.

Given the graph of a base function $y = f(x)$ and the graph of $y-k=f\left(x-h\right)$, students should identify the horizontal and vertical translations, and write the equation of the translated function. Students could be given the two graphs below and asked to describe the horizontal and vertical translations. They should then write the equation for the translated function as $y - 3 = f(x + 2)$ or $y = f(x + 2) + 3$.

(a)



(b)



Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

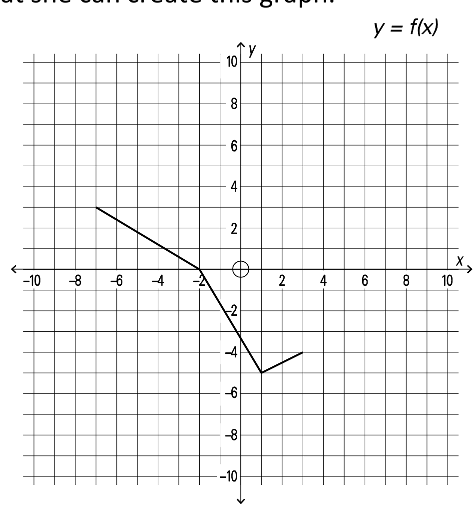
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- State the vertex for the function $f(x)=(x-1)^2-5$.
- Graph each of the following:
  (a) $y=x$
  (b) $y=x-2$
  (c) $y=x^2$
  (d) $y=x^2-1$
  (e) $y=(x-1)^2$

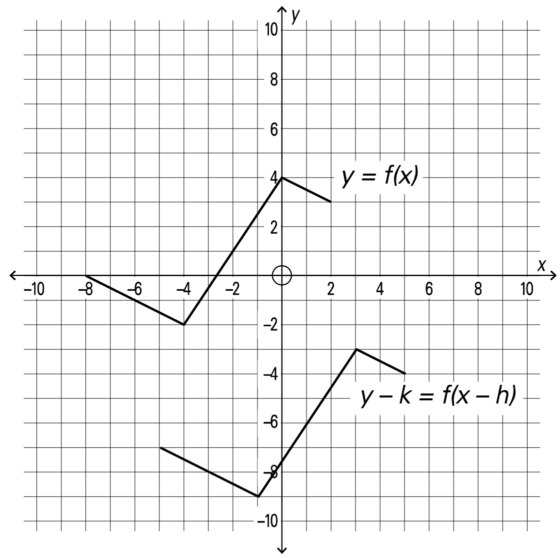
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

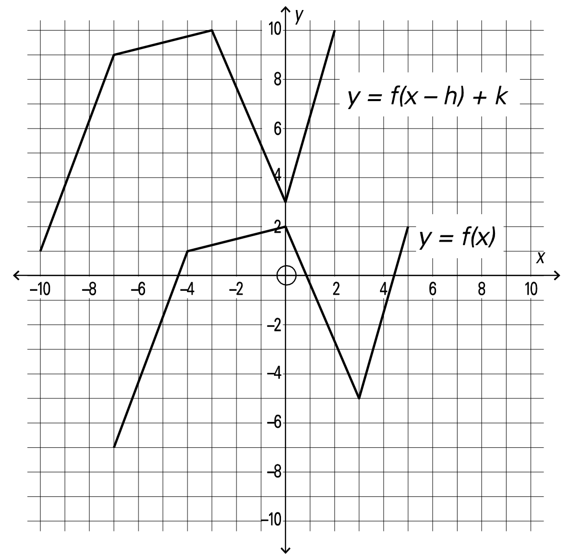
- Describe the similarities and differences among the graphs of the following functions if they are transformations of the base function $y=f(x)$.
  (a) $y=f(x)+10$
  (b) $y=f(x+10)$
- Describe the translations of each function when compared to $y=f(x)$.
  (a) $y=f(x-2)$
  (b) $y=f(x)-9$
  (c) $y=f(x+3)-7$
  (d) $y-12=f(x+4)$
- A friend phones for help with her homework. She has a function with four key points and wants to graph $y=f(x+5)-7$. Ask students to describe two ways that she can create this graph.
- Given the graph of $y=f(x)$, ask students to create a mapping rule and a table of values for each of the transformations below and graph the transformed functions.
  (a) $y+2=f(x-6)$
  (b) $y=f(x+2)+5$
  (c) $y=f(x-4)-7$



- Determine the values of *h* and *k* and write the equation for the translated graph for each of the following:

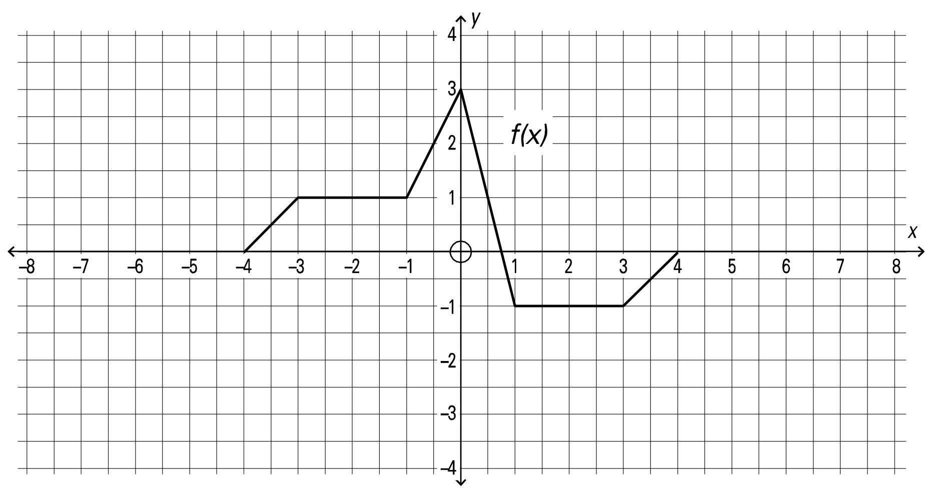


(a)



(b)

- Given the graph of $y = f(x)$, sketch the graph of each of the following.



(a) $y = f(x + 2)$

(b) $y = f(x) + 2$

(c) $y = f(x - 3) + 2$

- Use mapping notation to describe how the graph of each function can be found from the graph of $y = x^2$.

(a) $y = (x + 4)^2 - 3$

(b) $y = (x - 6)^2 + 2$

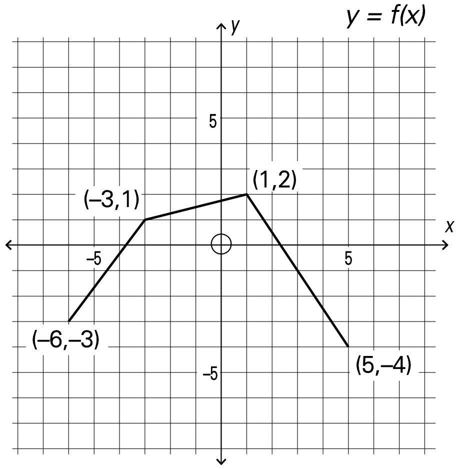
- The function $y = x + 3$ could be a vertical translation or a horizontal translation of the line $y = x$. Explain.

Planning for Instruction

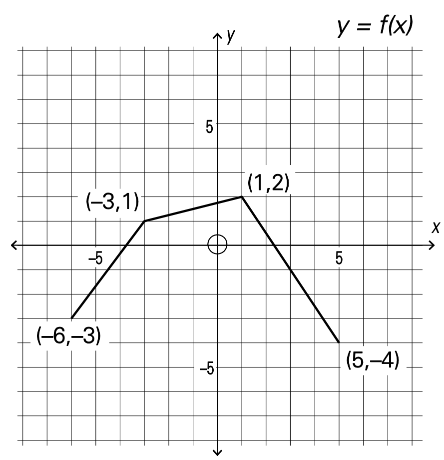
SUGGESTED LEARNING TASKS

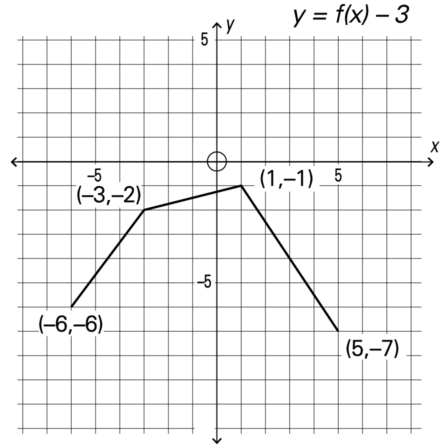
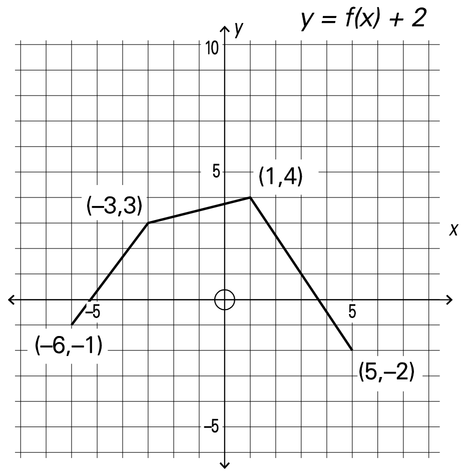
Consider the following sample instructional strategies when planning lessons.

- Provide students with a base graph, such as the graph of $y = f(x)$ below:



Ask students to identify the key points of $y = f(x)$ and create a table of values. They should then create new tables of values and respective graphs for functions such as $y + 3 = f(x)$, and $y = f(x) + 2$, which have a vertical translation, k, using a chart such as the one below.

<table>
<tr><th>x</th><th>–6</th><th>–3</th><th>1</th><th>5</th><th>Sketch</th></tr>
<tr><td>$y = f(x)$</td><td>–3</td><td>1</td><td>2</td><td>–4</td><td>
</td></tr>
</table>

x	**−6**	**−3**	**1**	**5**	**Sketch**
$y+3=f(x)$ $y=f(x)-3$	−6	−2	−1	−7	
$y=f(x)+2$	−1	3	4	−2	

Discuss with students how this vertical translation affects the position but not the shape nor the orientation of the graph. Students should then use mapping notation $(x, y) \rightarrow (x, y + k)$ to describe the applicable vertical translation for each graph.

- In a similar fashion, students are expected to explore the corresponding mapping notation $(x, y) \rightarrow (x + \text{h}, y)$.
- Provide each student with a piece of graph paper and a sticky note. On the graph paper, ask students to construct a function consisting of at least four points. On the sticky note, they should write an equation in the form $y - k = f(x)$ or $y = f(x) + k$. Students should trade sticky notes and apply the transformation on the new sticky note to their own graphs.
- Students could work in small groups for this activity. Provide each group with a base function $y = f(x)$ and ask each student to create their own translated graph of $y = f(x)$. They should then trade their graphs within their group and determine the equation of the function for each graph.
- Use graphing technology and a function such as $y=|x|$ with students to explore how $y = f(x) + k$ is related to $y = f(x)$ and how $y = f(x - h)$ is related to $y = f(x)$. Students should be able to determine how the parameters h and k affect the graph of $y = f(x)$.

- Some students may get confused, because the sign that appears in a horizontal translation may be different from what they expect. If $y = f(x) + k$ is rewritten as $y - k = f(x)$ they should be able to see the consistency in the notation between vertical and horizontal translations.
- Use technology to explore the effect of h and k on the graph of a given equation, $f(x) = |x - h| + k$, to develop a rule about the effect of various values of h and k, and repeat with a variety of different equations.

Suggested Models and Manipulatives

- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- image point
- mapping rule
- transformation
- translation

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 1.1 Horizontal and Vertical Translations
    - > Student Book: pp. 6–15

SCO RF03 Students will be expected to demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. [C, CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF03.01 Compare the graphs of a set of functions of the form $y = af(x)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of *a*.

RF03.02 Compare the graphs of a set of functions of the form $y = f(bx)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of *b*.

RF03.03 Compare the graphs of a set of functions of the form $y = af(bx)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effects of *a* and *b*.

RF03.04 Sketch the graph of $y = af(x)$, $y = f(bx)$, or $y = af(bx)$ for given values of *a* and *b*, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

RF03.05 Write the equation of a function, given its graph which is a vertical and/or horizontal stretch of the graph of the function $y = f(x)$.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. (PC11)** **RF03** Students will be expected to analyze quadratic functions of the form $y = a(x-p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, and *x*- and *y*-intercepts. (PC11)**	**RF03** Students will be expected to demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

* M11—Mathematics 11

** PC11—Pre-calculus 11

Background

In Pre-calculus 11, students explored the effect of a vertical stretch on quadratic functions (RF03). They will now work with both vertical and horizontal stretches for general functions. Students should describe how these stretches change the shape of the graph and determine these stretches when given the graphs of the base function and the transformed function. This outcome focuses on horizontal and vertical stretches, so at this point, students should not encounter functions that have been both stretched and translated.

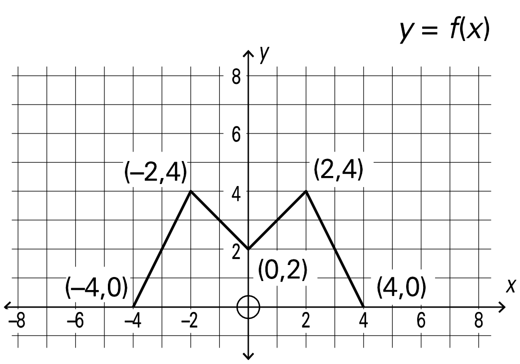
Specifically, for this outcome students are expected to investigate the effect of *a* and *b* on the graph of the function $y = af\left(bx\right)$.

- The value of *a* indicates the magnitude of the **vertical stretch** of $f\left(x\right)$ about the *x*-axis by a factor of $|a|$.
- The reciprocal of the value of *b* indicates the magnitude of the **horizontal stretch** of $f\left(x\right)$ about the *y*-axis by a factor of $\frac{1}{|b|}$.

If students have not completed the outcomes on trigonometric functions prior to this, horizontal stretch will be a new concept for them. Horizontal stretches are difficult to see unless a function has a period or a restricted domain.

Students should understand the effect of *a* as a vertical stretch of $|a|$ on the function $y = f\left(x\right)$ in the form $y = af\left(x\right)$. [Note that this could also be written as $\frac{1}{a}y = f\left(x\right)$.]

When given a graph of *f*(*x*), such as the one shown below, students would be expected to use key points and generate the table of values for functions, such as $y = 2f\left(x\right)$ and $y = \frac{1}{2}f\left(x\right)$.



x	$y = 2f\left(x\right)$
–4	0
–2	8
0	4
2	8
4	0

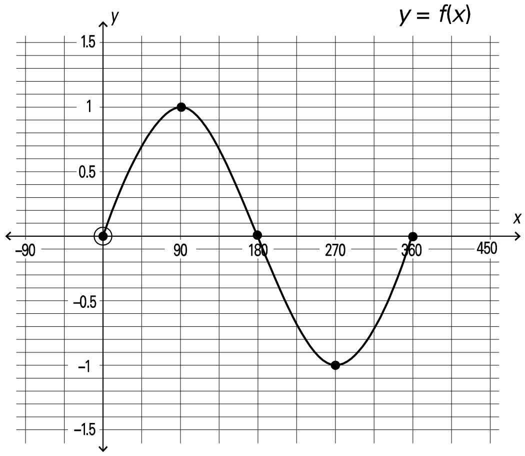
x	$y = \frac{1}{2}f\left(x\right)$
–4	0
–2	2
0	1
2	2
4	0

Students are expected to notice that *a* changes the shape of the function by stretching the graph vertically. They should compare the key points of the original function to the points of the transformed function to generalize the mapping rule (*x*, *y*) → (*x*, *ay*).

When creating the table of values and graph of the transformed function, students may notice that some points did not change even after a vertical stretch is applied. Points that do not change after a transformation has been applied are called **invariant points**. Students were exposed to these points in Pre-calculus 11 when they worked with reciprocal functions (RF11).

For the outcome RF05, students are expected to work with values of a that are negative. It should be noted that a negative a value will cause a reflection in the x-axis, and that the vertical stretch is positive; that is, the vertical stretch is $|a|$. Students should compare relations where $|a| > 1$ to those where $|a| < 1$ to see how these values of a influence the graph.

This may be students' first exposure to the concept of horizontal stretch. Graphing technology could be used to display a base function such as $y = \sin(x)$ with a restricted domain of [0°, 360°]. It is not necessary for students to know the equation of the base function.



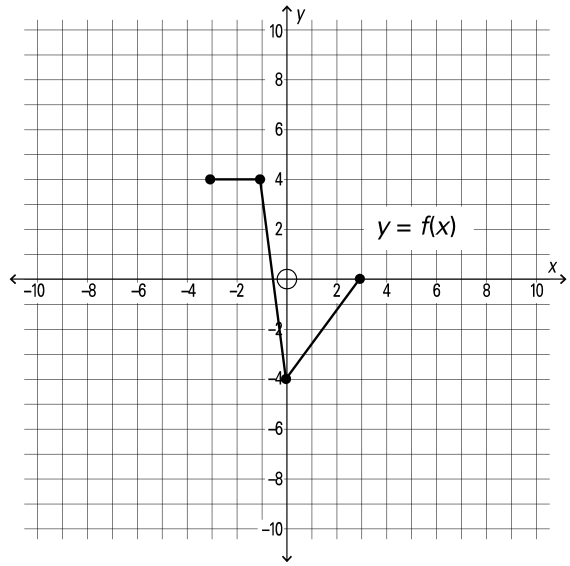
Providing students with the graphs of $y = 3f(x)$ and $y = f\left(\frac{1}{2}x\right)$, they can examine the effect on the graph and the table of values. Students should discuss the general effects of a horizontal stretch on the graph of the base function and the table of values. The general mapping $(x, y) \rightarrow \left(\frac{1}{b}x, y\right)$ from the base function to the transformed function can be inferred from the example. Students should compare relations where $|b| < 1$ to those for which $|b| > 1$. It should also be noted that if $b < 0$, the graph will be stretched as well as reflected in the y-axis. As with vertical stretches, horizontal stretches are always positive.

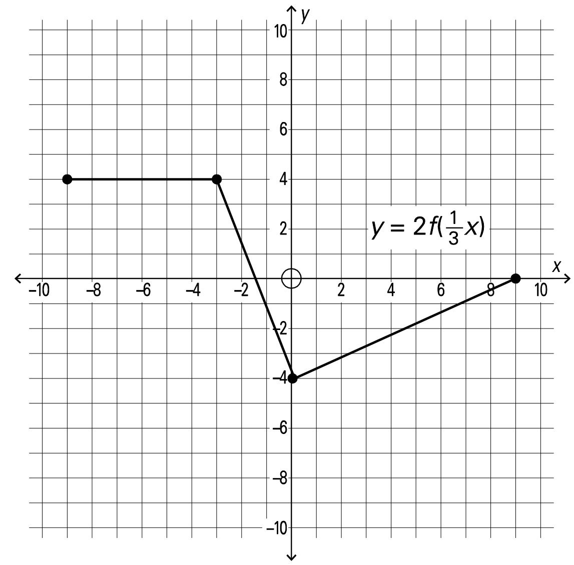
Students should also work with functions that have both a vertical and horizontal stretch. By comparing two graphs, they should be able to determine the effects that the values of a and b have on the graph of $y = g(x) = af(bx)$ when compared to $y = f(x)$.

For example, when given the two graphs below, students are expected to answer questions, such as

- What effect did the 2 have on the graph?
- What effect did the 2 have on the table of values?
- What effect did the $\frac{1}{3}$ have on the graph?

- What effect did the $\frac{1}{3}$ have on the table of values?





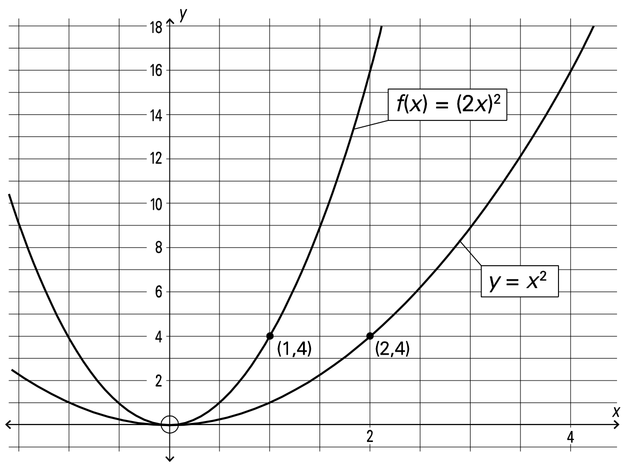
Students are expected to generalize a mapping rule for functions that have a vertical stretch of $|a|$ and a horizontal stretch of $\left|\frac{1}{b}\right|$ as $(x, y) \rightarrow \left(\frac{1}{b}x, ay\right)$.

It should be noted that the focus on vertical and horizontal stretch should be with functions that are bounded.

Students could be introduced to unbounded functions such as $y = x^2$ and $y = |x|$ and explore transformations of these functions with both the vertical and horizontal stretch. For many of the unbounded functions that students have already seen, stretches can be described as either a horizontal stretch or a vertical stretch.

For example $f(x) = (2x)^2 = 4x$ can be considered a horizontal stretch of $\frac{1}{2}$ since it is half as wide as $y = x^2$, or it can be viewed as a vertical stretch of 4 since it can be seen as four times as high as $y = x^2$.

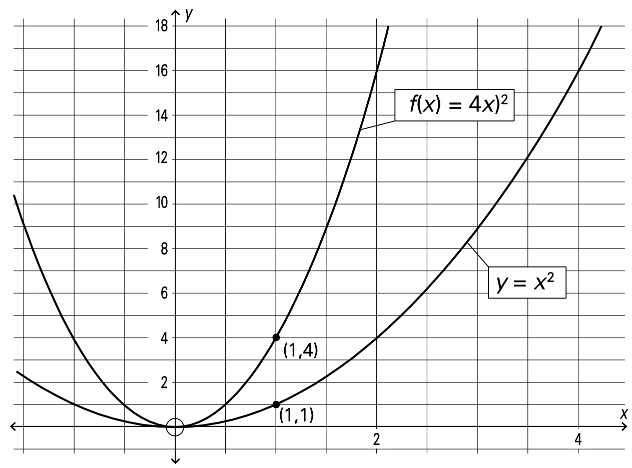
$f(x) = (2x)^2$



Horizontal stretch of $\frac{1}{2}$

$(x, y) \rightarrow \left(\frac{1}{2}x, y\right)$

$f(x) = 4x^2$



Vertical stretch of 4

$(x, y) \rightarrow (x, 4y)$

Given the graph of a function, students should apply a horizontal stretch and/or a vertical stretch to produce the graph of the transformed function $y = af(bx)$.

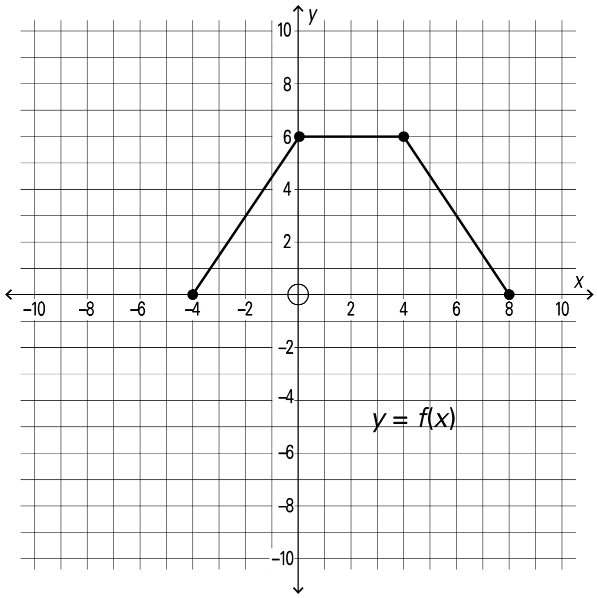
Two possible ways to produce the graph are to

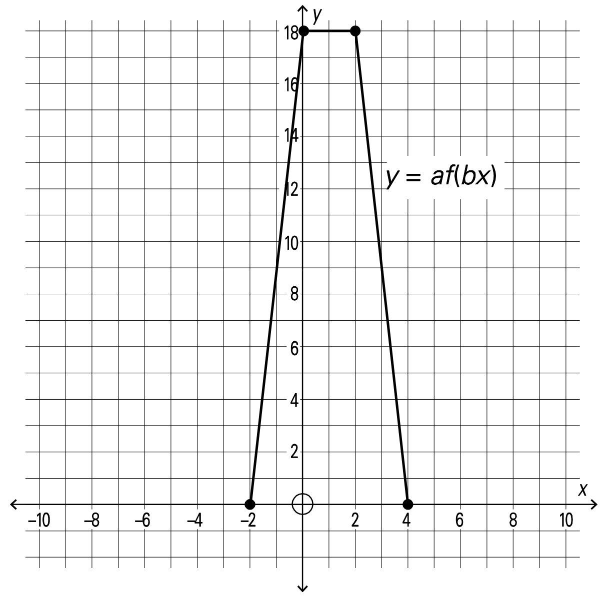
- identify key points and use a mapping rule to create a new table of value
- use knowledge of stretches to transform the graph directly

Given the graph of a transformed function and a base function to compare it with, students should determine the horizontal and vertical stretch, and then write the equation of the transformed function in terms of $y = af(bx)$ or $\frac{1}{a}y = f(bx)$.

The focus of this outcome is not to determine the specific equation for the base function or transformed function such as $\frac{1}{7}y = |4x|$, but to work with general equations like $\frac{1}{7}y = f(4x)$.

The stretches can be determined by comparing the domain and range of the functions. Students should be exposed to graphs such as the following:





The domain of $y = f(x)$ is [−4, 8], which has a span of 12 units. The domain of $y = af(bx)$ is [−2, 4], which has a span of six units. Students should understand that the horizontal stretch is $\frac{1}{2}$, which means $|b| = 2$.

Similarly, the range of $y = f(x)$ is [0, 6], a span of six units, and the range of $y = af(bx)$ is [0, 18], a span of 18 units. Students should understand that the vertical stretch is three, which means $|a| = 3$.

Since there are no reflections, students can conclude that the equation of the transformed graph is $y = 3f(2x)$.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Explain how the graphs of $y = x^2$ and $y = 2x^2$ compare to each other.

- Describe how the functions $f(x)$ and $g(x)$ compare to each other.

x	$f(x)$
1	6
2	7
3	–3
4	4

x	$g(x)$
1	12
2	14
3	–6
4	8

- Describe how the functions $f(x)$ and $g(x)$ compare to each other.

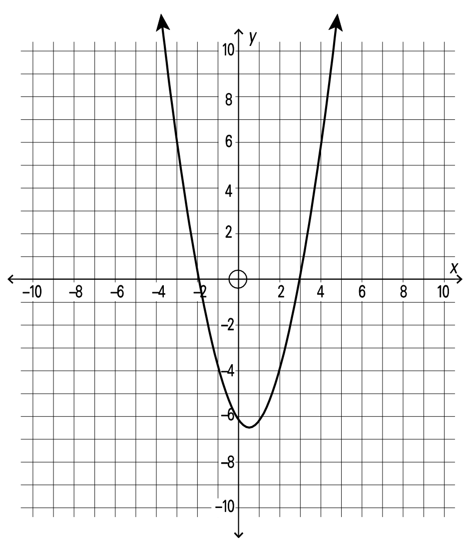
x	$f(x)$
1	6
2	7
3	–3
4	4

x	$g(x)$
2	6
4	7
6	–3
8	4

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

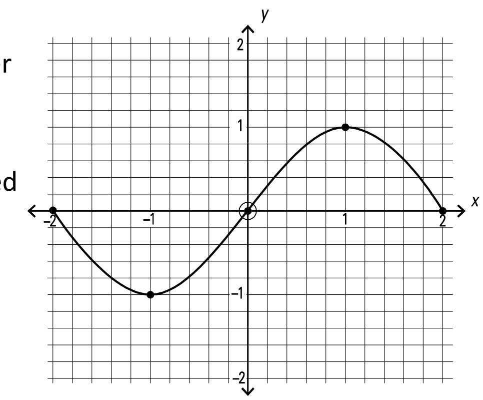
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Explain why the vertical stretch factor for the function $y = af(x)$ is determined using $|a|$.
- Given the graph of the function below, explain which points are invariant points and why they do not change after the application of a vertical stretch.

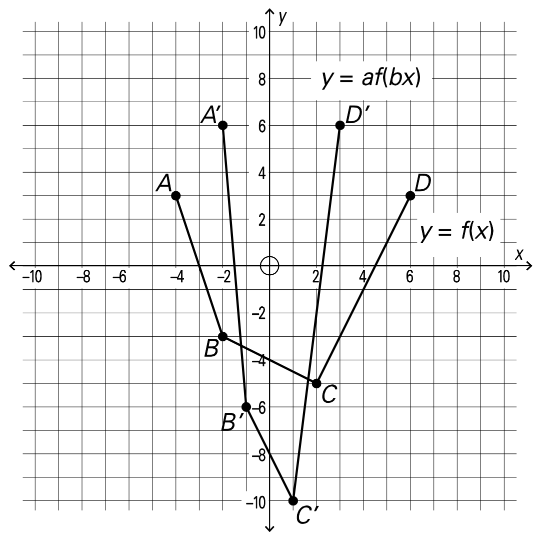


- Explain why the horizontal stretch factor is given by $\left|\frac{1}{b}\right|$ for the function $y = f(bx)$.

- Given the graph of the function to the right, explain which points are invariant points and why they do not change after the application of a horizontal stretch.



- Write the general equation, in terms of *f* (*x*) for the stretched image shown on the same set of axes, by determining the values of *a* and *b* from the graph below.

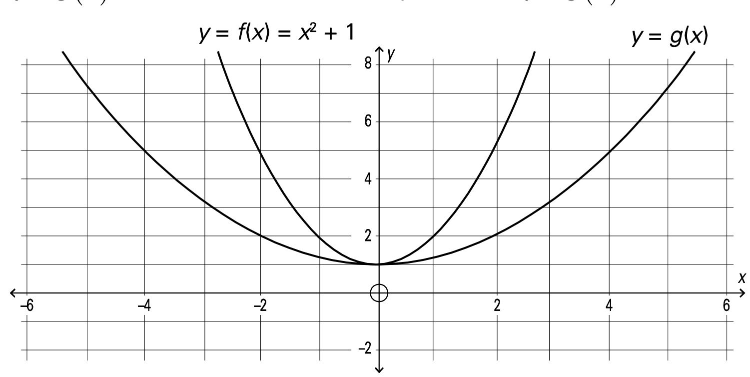


- Explain when the *x*-intercepts are invariant for a stretch, and when the *y*-intercepts are invariant for a stretch. Describe the circumstances under which a point would be invariant for both types of stretches.
- Use mapping notation to describe how the graph of each function can be found from the graph of $y = f(x)$.

  (a) $y = \frac{1}{2}f(x)$

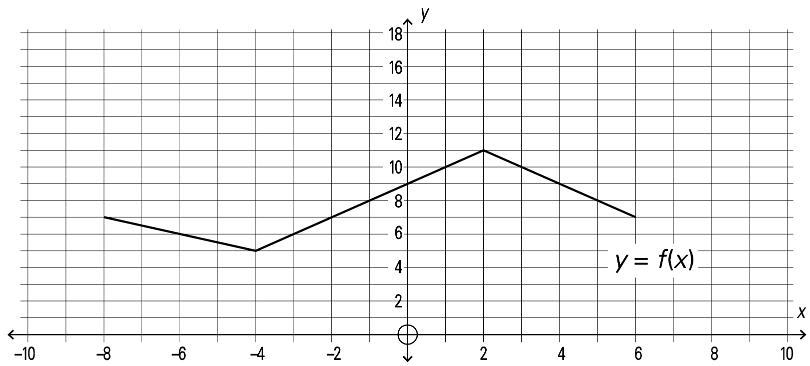
  (b) $y = f(2x)$

- Describe the transformation that must be applied to the graph of $f(x) = x^2 + 1$ to obtain the graph of $y = g(x)$. Then, determine the equation of $y = g(x)$.



- How are the graphs of $y = x^2 - 2x$ and $f(x) y = 3x^2 - 6x$ related? Explain.

- For the following graph of $f(x)$, draw and label the graph of $g(x)$ that shows
  - (a) a horizontal stretch of 2
  - (b) a vertical stretch of 2
  - (c) both a horizontal stretch of 3 and a vertical stretch of $\frac{1}{2}$
  - (d) $g(x) = 3f\left(\frac{1}{2}x\right)$



Planning for Instruction

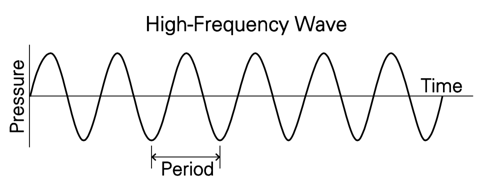
Suggested Learning Tasks

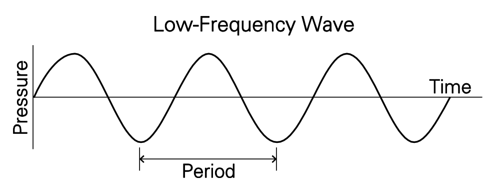
Consider the following sample instructional strategies when planning lessons.

- Using a context, such as the following, students can explore the impact of stretches on a periodic function without any expectation of graphing trigonometric functions except using technology.

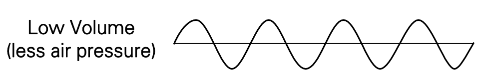
  Sound is a form of energy produced and transmitted by vibrating matter that travels in waves.

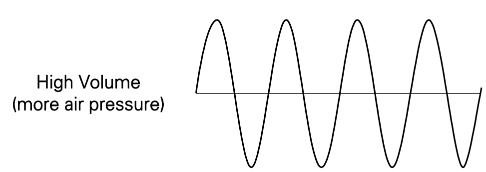
  High-frequency waves have a higher pitch (a soprano voice for example) and low frequency waves a lower pitch (a base voice for example).



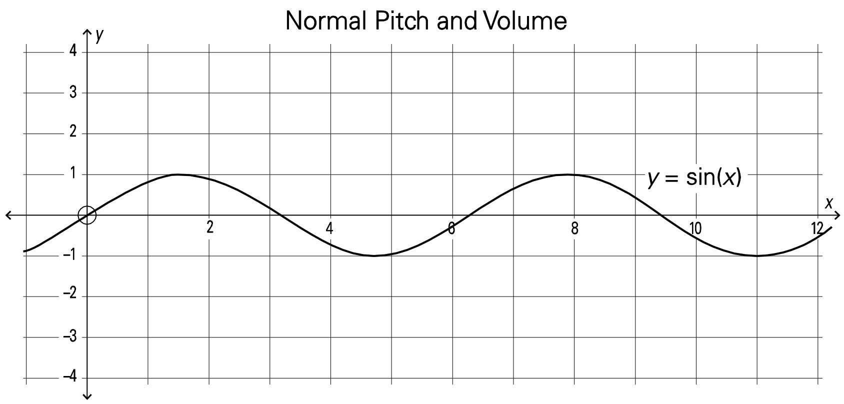


  Higher amplitude waves have a louder volume and lower amplitude waves a softer volume.

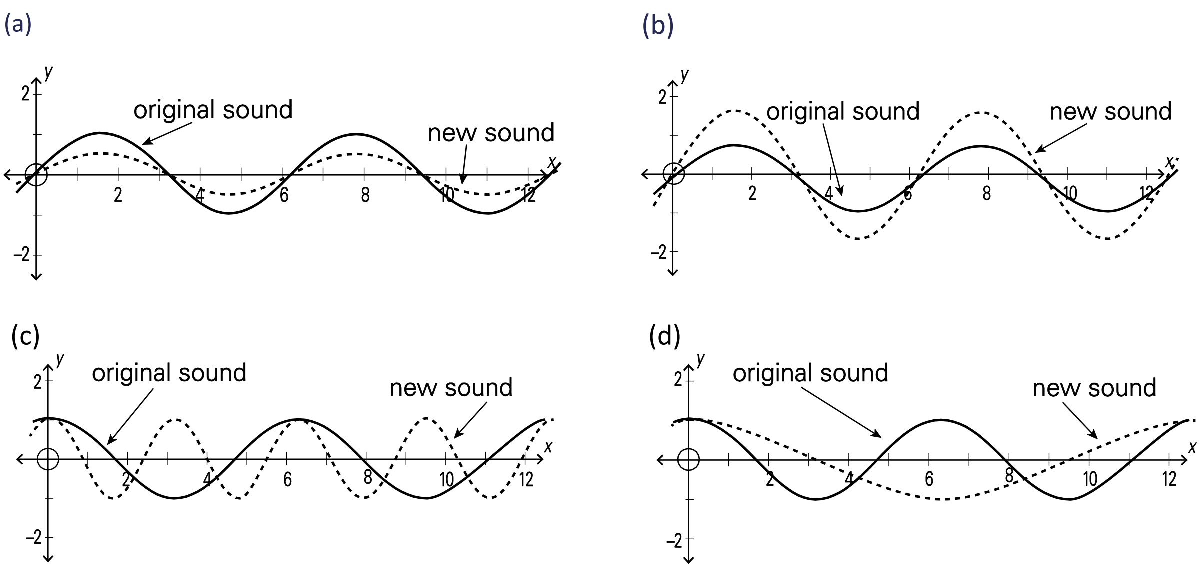




- Assuming the graph of $y = \sin(x)$, illustrated below, represents a normal pitch and volume for a note, ask students to use a graphing utility to (Ensure that graphing utility is set to radian measure.)
  - Graph $y = a\sin(x)$ for different values of *a* and decide what impact the value of *a* has on volume and/or pitch of the note.
  - Graph $y = \sin(bx)$ for different values of *b* and decide what impact the value of *b* has on the volume and/or pitch of the note.



- Using technology, demonstrate the effects of stretching an image for different values of *a* and *b* in various combinations.
- Provide students with pairs of sound waves, such as those shown in the chart below, and ask them to describe how the graphs compare, how the sounds compare, and predict the equation of the function. They could then check their answer using a graphing utility.



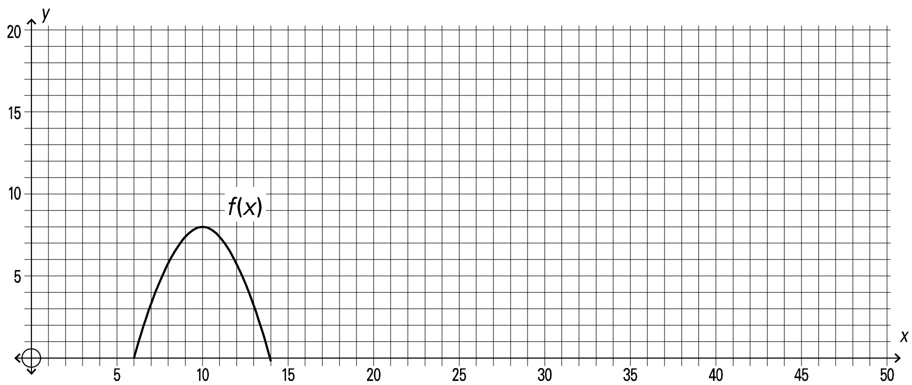
- Provide students with a restricted domain function, *f*(*x*), such as the one shown below (could represent a bridge or a tunnel), and ask them sketch graphs of

  (a) $y = g(x) = 2f(x)$

  (b) $y = h(x) = f\left(\frac{1}{2}x\right)$

  (c) $y = \frac{1}{2}f(2x)$

  (d) $\frac{1}{2}y = f\left(\frac{1}{3}x\right)$



Suggested Models and Manipulatives

- graphing utility
- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- horizontal stretch
- invariant point
- vertical stretch

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 1.2 Reflections and Stretches
    - > Student Book: pp. 16–31

SCO RF04 Students will be expected to apply translations and stretches to the graphs and equations of functions. [C, CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF04.01 Sketch the graph of the function $y-k=af[b(x-h)]$ for given values of *a*, *b*, *h*, and *k*, given the graph of the function $y=f(x)$, where the equation of $y=f(x)$ is not given.

RF04.02 Write the equation of a function, given its graph that is a translation and/or stretch of the graph of the function $y=f(x)$.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. (PC11)** **RF11** Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). (PC11)**	**RF04** Students will be expected to apply translations and stretches to the graphs and equations of functions.

* M11—Mathematics 11

** PC11—Pre-calculus 11

Background

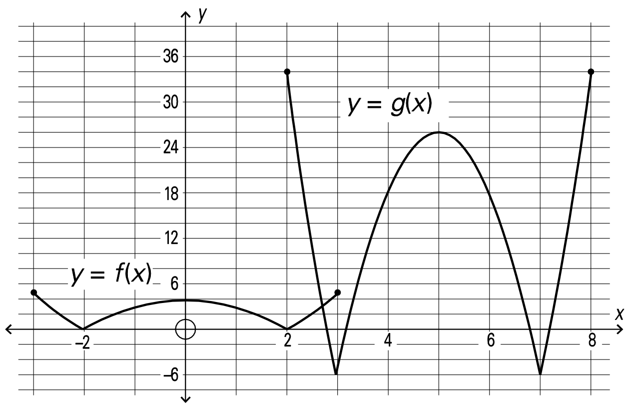
This outcome will focus on using the notation $y=af\left\lfloor b\left(x-h\right)\right\rfloor+k$ or $y-k=af\left\lfloor b\left(x-h\right)\right\rfloor$ to sketch graphs of translations and stretches of functions.

Students are expected to compare the graph of a base function with the graph of a transformed function, identify all transformations, and state the equation of the transformed function. The focus should be on functions that have a bounded domain and range.

To determine the respective stretch factors, students will compare the distances between key points both vertically and horizontally.

When students sketch or analyze graphs of functions with translations and stretches, they will need to apply or **consider the stretch first**, and then the translation. Invariant points will need to be examined.

In an example such as the following, students should find the equation of $y = g(x)$ as a transformation of $y = f(x)$.



Domain of $f(x)$ is [–3, 3] for a width of 6.
Domain of $g(x)$ is [2, 8] for a width of 6.
Therefore, there is no horizontal stretch.

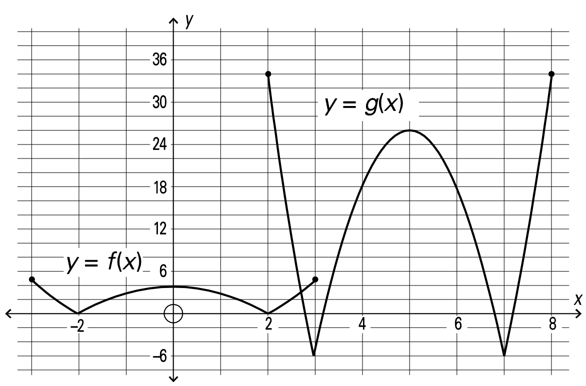
Range of $f(x)$ is [0, 5] for a range of 5.
Range of $g(x)$ is [–6, 34] for a range of 40.
Therefore, there is a vertical stretch of 8.

Considering the point (3, 5) on $f(x)$, its counterpart on $g(x)$ is (8, 34). The vertical stretch of 8 would have created the point (3, 40). Therefore, there was a horizontal translation of +5 and a vertical translation of –6.

$$g(x) = 8f(x-5) - 6$$

When students analyze a graph and a transformed graph, they may choose points that are located on both axes to determine the horizontal and vertical translations.

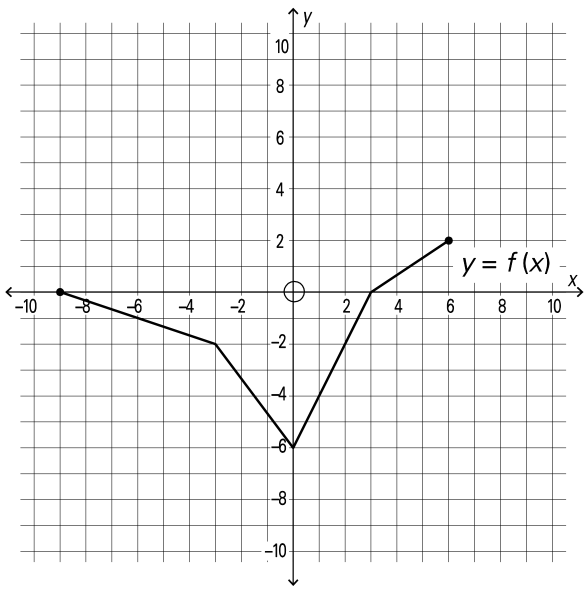
Example:



The points (–2, 0) and (2, 0) on $f(x)$ will not be impacted by a vertical stretch, so any vertical change will be due to a vertical translation. Therefore, there is a vertical translation of –6.

The point (0, 4) on $f(x)$ will not be impacted by a horizontal stretch, so any horizontal change will be due to a horizontal translation. Therefore, there is a horizontal translation of 5.

Using the graph of a function, such as $y=f\left(x\right)$, shown below, students are expected to identify the transformations at work and graph a transformed function such as $y=2f\left\lfloor 3\left(x-1\right)\right\rfloor+4$.



x	f(x)
–9	0
–3	–2
0	–6
3	0
6	2

$y=2f\left\lfloor 3\left(x-1\right)\right\rfloor+4$

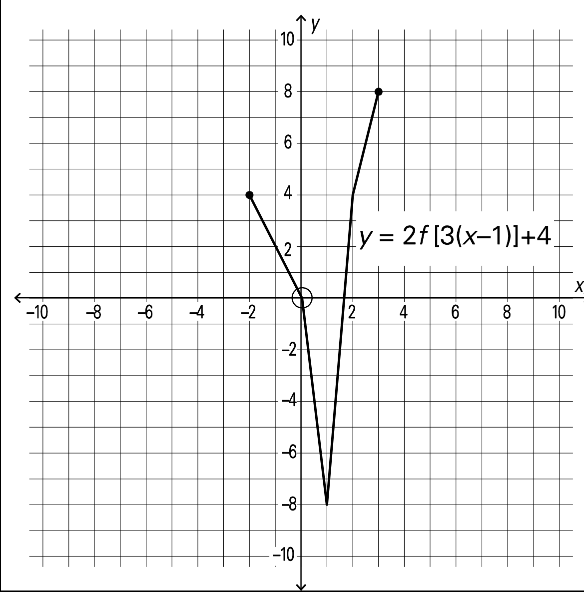
Horizontal stretch of $\frac{1}{3}$

Vertical stretch of 2

Horizontal translation of +1
Vertical translation of +4

$\left(x,y\right)\rightarrow\left(\frac{1}{3}x+1,2y+4\right)$

x	g(x)
–2	4
0	0
1	–8
2	4
3	8



Since stretches and reflections are the result of multiplication and translations are the result of addition, the stretches and reflections are applied first. They apply the transformations to each point to produce the transformed graph.

In some cases the horizontal stretch factor, *a*, must first be factored from the binomial before indicating the horizontal translation.

Example:

$y=\frac{1}{2}f(3x-6)+5$

$y=\frac{1}{2}f\left[3(x-2)\right]+5$

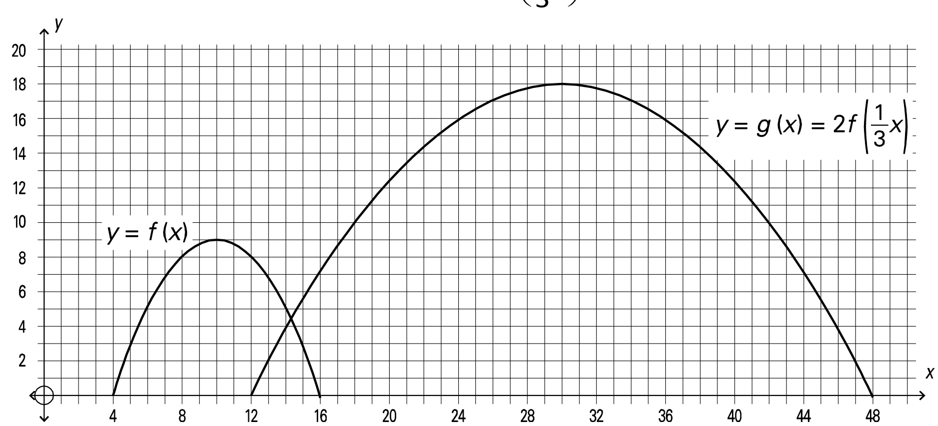
Horizontal translation: 2
Vertical translation: 5
Horizontal stretch $\left(\frac{1}{b}\right):\frac{1}{3}$

Vertical stretch: $\frac{1}{2}$

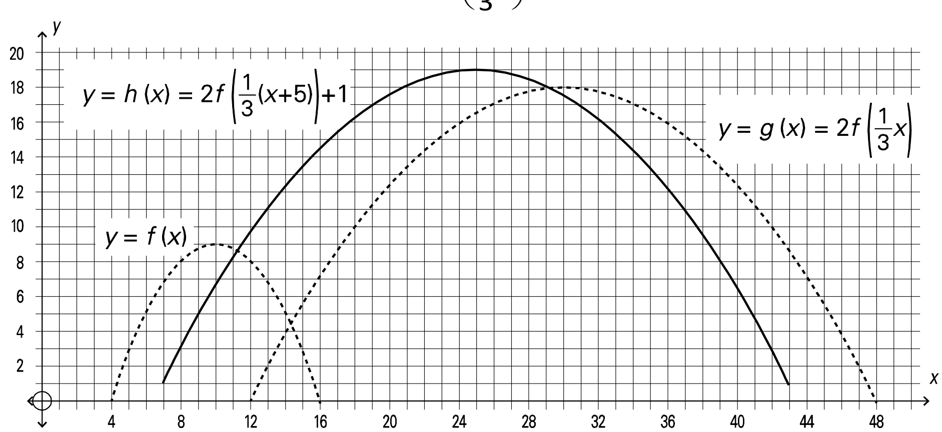
Students are expected to note that the stretches and reflections can be applied in any order, as long as it is before the translations. Similarly, the order in which the translations are applied is not important, as long as they are applied after the stretches and reflections.

Thus, for the example $y=h(x)=2f\left[\frac{1}{3}(x+5)\right]+1$, students would first apply a vertical stretch of two and a horizontal stretch of three. These stretches would then be followed by a vertical translation of one and a horizontal translation of –5.

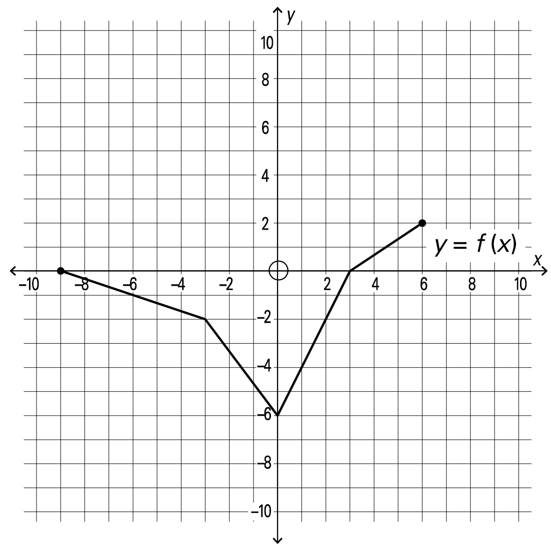
For example, the arch shown below, $y=f(x)$, would be first stretched to twice its height and three times its width. Obtaining the $y=g(x)=2f\left(\frac{1}{3}x\right)$.



Then this stretched version, $g(x)=2f\left(\frac{1}{3}x\right)$, would be translated left five and up one.



Students could also generate a table of values for $y = h(x) = 2f\left[3(x-1)\right] + 4$ based on the key points for a function, such as the one shown below, and use the mapping $(x, y) \rightarrow \left(\frac{1}{3}x + 1, 2y + 4\right)$ to generate a new table of values.

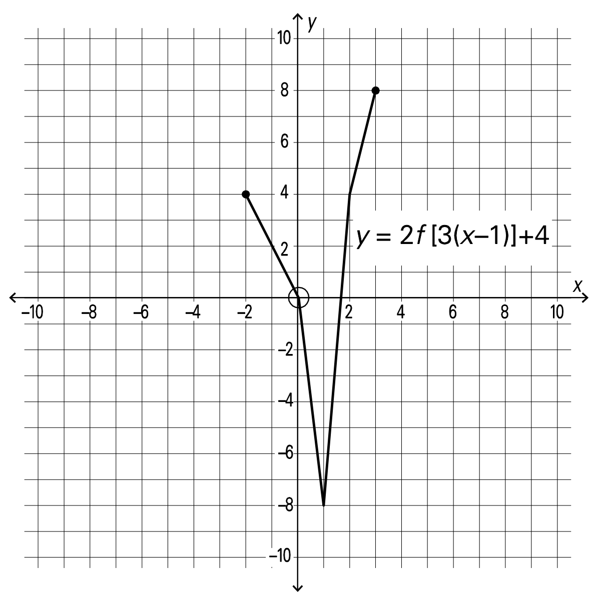


Key points on original graph

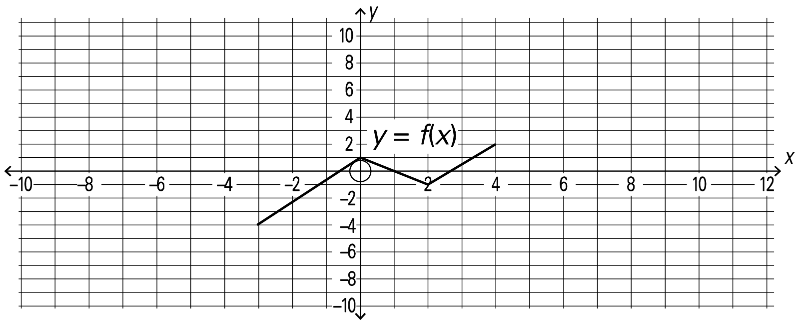
x	$y = f(x)$
–9	0
–3	–2
0	–6
3	0
6	2

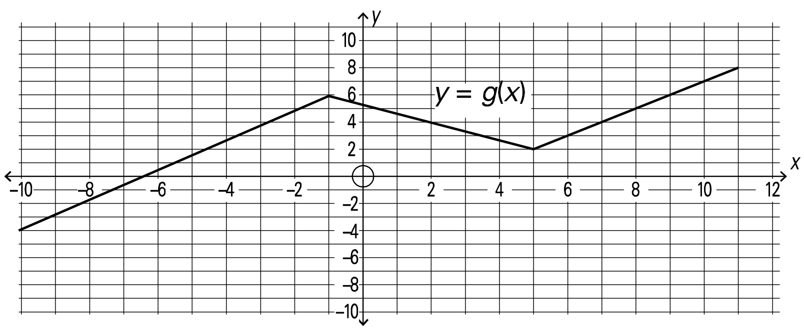
Key points on transformed graph

x	$y = h(x)$
$\frac{1}{3}(-9)+1=-2$	$2(0)+4=4$
$\frac{1}{3}(-3)+1=0$	$2(-2)+4=0$
$\frac{1}{3}(0)+1=1$	$2(-6)+4=-8$
$\frac{1}{3}(3)+1=2$	$2(0)+4=4$
$\frac{1}{3}(6)+1=3$	$2(2)+4=8$



Using a graph, such as the one shown below, students are expected to determine the specific equation for the image of $y=f(x)$ in the form $y=g(x)=af\lfloor b(x-h)\rfloor+k$ and to determine the mapping rule that describes $g(x)$ as a transformation of $f(x)$.





Assessment, Teaching, and Learning

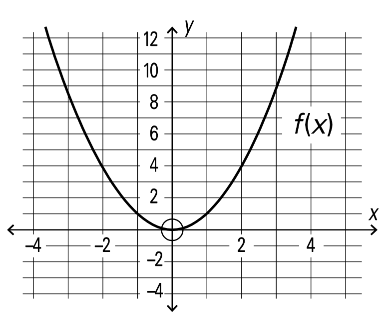
Assessment Strategies

Assessing Prior Knowledge

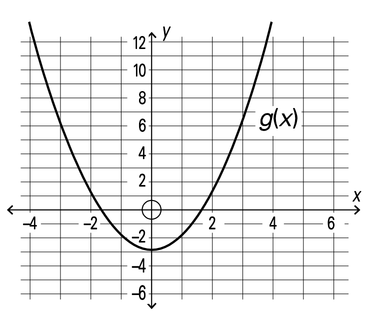
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

State the equation of each of the following:

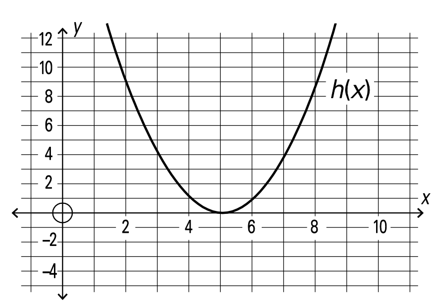
(a)



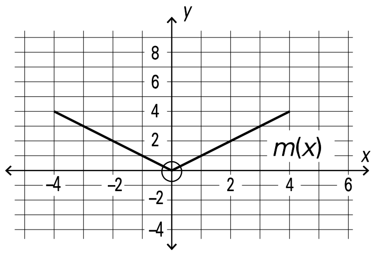
(b)



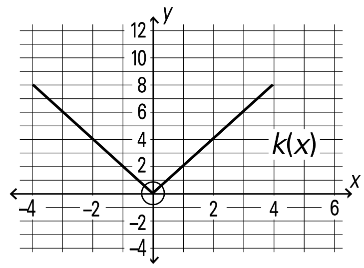
(c)



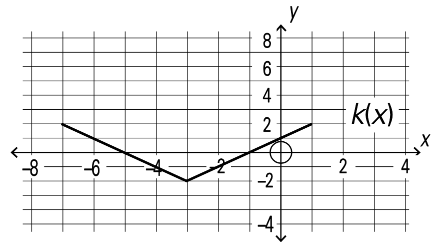
(d)



(e)



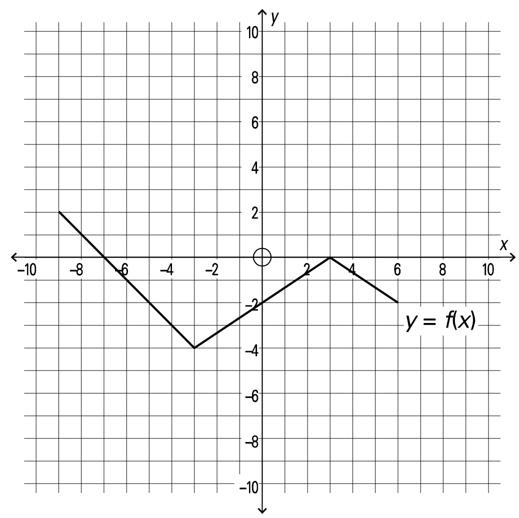
(f)



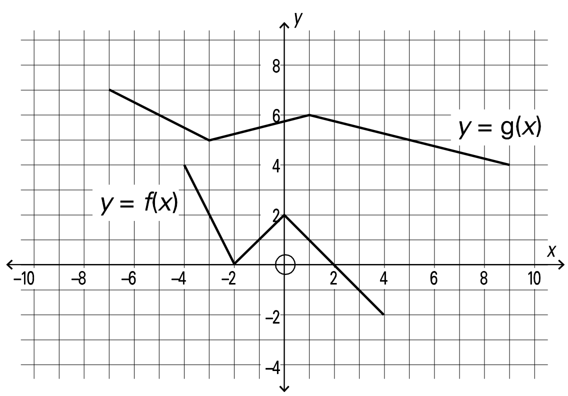
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Use the graph of $y = f(x)$, to the right, to draw the graph of $y = g(x) = 3f\left\lfloor \frac{1}{2}(x+1) \right\rfloor - 4.$



- Determine what transformations were done to $f(x)$ to obtain the graph of $g(x)$ below. Write your answer in the form of $y = g(x) = af\lfloor b(x-h) \rfloor + k.$



- The function $y = f(x)$ is transformed to the function $y = g(x) = 3f(2x+8) - 1.$ Describe the transformation(s) in words.
- The point (2, 5) is on the graph of $y = f(x)$. What is its image point for each of the following?

  (a) $y = g(x) = 4f(x-6) + 2$

  (b) $y = p(x) = f\left(\frac{1}{3}x\right) - 4$

- Write an equation of the function, $k(x)$, that results by applying the following set of transformations to the graph of $y=f(x)=\frac{x}{x+1}$.

  (a) horizontal stretch of $\frac{1}{5}$

  (b) vertical stretch of 4

  (c) translated 2 unit to the right

  (d) translated 4 units down

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Use technology to explore the effect of *a*, *b*, *h*, and *k* on the graph of a given equation $y=af\lfloor b(x-h)\rfloor+k$.
- For a few different equations, such as the ones shown below, have students identify $f(x)$, *a*, *b*, *h*, and *k*. Then have them sketch the graph of the given equation and the graph of $f(x)$ on the same set of axes.

  (a) $y=\left|\frac{1}{4}(x+2)\right|-1$

  (b) $y=5|3x|+4$

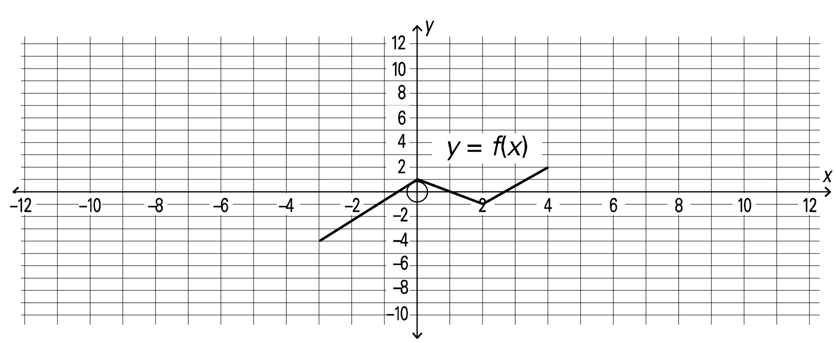
  (c) $y=\frac{1}{2}|x-1|$

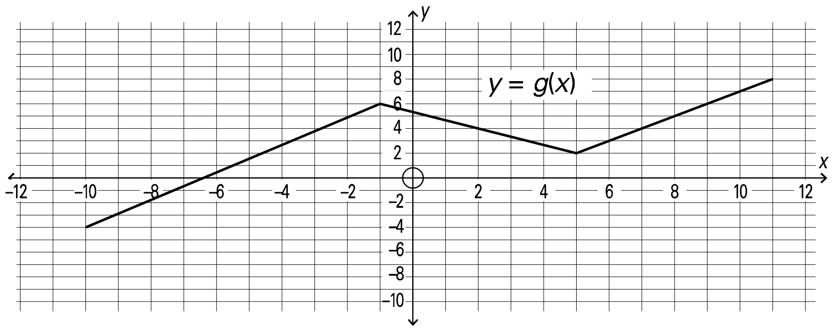
  (d) $y=2(x+3)^2$

  (e) $y=\left(\frac{1}{2}x\right)^2+5$

  (f) $y=\left[\frac{1}{2}(x-3)\right]^2-1$

- Remind students of the importance of the order of operations. Since stretches are the result of multiplication and translations are the result of addition, the stretches are applied first. They apply the transformations to each point to produce the transformed graph. Once students have considered reflections as well as stretches and translations, they may find the Acronym RST helpful.
- Ask students to describe the steps in graphing a function of the form $y=af\lfloor b(x-h)\rfloor+k$, where the shape of the graph of $f(x)$ is well-known.
- The following prompts could be used to initiate discussion when considering a bounded function and its transformations such as shown below.





- What is the domain of each function? How will this help in identifying the horizontal stretch?

  Example:

  Students would be expected to compare the domain of $f(x)$ with that of the transformed function $g(x)$. If the domain of $f(x)$ is [−3, 4] which has a span of 7 units, and that the domain of $g(x)$ is [−10, 11] which has a span of 21 units, then there has been an horizontal stretch of 3.
- What is the range of each function? How will this help in identifying the vertical stretch?

  Example:

  Students would be expected to compare the range of $f(x)$ with that of the transformed function $g(x)$. If the range of $f(x)$ is [−4, 2] which has a span of 6 units, and that the domain of $g(x)$ is [−4, 8] which has a span of 12 units, then there has been a vertical stretch of 2.
- What translations are implied by the key points?

Key points of original function $y = f(x)$

x	***f*(*x*)**
–3	–4
0	1
2	–1
4	2

Key points where stretches have occurred (vertical stretch of 2, horizontal stretch of 3)

$y = k(x) = 2f\left(\frac{1}{3}x\right)$

3*x*	**2*y***
–9	–8
0	2
6	–2
12	4

Key points from transformed function

x	**y = g(*x*)**
–10	–4
–1	6
5	2
11	8

Comparing the key points from the transformed function, $y = g(x)$, with the key points where stretches have already occurred, $y = k(x)$ reveals that 1 is subtracted from the *x*-values (horizontal translation of –1) and that 4 is added to the *y*-values (vertical translation of +4).

Thus the mapping rule for transforming $y = f(x)$ to $y = g(x)$ is $(x, y) \rightarrow (3x - 1, 2y + 4)$ and

$y = g(x) = 2f\left[\frac{1}{3}(x+1)\right] + 4.$

- Time must be spent explaining why parentheses and brackets are in the equation and not in the mapping rule.

Suggested Models and Manipulatives

- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- horizontal stretch
- horizontal translation
- image point
- mapping rule
- transformations
- vertical stretch
- vertical translation

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 1.3 Combining Transformations
    - Student Book: pp. 32–43

SCO RF05 Students will be expected to demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections in the, *x*-axis, *y*-axis, and line *y* = *x*. [C, CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF05.01 Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection in the *x*-axis, the *y*-axis, or the line *y* = *x*.

RF05.02 Sketch the reflection of the graph of a function $y = f(x)$ in the *x*-axis, the *y*-axis, or the line *y* = *x*, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

RF05.03 Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$ in the *x*-axis, the *y*-axis, or the line *y* = *x*.

RF05.04 Sketch the graphs of the functions $y = -f(x)$, $y = f(x)$, and $x = f(\mathrm{y})$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$, is not given.

RF05.05 Write the equation of a function, given its graph that is a reflection of the graph of the function $y = f(x)$ in the *x*-axis, the *y*-axis, or the line *y* = *x*.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. (PC11)**	**RF05** Students will be expected to demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections in the, *x*-axis , *y*-axis, and line *y* = *x*.

* M11—Mathematics 11

** PC11—Pre-calculus 11

Background

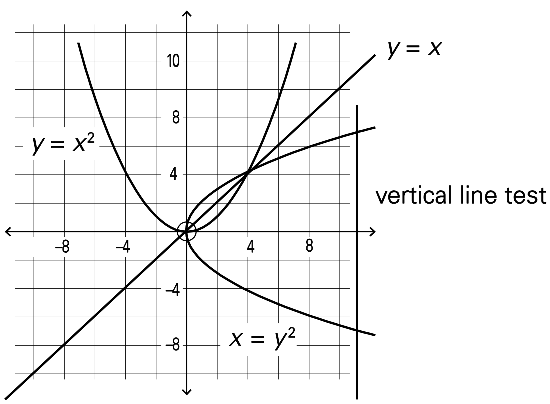
This outcome will focus on exploring reflections in the *x*-axis, in the *y*-axis, and in the line *y* = *x*.

Students should analyze the relationship between the coordinates of an ordered pair and those of its given reflection and develop the following rules:

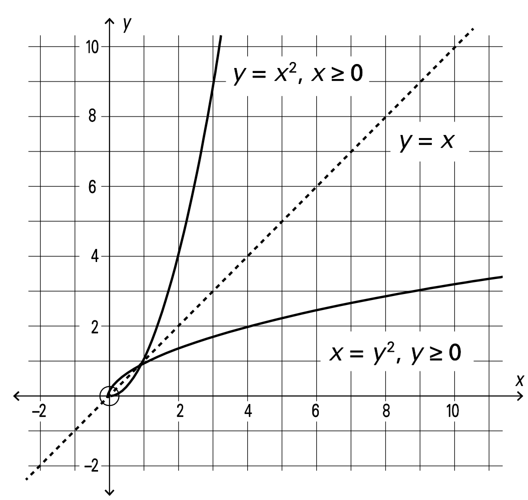
- A reflection in the *x*-axis will be represented as $y=-f(x)$, since the values for *y* will be additive inverses of each other or $(x,y)\to(x,-y)$.
- A reflection in the *y*-axis will be represented as $y=f(-x)$, since the values for *x* will be additive inverses of each other or $(x,y)\to(-x,y)$.
- A reflection in the line *y* = *x* will be represented as $x=f(y)$, since the *x*- and *y*-values will be reversed and $(x,y)\to(\text{y,x})$. This is the inverse function, explored further in the next outcome. Technically the notation requires drawing a new set of axes, with *x*-values on the vertical axis and *y*-values on the horizontal axis. Some of this confusion can be alleviated by introducing the notation $f^{-1}(x)$ as soon as possible.

For a relation to be a function, one *x*-value can correspond to only one *y*-value. This can be checked quickly by drawing a vertical line through the graph to determine if at any point, more than one point lies on the line (vertical line test). Students were introduced to this concept in Mathematics 10 when functions were introduced.

The reflection of a function is not always a function. For example, reflecting the graph of $y=x^2$ in the line will give the graph of $x=y^2$. A vertical line drawn anywhere through this graph crosses two points indicating that each *x*-value corresponds to two *y*-values. Therefore, $x=y^2$ is not a function.



If the domain of $y=x^2$ is restricted to values that are greater than or equal to zero, its inverse will be a function with each *x*-value corresponding to only one *y*-value as illustrated in the graph to the right.



The **vertical line test** is extended in this outcome to the **horizontal line test** in which a horizontal line is drawn through the original function, and if the line crosses more than one point, the function's inverse will fail a vertical line test and will not be a function.

Students should be exposed to multiple transformations applied to a single function. When sketching, order of applying these multiple transformations is important. The **reflections and stretches** *must* be applied first, followed by any **translations**. Similarly, when analyzing graphs, the transformations should be considered in this same order.

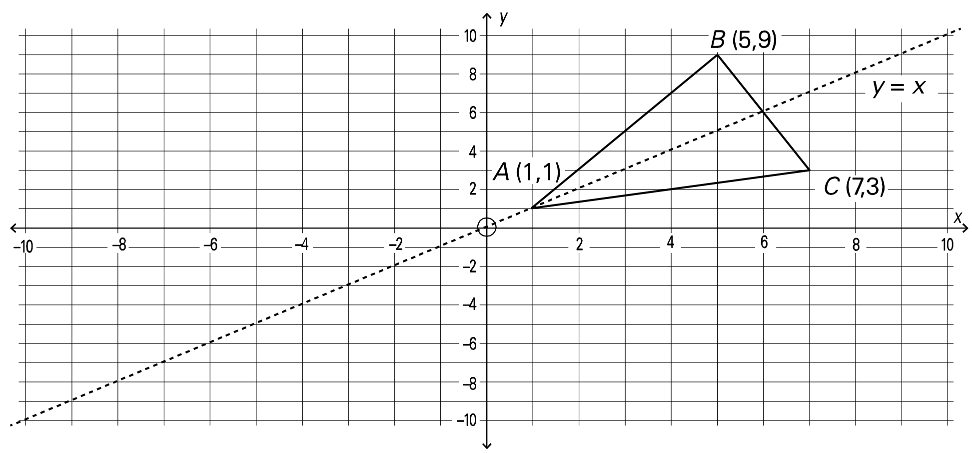
Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

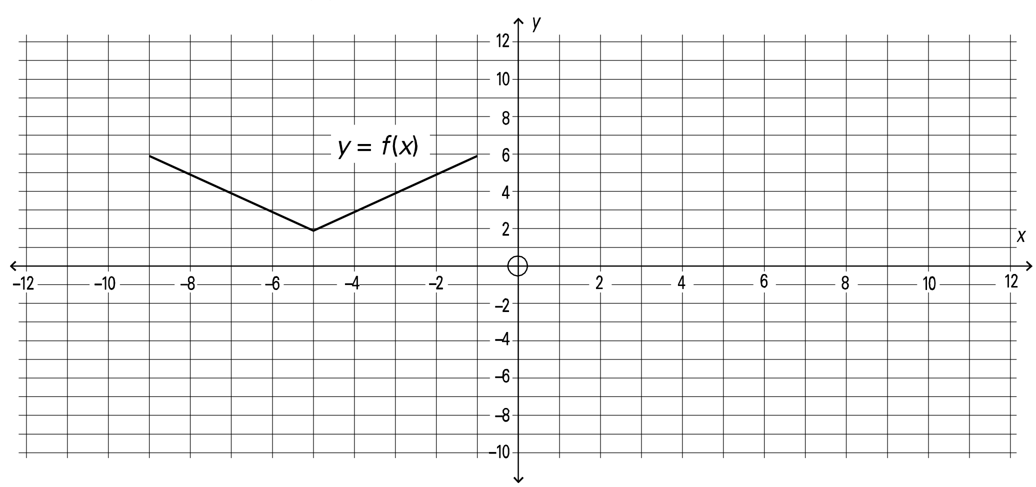
- Draw the image of the triangle ΔABC and label the vertices, when
  - (a) reflected in the y-axis
  - (b) reflected in the x-axis



Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

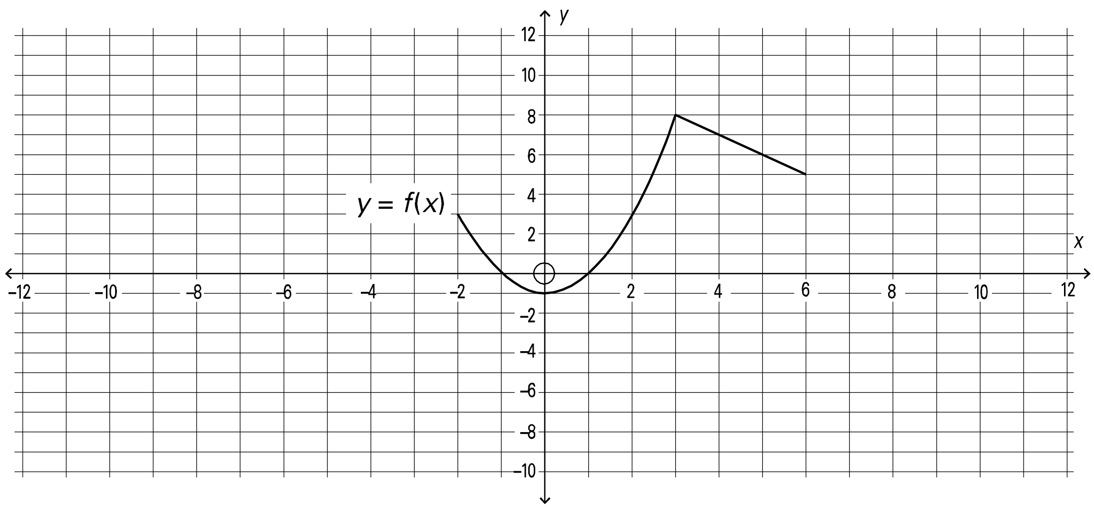
- Use the graph of $y = f(x)$ below to draw the graph of



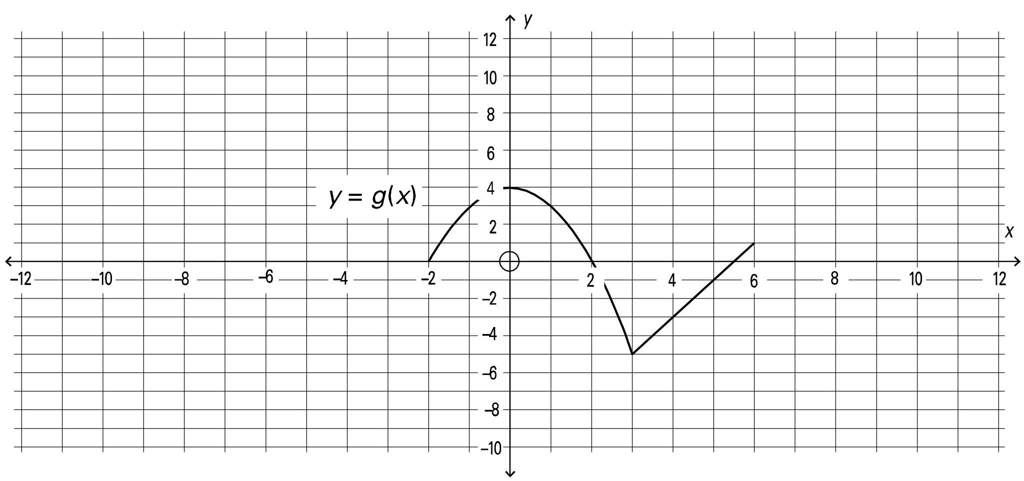
(a) $g(x) = -f(x)$ (c) $k(x) = -f(-x)$

(b) $h(x) = f(-x)$ (d) $x = f(y)$

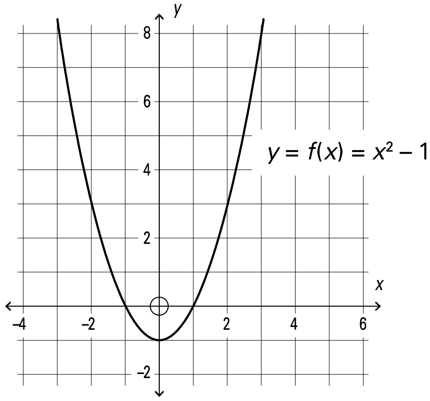
- Sketch the graph of $y = 2 - 3x$, then sketch its reflection in the line $y = x$. Is the new graph a function? Find the equation of the new graph.
- Sketch the graph of $y = |x|$, then sketch its reflection in the line $y = x$. Is the new graph a function? Find the equation for the new graph.
- Use the graph of $y = f(x)$ below to draw the graph of $y = g(x) = -2f\left(\frac{1}{3}x\right)$.



- Use the graph of $y = g(x)$ below to draw the graph of $y = h(x) = g(-2x) + 1$.



- The graph of the function $f(x)=x^2-1$ is shown below. Sketch its reflection in the x-axis and state the equation of the reflection.

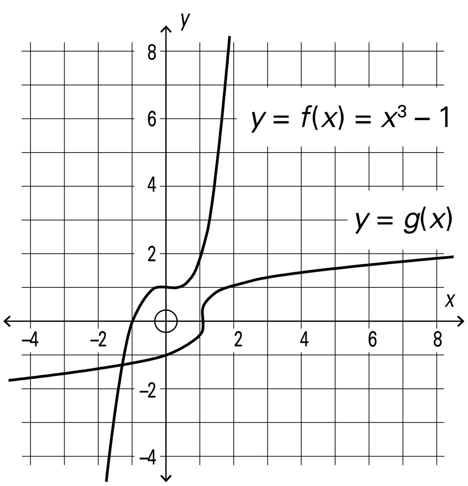


- Use mapping notation to describe how the graph of each function can be found from the graph of $y=f(x)$.

  (a) $y=g(x)=-\frac{1}{2}f(x)$

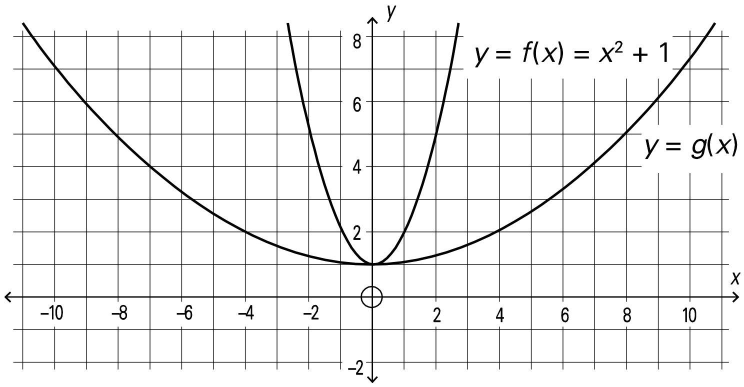
  (b) $h(x)=f(-3x)$

- Describe the transformation that must be applied to the graph of $f(x)=x^3+1$ to obtain the graph of $g(x)$. Then, determine the equation of $g(x)$.



- Describe the transformation(s) that must be applied to the graph of $f(x)=x^2+1$ to obtain the graph of $g(x)$. Then, determine the equation of $g(x)$.

(a)



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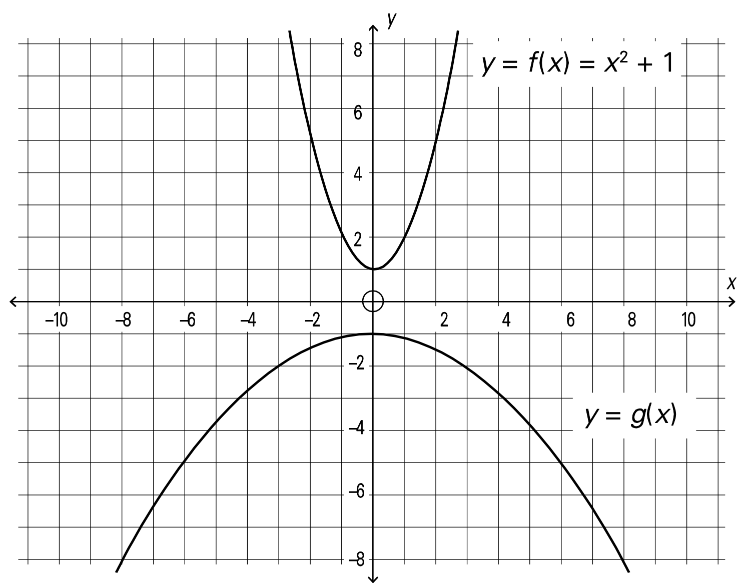


- How are the graphs of $f(x)=2x^2-x$ and $g(x)=x-2x^2$ related? Explain.

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Provide graphs of various types of functions and have students run the horizontal line test on each to determine if the inverse will be a function or not. Have students find the inverse of the relation by interchanging the *x*- and *y*-coordinates of key points on the graph and run the vertical line test on the new graph to confirm their prediction.

- Use a table of values to make the connection between a function and its reflection in either the x-axis or the y-axis. Have students make a conjecture about the mapping that takes place for each type of reflection.
- Use a reflection in the line $y = x$ as an introduction to the outcome on inverse relations. Suggest order of transformations (RST or SRT) when teaching.
- Use graphing technology with students to explore how $f(x) = af(x)$, $a < 0$ is related to $y = f(x)$, and how $f(x) = f(bx)$, $b < 0$ is related to $y = f(x)$.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- function
- reflection in line $y = x$
- reflection in x-axis
- reflection in y-axis

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 1.2 Reflections and Stretches
    - > Student Book: pp. 16–31

SCO RF06 Students will be expected to demonstrate an understanding of inverses of relations. [C, CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF06.01 Explain how the graph of the line *y* = *x* can be used to sketch the inverse of a relation.
RF06.02 Explain how the transformation $(x, y) \to (y, x)$ can be used to sketch the inverse of a relation.
RF06.03 Sketch the graph of the inverse relation, given the graph of a relation.
RF06.04 Determine if a relation and its inverse are functions.
RF06.05 Determine restrictions on the domain of a function in order for its inverse to be a function.
RF06.06 Determine the equation and sketch the graph of the inverse relation, given the equation of a linear or quadratic relation.
RF06.07 Explain the relationship between the domains and ranges of a relation and its inverse.
RF06.08 Determine, algebraically or graphically, if two functions are inverses of each other.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **RF03** Students will be expected to analyze quadratic functions of the form $y = a(x-p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, and *x*- and *y*-intercepts. (PC11)** **AN03** Students will be expected to solve problems that involve radical equations (limited to square roots). (PC11)**	**RF06** Students will be expected to demonstrate an understanding of inverses of relations.

* M11—Mathematics 11
** PC11—Pre-calculus 11

Background

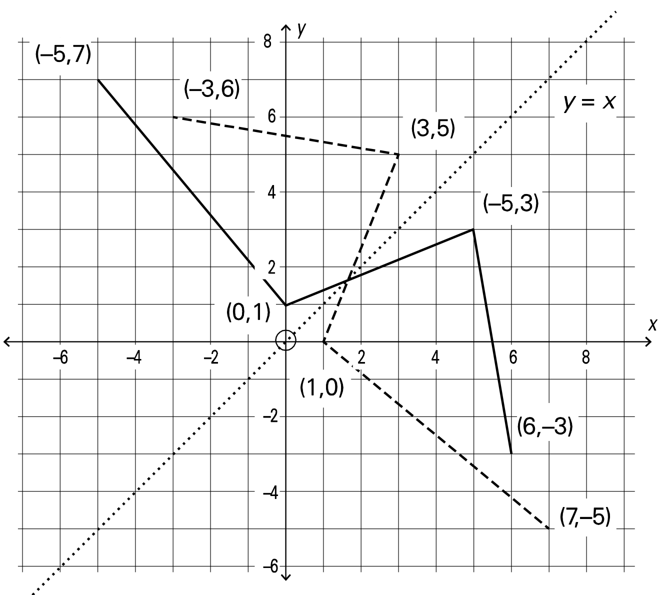
As students begin work with inverse relations, they will explore the relationship between the graph of a relation and its inverse, and determine whether a relation and its inverse are functions. Students produce the graph of an inverse from the graph of the original relation, restrict the domain of a function so that its inverse is also a function, and determine the equation for f^{-1} given the equation for *f*.

When the graph of a function is reflected in the line $y = x$, the resulting graph is the inverse of the given function. When the *x*- and *y*-values of a function are interchanged, they become the *x*-values and *y*-values of the inverse. If (3, 4) is a point on the function, (4, 3) is a point on the inverse. (*x*, *y*) of the function maps onto (*y*, *x*) of the inverse.

The domain of the original graph becomes the range of the inverse, and the range of the original graph becomes the domain of the inverse. For a function f with domain *A* and range *B*, the inverse function, if it exists, is denoted by f^{-1} and has domain *B* and range *A*.

The inverse equation can be found by interchanging *x* and *y*; that is, the inverse of $y = x^2$ is $x = y^2$. The inverse of $y = x^2$ is not a function because for each value of the independent variable *x*, other than 0, there are two values for the dependent variable $y = x^2$.

If a function is already graphed, its inverse relation can be graphed by plotting the ordered pairs with *x*-values and *y*-values interchanged. This can be done by hand, or by using technology.



The inverse equation for the quadratic $y = x^2$ is $x = y^2$. It can also be written as $y = \pm\sqrt{x}$, where $y = +\sqrt{x}$ is the equation of the upper branch of its graph and $y = -\sqrt{x}$ is the equation of the lower branch. If the domain of the original graph, $y = x^2$, is restricted to either $x \geq 0$ or $x \leq 0$, then the inverse will be $y = +\sqrt{x}$ or $y = -\sqrt{x}$, respectively, and each will be a function. The inverse of $y = x^2$ is not a function unless it is expressed as two separate functions.

In Pre-calculus 11 students solved radical equations (A03) but were not introduced to the graphs of radical functions.

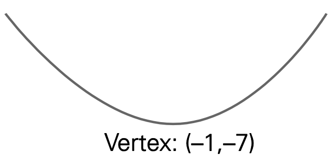
After seeing the graphical relationship between a relation and its inverse, the goal is for students to take a graph of a relation and produce a graph of its inverse on the same set of axes without having to go through the process of producing tables of values. To graph the inverse, students may take the inverse of key points of the relation or they may fold the graph along the line $y = x$.

Students are required to restrict the domain of functions to ensure that the inverse is also a function. Examples should be limited to linear and quadratic functions. Students should understand that the only linear functions whose inverses are not functions are those that have graphs that are horizontal lines.

In Pre-calculus 11, students converted the equations of quadratic functions from standard form $y = ax^2 + bx + c$, to vertex form, $y = a(x-p)^2 + q$, by completing the square (RF04) and graphed quadratic functions in vertex form (RF03). Students are expected to convert forms of quadratic equations, in order to identify the vertex of the function so that its domain may be restricted. For example, to find the inverse of $y = 2x^2 + 4x - 5$, it would be best for students to change this to vertex form before finding the inverse of the function.

$$y = 2x^2 + 4x - 5$$
$$y = 2(x^2 + 2x + 1) - 5 - 2(1)$$
$$y = 2(x+1)^2 - 7$$

Sketching a quick graph of this function:



Considering this sketch of $y = 2(x+1)^2 - 7$, we note that its inverse would not be a function unless its domain is restricted. Specifically, the domain of the function $y = 2x^2 + 4x - 5$ must be restricted to $x \le -1$ or $x \ge -1$ in order for its inverse to be a function. The graph also reveals that the range of $y = 2x^2 + 4x - 5$ is $\{y \mid y \ge -7, y \in R\}$, which means that the domain of the inverse function will be $\{x \mid x \ge -7, x \in R\}$. (Knowing the domain of $f^{-1}(x)$ before determining the equation of $f^{-1}(x)$ provides a method of checking the work.)

By restricting the domain of the function, two possible inverse functions can be written.

For $f(x)$:

$y = 2x^2 + 4x - 5,\ x \le -1$

or $f(x) = y = 2(x+1)^2 - 7,\ x \le -1$

$f^{-1}(x)\!: x = 2(y+1)^2 - 7,\ y \le -1$

$x = 2(y+1)^2 - 7$

$\frac{x+7}{2} = (y+1)^2$

$-\sqrt{\frac{x+y}{2}} = y+1$

$\therefore f^{-1} = -1 - \sqrt{\frac{x+7}{2}};\ x \ge -7$

For $f(x)$:

$y = 2x^2 + 4x - 5,\ x \ge -1$

or $f(x) = y = 2(x+1)^2 - 7,\ x \ge -1$

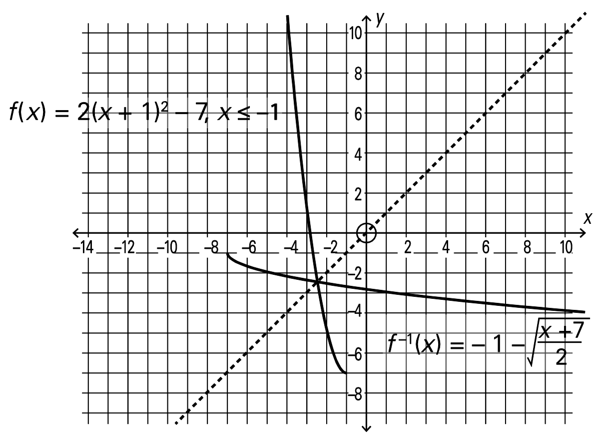
$f^{-1}(x)\!: x = 2(y+1)^2 - 7,\ y \ge -1$

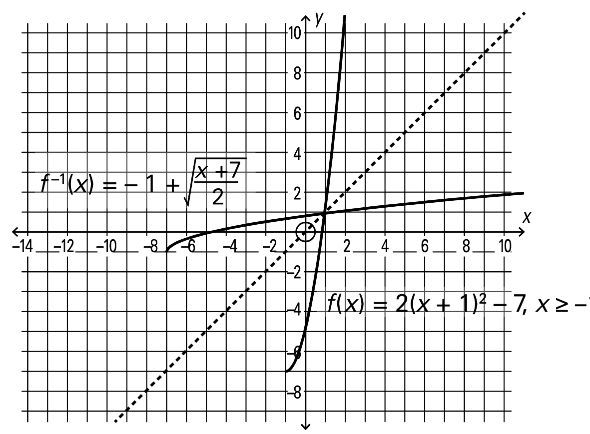
$x = 2(y+1)^2 - 7$

$\frac{x+7}{2} = (y+1)^2$

$+\sqrt{\frac{x+7}{2}} = y+1$

$\therefore f^{-1} = -1 + \sqrt{\frac{x+7}{2}};\ x \ge -7$



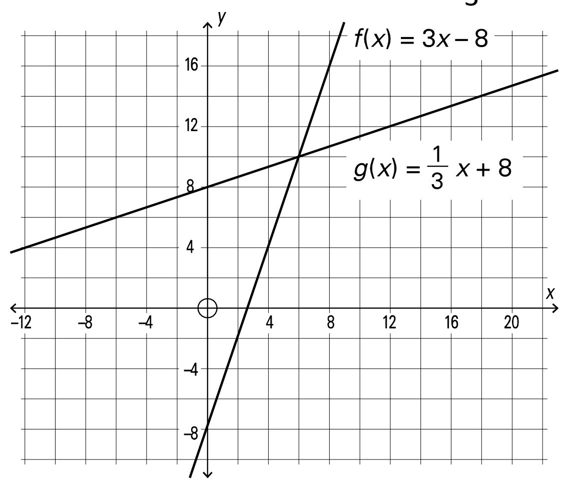


When presented with two functions, students are expected to determine whether or not they are inverses of one another.

Method 1
This can be done on a graph that displays both functions by sketching the line $y = x$ and deciding if the functions are mirror images of one another.

Example:

To determine if $f(x) = 3x - 8$ and $g(x) = \frac{1}{3}x + 8$ are inverse functions:



If students do not use equal aspects to graph the functions, the resulting distortion may make it difficult to judge if the functions are inverses of each other.

In this case, it is clear that they are not since they are not reflections in the line $y = x$.

Method 2
Algebraically, students should be given the equation representing each function. They determine the equation of the inverse of one of the given functions, and then decide if it is equivalent to the other given function. This algebraic approach is efficient when finding the inverse of one of the functions is fairly straight forward.

Example:

To determine if $f(x) = 3x - 8$ and $g(x) = \frac{1}{3}x + 8$ are inverse functions:

$f(x)$: $y = 3x - 8$

$f^{-1}(x)$: $x = 3y - 8$

$\frac{x+8}{3} = y$

$y = \frac{1}{3}(x+8)$

$y = \frac{1}{3}x + \frac{8}{3}$

$\therefore f^{-1}(x) = \frac{1}{3}x + \frac{8}{3}$

Therefore, $f(x) = 3x - 8$ and $g(x) = \frac{1}{3}x + 8$ are *not* inverse functions since $f^{-1}(x) \neq g(x)$.

Method 3
(Consider after students have studied composition of functions.)

Algebraically, students can also determine if two functions are inverses if $(f \circ g)(x) = x$ or if $(g \circ f)(x) = x$.

Example:

To determine if $f(x) = 3x - 8$ and $g(x) = \frac{1}{3}x + 8$ are inverse functions, compute one of the following compositions to determine if $(f \circ g)(x) = x$ or $(g \circ f)(x) = x$.

$\boldsymbol{(f \circ g)(x)}$	$\boldsymbol{(g \circ f)(x)}$
$f(x) = 3x - 8;\ g(x) = \frac{1}{3}x + 8$ $f[g(x)] = 3\left(\frac{1}{3}x + 8\right) - 8$ $f[g(x)] = x + 24 - 8$ $f[g(x)] = x + 16$	$f(x) = 3x - 8;\ g(x) = \frac{1}{3}x + 8$ $g[f(x)] = \frac{1}{3}(3x + 8) + 8$ $g[f(x)] = x - \frac{8}{3} + 8$ $g[f(x)] = x + \frac{16}{3}$
Therefore, $f(x) = 3x - 8$ and $g(x) = \frac{1}{3}x + 8$ are *not* inverse functions since $(f \circ g)(x) \neq x$ or since $(g \circ f)(x) \neq x$.	

It should be noted that proving that complicated functions are inverses is efficiently done by considering if $(f \circ g)(x) = x$ or if $(g \circ f)(x) = x$.

For example, to determine if $f(x) = 3x^2 - 24x + 12;\ x \geq 4$ and $g(x) = 4 + \sqrt{\frac{x}{3} + 12}$ are inverse functions:

Consider if either $(f \circ g)(x) = x$ or if $(g \circ f)(x) = x$.

$\boldsymbol{(f \circ g)(x)}$	$\boldsymbol{(g \circ f)(x)}$
$f[g(x)] = 3\left(4 + \sqrt{\frac{x}{3} + 12}\right)^2 - 24\left(4 + \sqrt{\frac{x}{3} + 12}\right) + 12$ $f[g(x)] = 3\left[16 + 8\sqrt{\frac{x}{3} + 12} + \left(\frac{x}{3} + 12\right)\right] - \left(96 + 24\sqrt{\frac{x}{3} + 12}\right) + 12$ $f[g(x)] = 48 + 24\sqrt{\frac{x}{3} + 12} + x + 36 - 96 - 24\sqrt{\frac{x}{3} + 12} + 12$ $f[g(x)] = x$	$g[f(x)] = 4 + \sqrt{\frac{(3x^2 - 24x + 12)}{3} + 12}$ $g[f(x)] = 4 + \sqrt{x^2 - 8x + 4 + 12}$ $g[f(x)] = 4 + \sqrt{x^2 - 8x + 16}$ $g[f(x)] = 4 + \sqrt{(x - 4)^2}$ $g[f(x)] = 4 + (x - 4)$ $g[f(x)] = x$

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

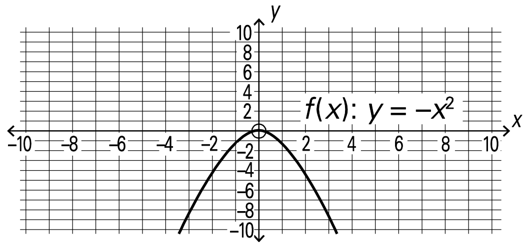
- Sketch each of the following and state their domain and range.
  (a) $y=2(x+1)^2-7$
  (b) $y=-3(x+1)^2+5$
- Isolate x in each of the following equations.
  (a) $y=2x-6$
  (b) $y=2x^2-6$
  (c) $y=\frac{1}{2}(x-5)^2-6$

Whole-Class/Group/Individual Assessment Tasks

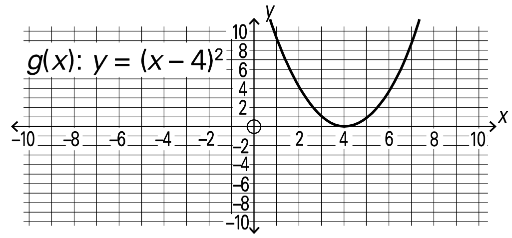
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Explain why the inverse of $y=x^2$ is not a function, but the inverse of $y=x^2, x\geq 0$ is a function.
- Find the equation of the inverse for each of the following.
  (a) $f(x)=6x$
  (b) $f(x)=\frac{x-3}{2}$
  (c) $f(x)=\frac{1}{3}(x-4)$
- For each of the following graphs of quadratics,
  (a) draw its inverse
  (b) describe the domain and range of the function and its inverse
  (c) explain how would you restrict the domain of the original function in order to guarantee that the inverse relation is a function
  (d) state the equation for the inverse function

(i)



(ii)



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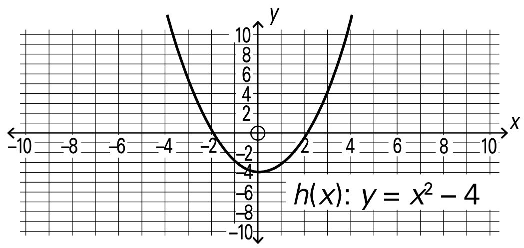

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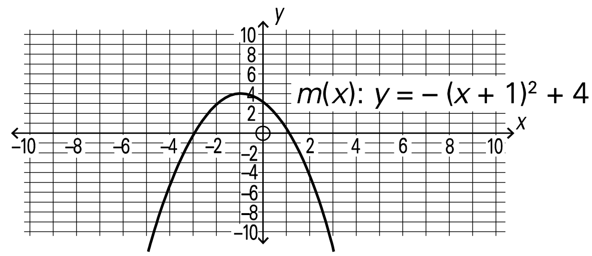
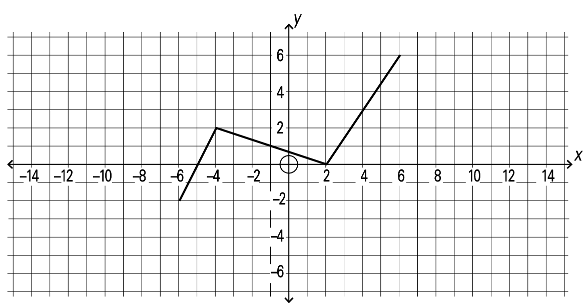
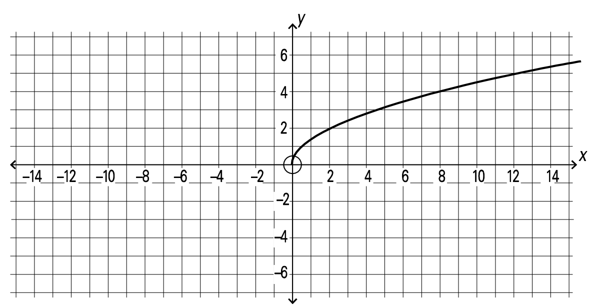


- Sketch the graph of the inverse of each of the following functions by reflecting the graph in the line $y = x$. Then, determine if the inverse is a function.

(a)



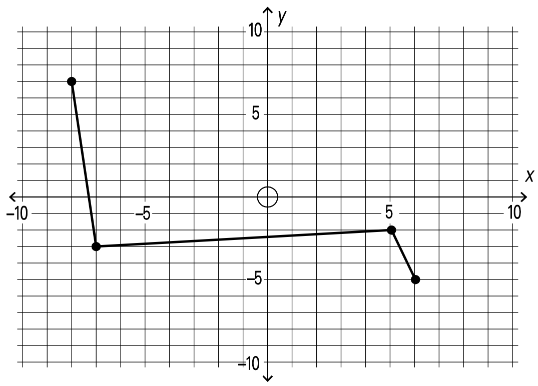
(b)



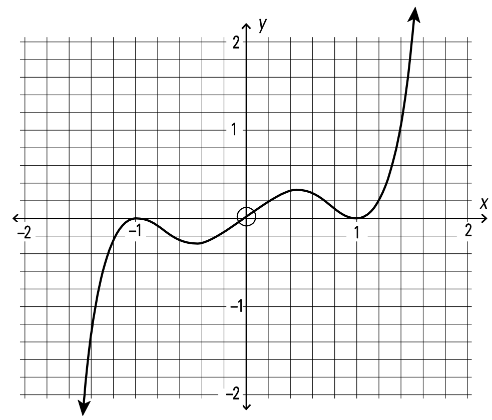
- Write an equation for the line obtained by reflecting the line $x = 2$ in the line $y = x$.
- Explain what it means for one function to be the inverse of another in terms of their domains and ranges.

- Sketch the inverse relation on the same set of axes.

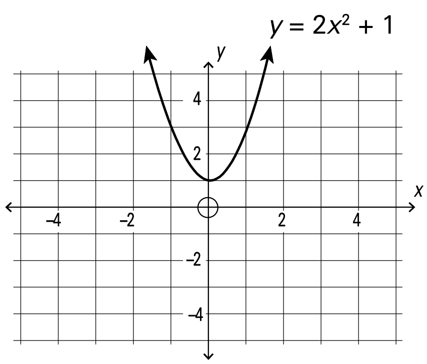
(a)



(b)

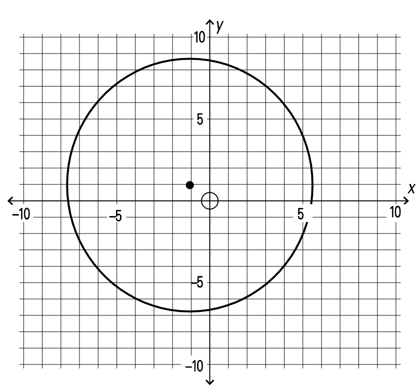


- Discuss how you could sketch the graph of the inverse of the relation graphed below, using both indicated approaches:
  (a) the line $y = x$
  (b) the mapping rule $(x, y) \rightarrow (y, x)$

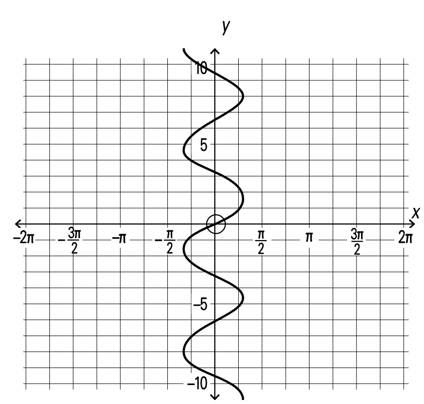


- Explain the difference between a vertical line test and a horizontal line test in terms of what each one is used to determine, and why they work.
- Determine whether the inverse of each relation graphed here is a function, without actually sketching it.

(a)



(b)



- For each of the following,
  (a) state the inverse relation
  (b) state the restriction(s) necessary so that the inverse relation could be written as two possible inverse functions
    (i) $y = x^2 - 6x + 10$
    (ii) $y = 5x^2 + 20x - 9$
    (iii) $y = 2x^2 - 8x + 1$
- Determine if the inverses of the following functions are functions. If the inverse is not a function, describe how each can be modified to become a function.
  (a) $f(x) = 3x^2$
  (b) $f(x) = x^2 + 2x$
  (c) $f(x) = 2x^2 + 4$
- Match each of the equations from the first list with its inverse in the second list:

Function	**Inverse**
$y = 4x - 1$	$y = \dfrac{x-16}{4}$
$y = x^2 + 8x + 2,\ x \geq -4$	$y = \dfrac{x+1}{4}$
$y = 3x^2 - 12x + 15,\ x \geq 2$	$y = \sqrt{x+14} - 4$
	$y = \sqrt{\dfrac{x-3}{3}} + 2$

- Determine if the following are inverses.
  (a) $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 3$
  (b) $y = \dfrac{-3x+6}{7}$ and $y = -\dfrac{7}{3}x - 2$
  (c) $y = 2x^2 + 12x$ and $y = \sqrt{\dfrac{x+18}{2}} - 3$

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- When students know the shape of a graph, they can use the horizontal line test to determine if the domain of a function needs to be restricted in order for its inverse to be a function.
- Discuss with students how an inverse of a relation "undoes" whatever the original relation did. Using examples from non-mathematical situations may provide a better understanding of a topic that is otherwise very abstract. The inverse of opening a door is closing the door; the inverse of

wrapping a gift is unwrapping a gift. From a mathematics perspective, encourage students to think of inverse relations as undoing all of the mathematical operations. The following table helps to illustrate this concept.

Function	**Inverse**
$f(x)=x+2$	$f^{-1}(x)=x-2$
$f(x)=5x$	$f^{-1}(x)=\dfrac{x}{5}$
$f(x)=x^2$	$f^{-1}(x)=\sqrt{x}$

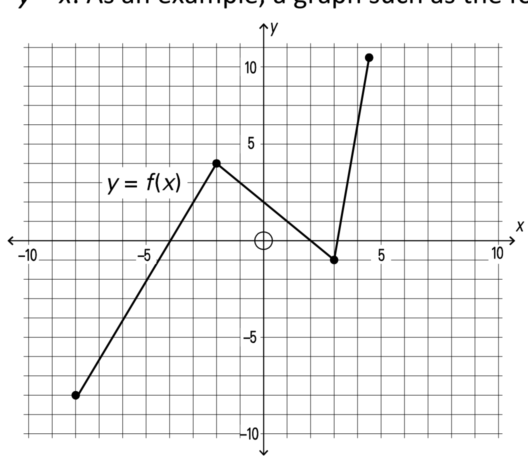
Students should compare a function to its inverse using a table of values. Examining a table of values for $y=x+4$ and its inverse, $y=x-4$, shows that the *x*- and *y*-values are interchanged.

x	***y***
−2	2
−1	3
0	4
1	5
2	6

x	***y***
2	−2
3	−1
4	0
5	1
6	2

From this, students should see the mapping notation (*x*, *y*) → (*y*, *x*) as a reversal of the *x*- and *y*-values in order to represent an undoing of a process. The input of the function is the output of the inverse, and vice versa. This leads to the relationship between the domains and ranges of a relation and its inverse. Students may have difficulty generalizing this relationship for linear functions, as the domain and range are all real numbers. Use the graph of a quadratic function, such as $y=(x-2)^2+1$, and the graph of its inverse to highlight the relationship. The original function has all real numbers in its domain and the range is $\{y \mid y\geq 1, y\in R\}$. They should note that the domain and range are interchanged for the inverse.

- By having students use $(x,y)\rightarrow(y,x)$ to create a table of values for the inverse of a function, they should quickly understand that there is reflective symmetry when the graphs of *f* and $f-1$ are sketched on the same set of axes. The axis of symmetry becomes apparent when they draw the line *y* = *x*. As an example, a graph such as the following could be presented.



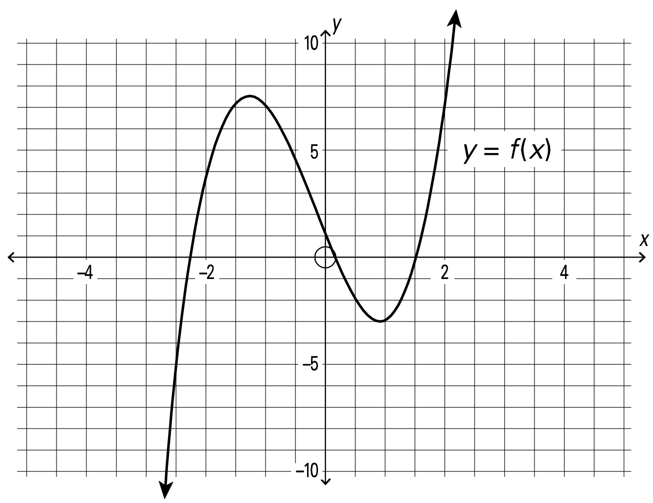
Based on the graph, ask students to complete the following:

- Create a table of values for the function using the ordered pairs for the four key points shown.
- Transform the table by applying the mapping $(x, y) \rightarrow (y, x)$.
- Graph the relation represented by the new table.
- Look for a relationship between the graphs of f and f^{-1}.
- Generalize the pattern to a relationship between graphs of functions and their inverses in general.

▪ Have students explore the transformations of $f(x) = 2x + 1$, complete a table of values, then graph the function. Find the inverse of the function, complete a new table of values for the inverse, and then graph the inverse. Have students reflect and respond on what has actually taken place.

Repeat the above process with quadratic functions, $f(x) = x^2 + 1$, which they have studied previously. Have them reflect this in the line $y = x$, and write the equation of the image. Some students will take $x = y^2 + 1$, solve for y and try to graph this using technology. Many will graph and expect to see a sideways parabola. Have them explain what is happening here.

▪ Students could be given a graph such as the following and asked to complete the tasks below.



- Use a vertical line test to determine if $y = f(x)$ is a function.
- Construct the graph of $y = f^{-1}(x)$ by reflecting the graph of $y = f(x)$ in the line $y = x$.
- Use the vertical line test to determine if $y = f^{-1}(x)$ is a function.
- What kind of line test could be used with the graph of $y = f(x)$ to determine if its inverse would be a function?

Students should conclude that the horizontal line test is used to determine if an inverse relation is a function.

▪ When students are graphing the inverse of a function is important for the grid to be a square grid. When using graphical software this may not be the default.

- Ensure that students understand that $f^{-1}(x)$ and $\dfrac{1}{f(x)}$ are not equivalent expressions. $f^{-1}(x)$ is the inverse of $f(x)$ and $\dfrac{1}{f(x)}$ is the reciprocal of $f(x)$.

- When reflecting a relation in the line $y = x$, some students will incorrectly use the mapping rule $(x,y)\to(-y,-x)$. Remind students that a reflection in the line $y = x$ does not change the signs of the coordinates.
- To avoid confusion for students when determining the equation of the inverse of a function, suggest that they label their work carefully.

  Example:

  To find the inverse of $f(x)=2x-6$

  $$f(x)=2x-6$$
  $$f(x):\ y=2x-6$$
  $$f^{-1}(x):\ x=2y-6$$
  $$\frac{x+6}{2}=y$$
  $$\therefore f^{-1}(x)=\frac{x+6}{2}$$

  To find the inverse of $f(x)=3x^2-1, x\le 0$

  $$f(x)=3x^2-1, \qquad x\le 0,\ y\ge -1$$
  $$f(x):\ y=3x^2-1, \qquad x\le 0$$
  $$f^{-1}(x):\ x=3y^2-1 \qquad y\le 0, x\ge -1$$
  $$\frac{x+1}{3}=y^2$$
  $$\therefore f^{-1}(x)=-\sqrt{\frac{x+1}{3}}; \qquad x\ge -1$$

- Students respond to three reflective prompts that describe what they learned about inverse relations. Provide students with a copy of the reflection sheet and time to complete their reflections. They could also be paired up to share their 3-2-1 reflections.

  Example of a 3-2-1 reflection sheet:

Three new things I learned: 1. 2 3. Two things I am still struggling with: 1. 2. One thing that will help me tomorrow: 1.

- Students could be guided through the following process for the quadratic function represented by $f(x) = 2x^2 - 8x + 11$.
  - (a) Write the equation in vertex form and sketch its graph.
  - (b) Use the graph of the quadratic to sketch its inverse on the same set of axes.
  - (c) Determine if the inverse relation is also a function. If not, consider what would need to change about the graph of the original quadratic in order for its inverse to be a function.
  - (d) Restrict the domain of the function $y = f(x)$ so that $y = f^{-1}(x)$ is also a function.
- Restricting the domain of a parabola, $y = a(x-h)^2 + k$, so that $x > h$ produces a function whose inverse is also a function. Restricting the domain so that $x < h$ is also acceptable, although not as common. Students should then be shown the process by which interchanging the *x*- and *y*-variables in the equation and then rearranging to isolate *y* produces the equation for $y = f^{-1}(x)$.
- Create square sheets of paper with graphs of pairs of functions. Create the pairs in such a way that some are inverses of one another while others are not. Construct the grid on each graph so that the scales are identical on each axis, and so that the *x*- and *y*-axes have the same limits. Distribute one square to each student and ask them to fold their papers in a way that would determine whether or not the graphs are inverses of one another. Ask them to explain their method.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- horizontal line test
- inverse of a function

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 1.4 Inverse of a Relation
    - Student Book: pp. 44–55
    - Teacher Resource: pp. 25–30

SCO RF07 Students will be expected to demonstrate an understanding of logarithms. [CN, ME, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF07.01 Explain the relationship between logarithms and exponents.
RF07.02 Express a logarithmic expression as an exponential expression and vice versa.
RF07.03 Determine, without technology, the exact value of a logarithm, such as $\log_2 8$ and ln e.
RF07.04 Estimate the value of a logarithm, using benchmarks, and explain the reasoning.

Scope and Sequence

Pre-calculus 11	Pre-calculus 12
RF10 Students will be expected to analyze geometric sequences and series to solve problems.	**RF07** Students will be expected to demonstrate an understanding of logarithms.

Background

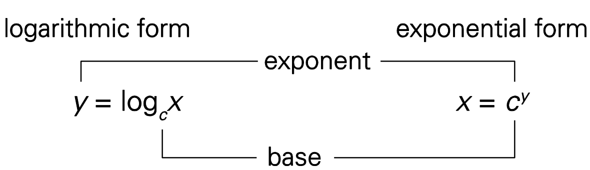
In Mathematics 10, students were expected to demonstrate an understanding of powers with integral and rational exponents (AN03). In Pre-calculus 11, students worked with geometric sequences and series and solved exponential equations by equating bases (RF10).

It is expected that students will be introduced to logarithms as a different form of an exponential statement. The statement $3^2 = 9$, for example, can be written as $\log_3 9 = 2$. The base of the exponent is the same as the base of the logarithm.

Students are expected to understand that the mathematical statement $\log_c x = y$ is asking, What exponent, y, is needed so that $c^y = x$?

The **argument** of a function is an input to a function or a variable that affects a function's result. For the function $y = \log(x)$, x is the argument. For the function $y = \log(4x + 3)$, $4x + 3$ would be the argument.

Given a statement in exponential form, students are expected to write it in logarithmic form, and vice versa.



This outcome introduces students to common logarithms and natural logarithms. Common logarithms have a base of 10 and can be written with or without the base as $y = \log_{10} x$ or simply as $y = \log x$. Students should be familiar with this convention.

Students may have not been introduced to the number *e*, Euler's number, prior to this unit. This constant can be used to describe naturally occurring constant growth rates. ($e \doteq 2.718281828$)

Natural logarithms have a base of *e* and can be written as $y = \log_e x$ or in its more common abbreviated form simply as $y = \ln x$. Students should also be familiar with this convention.

When the value of a logarithm is a rational number, such as $\log_{64}(4)$, $\log_9\left(\frac{1}{3}\right)$, $\log_{\frac{1}{2}}(32)$, students should be able to determine the exact value without technology.

$\log_{64}(4) = x$

$64^x = 4$

$\left(4^3\right)^x = 4^1$

$4^{3x} = 4^1$

Equating exponents:

$3x = 1$

$x = \frac{1}{3}$

$\log_9\left(\frac{1}{3}\right) = x$

$9^x = \frac{1}{3}$

$\left(3^2\right)^x = 3^{-1}$

$3^{2x} = 3^{-1}$

Equating exponents:

$2x = -1$

$x = -\frac{1}{2}$

$\log_{\frac{1}{2}}(32) = x$

$\left(\frac{1}{2}\right)^x = 32$

$\left(2^{-1}\right)^x = \left(2^5\right)$

$2^{-x} = 2^5$

Equating exponents:

$-x = 5$

$x = -5$

As well, students are expected to determine an unknown value in logarithmic equations.

The following examples can be solved by rewriting in exponential form.

$\log_3(x) = -2$

$3^{-2} = x$

$\frac{1}{9} = x$

$\log_x\left(\frac{81}{16}\right) = 2$

$x^2 = \frac{81}{16}$

$x = \pm\sqrt{\frac{81}{16}};\ x = \pm\frac{9}{4}$

Since the base of logarithms have to be positive, $x = \pm\frac{9}{4}$.

When working with logarithms, the base is restricted to positive values other than 1; that is, for $y = \log_c x$, $c > 0$, $c \neq 1$. Students are expected to understand the reasons for these restrictions on $c\left(c > 0,\ c \neq 1\right)$.

When *c* = 1

$\log_1(4) = x$

$1^x = 4$

This equation cannot be solved since $1^x = 1$ for all values of *x*.

When *c* < 0

$\log_{-2}(8) = x$

$(-2)^x = 8$

This equation is not solvable.

However, some equations with negative bases, such as $(-2)^x = 4$, can be solved. Since equations of the form $c^x = y$, $c < 0$ only exist under certain conditions and do not form a function, its inverse is not a function. For that reason, the base of logarithms must always be positive.

Students should also use benchmarks to estimate the value of a logarithm. To estimate $\log_5(106)$, for example, they should notice that $5^2 = 25$ and $5^3 = 125$. Therefore, the answer must be between 2 and 3, and is closer to 3. Through systematic trial, students should be able to determine the value accurately to a minimum of one decimal place.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Without using a calculator, evaluate each of the following:
  - (a) 3^2
  - (b) 2^{-3}
  - (c) $25^{\frac{1}{2}}$
  - (d) $\left(\frac{27}{125}\right)^{\frac{1}{3}}$
  - (e) 12^0
- Write each of the following as a power of 2, 2^n.
  - (a) $\sqrt{2}$
  - (b) 16
  - (c) $\frac{1}{8}$
  - (d) $\sqrt[3]{4}$
  - (e) $\frac{\sqrt{2}}{2}$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Evaluate the following without using technology.
  (a) $\log_{\frac{1}{5}}(25)$
  (b) $\log\left(\frac{1}{100}\right)$
  (c) $\log_4(32)$
  (d) $\ln\left(\sqrt{e}\right)$
- Use benchmarks to estimate the value of the following:
  (a) $\log_3(24)$
  (b) $\log_6(200)$
  (c) $\log_{\frac{1}{2}}(12)$
  (d) $\log_2(11)$
- Explain whether or not any positive real number can be the base of a logarithm.
- Express in logarithmic form.
  (a) $3^2 = 9$
  (b) $10^{-1} = 0.1$
  (c) $8^{\frac{2}{3}} = 4$
  (d) $5^{-2} = \frac{1}{25}$
  (e) $\sqrt[3]{e^2} = e^{\frac{2}{3}}$
- Express in exponential form.
  (a) $\log_5 25 = 2$
  (b) $\ln 1 = 0$
  (c) $\log 1000 = 3$
  (d) $\log_4 2 = \frac{1}{2}$
  (e) $\log 0.1 = -1$
- Change $y = 2\log\left(\frac{4}{x}\right)$ to exponential form.
- Evaluate each of the following without using technology.
  (a) $3\log_2 32$
  (b) $\log_6\left(\frac{1}{36}\right) + \log_2(8)$
  (c) $\frac{\log_{64}(2)}{\log(100)}$
  (d) $\log\left(\frac{1}{100}\right) \times \ln\left(\frac{1}{e^2}\right)$
  (e) $\log(1000) \div \ln\left(\sqrt{e}\right)$

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Ensure that students encounter situations where they need to rearrange an equation prior to converting from logarithmic to exponential form or from exponential to logarithmic form.
  - Students should understand that in order to change from logarithmic to exponential form, the equation must be in the form $c = \log_b(a)$. If an expression such as $25 = 4 \cdot \log_3 m$ is not in that form, it needs to be rewritten as $\frac{25}{4} = \log_3 m$ and then changed to exponential form as $3^{\frac{25}{4}} = m$.
  - Similarly, in order to change from exponential to logarithmic form, the equation must be in the form $b^c = a$. If an expression such as $500 = 3 \cdot (10)^m$ is not in that form, it needs to be rewritten as $\frac{500}{3} = 10^m$ and then changed to exponential form as $\log_{10} \frac{500}{3} = m$.
- If this is students' first exposure to Euler's number, it can be easily introduced by using compound interest. Suggest that a bank offers a promotion to have new customers open a savings account. They say that they will deposit the first dollar in the account and that they will offer 100% interest on that dollar and compound the interest continuously.

Number of compounding periods	**Amount in account** $\boldsymbol{A = P\left(1 + \frac{i}{n}\right)^{nt}}$
1 (annually)	$A = 1\left(1 + \frac{1}{1}\right)^{1} = 2$
2 (semi-annually)	$A = 1\left(1 + \frac{1}{2}\right)^{2} = 2.25$
4 (quarterly)	$A = 1\left(1 + \frac{1}{4}\right)^{4} = 2.44$
12 (monthly)	$A = 1\left(1 + \frac{1}{12}\right)^{12} = 2.61$
52 (weekly)	$A = 1\left(1 + \frac{1}{52}\right)^{52} = 2.69$
365 (daily)	$A = 1\left(1 + \frac{1}{365}\right)^{365} = 2.71$
8760 (hourly)	$A = 1\left(1 + \frac{1}{8760}\right)^{8760} = 2.72$
525 600 (every minute)	$A = 1\left(1 + \frac{1}{525\ 600}\right)^{525\ 600} = 2.72$

From this activity, students can observe that the more compounding periods they consider the closer they will get to Euler's number, *e*. They can then understand that this number helps to describe natural growth and decay rates, such as crystalline growth and radioactive decay, since these are continuous.

- The introduction to logarithms should be slow.
  - Use examples to motivate discussion of big and small numbers. (A good source of ideas is *A View from the Back of the Envelope* at www.vendian.org/envelope).
  - Ask questions that lead students to understand that the number of digits before the decimal and the number of zeroes following the decimal indicate the size of numbers, and that this information is conveyed with exponential notation. Review scientific notation.
  - Introduce the terminology and notation for logarithms. The relationship between logarithmic and exponential form should be explained, and students should practise converting from one form to the other.
  - Have students explore logarithms with base 10: $\log_{10}(10)=1$ because $10^1=10$; $\log_{10}(100)=2$ because $10^2=100$; $\log_{10}\left(\frac{1}{10}\right)=-1$ because $10^{-1}=\frac{1}{10}$; etc.
  - Have students explore logarithms with a smaller base, such as 2. Start with a review of powers of 2, leading to discussion of $\log_2(m)$. Have students determine a series of logarithms: $\log_2(2)=1$ because $2^1=2$; $\log_2(16)=4$ because $2^4=16$; $\log_2(1)=0$ because $2^0=1$; $\log_2\left(\sqrt{2}\right)=\frac{1}{2}$ because $2^{\frac{1}{2}}=\sqrt{2}$; etc.
  - Have students explore logarithms with base *e*. Have students determine a series of logarithms: $\log_e(e)=1$; $\log_e\left(e^3\right)=3$; $\ln(1)=0$ because $e^0=1$; $\ln\left(\frac{1}{e^3}\right)=-3$ because $\frac{1}{e^3}=e^{-3}$; etc.
- Students should evaluate logarithmic expressions to develop benchmarks with whole number solutions. Once established, benchmarks should be used to find approximate values of other logarithmic expressions.
- Asking students to consider the link between large and small numbers, scientific notation and the common logarithms, by completing a chart such as the one shown here, can provide an excellent context for understanding the role of logarithms.

Quantity	**Number (N)**	**Written as power of 10**	**Common log (N)**
One hundred miles	100	1×10^2	2
One thousand students	1000	1×10^3	3
Five million people	5 000 000	5×10^6	6.70
Twenty million dollars	20 000 000	2×10^7	7.30
Five-and-a-half billion people	5 500 000 000	5.5×10^9	9.74
One-tenth of a pound	$\frac{1}{10}=0.1$	1×10^{-1}	–1
One-thousandth of a gram	$\frac{1}{1000}=0.001$	1×10^{-3}	–3

Quantity	Number (N)	Written as power of 10	Common log (N)
Three-thousandths of a second	$\frac{3}{1000} = 0.003$	3×10^{-3}	–2.52
Five-millionths of a metre	$\frac{5}{1\ 000\ 000} = 0.000\ 005$	5×10^{-6}	–5.30

- Have students use their calculators to complete the following and then make a statement about the relationship between exponents and logarithms based on their observations.

  (a) $10^0 =$
  (b) $10^1 =$
  (c) $10^2 =$
  (d) $10^3 =$
  (e) $10^{-1} =$
  (f) $10^{-2} =$
  (g) $10^{-3} =$
  (h) $10^{\frac{1}{2}} =$
  (i) $\log(1) =$
  (j) $\log(10) =$
  (k) $\log(100) =$
  (l) $\log(1000) =$
  (m) $\log\left(\frac{1}{10}\right) =$
  (n) $\log\left(\frac{1}{100}\right) =$
  (o) $\log\left(\frac{1}{1000}\right) =$
  (p) $\log\left(\sqrt{10}\right) =$

- Ensure that students understand that logarithms and exponents are inverses of each other. As a result, many logarithmic problems can be solved by converting them to exponential problems and many exponential problems can be solved by converting them to logarithms.
- Write a common log as $\log_{10}(a)$ rather than $\log(a)$ and a natural log as $\log_e(a)$ rather than $\ln(a)$ for a few days until you are certain that students understand what the shortened versions represent.
- When solving an equation such as $\log_x\left(\frac{16}{81}\right) = 2$, students often forget that the base must be positive. Even though the quadratic $x^2 = \left(\frac{16}{81}\right)$ has two solutions, $\pm\frac{4}{9}$, the only solution for the logarithmic equation is $+\frac{4}{9}$.
- Including parentheses and brackets around the argument of the logarithm creates clarity. (In their study of calculus, students will consider the absolute value of the argument to extend the domain of the function when using logarithmic differentiation.)

Suggested Models and Manipulatives

- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- argument
- base
- common logarithm
- Euler's number, *e*logarithm
- logarithm function
- natural logarithm

Resources/Notes

Internet

- *A View from the Back of the Envelope* (Charity 1998): www.vendian.org/envelope

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 8.1 Understanding Logarithms
    - > Student Book: pp. 372–382

SCO RF08 Students will be expected to demonstrate an understanding of the product, quotient, and power laws of logarithms. [C, CN, R, T]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF08.01 Develop and generalize the laws for logarithms, using numeric examples and exponent laws.
RF08.02 Derive each law of logarithms.
RF08.03 Determine, using the laws of logarithms, an equivalent expression for a logarithmic expression.
RF08.04 Determine, with technology, the approximate value of a logarithmic expression, such as $\log_2 9$ and ln 10.

Scope and Sequence

Pre-calculus 11	Pre-calculus 12
RF10 Students will be expected to analyze geometric sequences and series to solve problems.	**RF08** Students will be expected to demonstrate an understanding of the product, quotient, and power laws of logarithms.

Background

In Mathematics 10, students worked with the laws of exponents, including integral and rational exponents. They also applied the exponent laws to expressions with rational and variable bases, as well as integral and rational exponents (AN03). The laws of logarithms will now be developed using both numerical examples and the exponent laws.

The laws of logarithms allow us to work easily with logarithmic expressions and equations. Since logarithms are exponents, the laws of logarithms are related to the laws of exponents. Students should be able to explain this relationship with examples.

Students will work with the following laws of logarithms, with the conditions $c > 0$, $M > 0$, $N > 0$, and $c \neq 1$ where $c, M, N \in$ R:

Product law: $\log_c MN = \log_c M + \log_c N$

Quotient law: $\log_c \frac{M}{N} = \log_c M - \log_c N$

Power law: $\log_c M^P = P \log_c M$

Students are expected to use several numerical examples, such as

$\log(5 \cdot 2)$, $\log(5)$, $\log(2)$, $\log\left(\frac{800}{8}\right)$, $\log(800)$, $\log(8)$, $\log\left(10^4\right)$, and $\log(10)$, to develop the laws of logarithms.

They are expected to evaluate each side of the equation to verify that the left-hand side of the equation is indeed the same as the right-hand side. For example, they should verify that

$\log(5\cdot 2)=\log(5)+\log(2)$, $\log\left(\frac{800}{8}\right)=\log(800)-\log(8)$, and $\log\left(10^4\right)=4\log(10)$.

Students are then expected to derive the general case for the quotient law, the product law, and the power law by using numerical examples.

Students should be exposed to the proofs of the laws of logarithms, such as those shown below, but they are not required to reproduce these proofs.

Quotient law	**Product law**
Let $A^m=C$ and $A^n=D$. Then $\log_A(C)=m$ and $\log_A(D)=n$.	Let $A^m=C$ and $A^n=D$. Then $\log_A(C)=m$ and $\log_A(D)=n$.
Dividing the equation $A^n=D$ by A^m results in the equation $\frac{A^n}{A^m}=\frac{D}{A^m}$ or $\frac{A^n}{A^m}=\frac{D}{C}$; therefore, $A^{n-m}=\frac{D}{C}$.	Multiplying the equation $A^n=D$ by A^m results in the equation $(A^n)(A^m)=(D)(A^m)$ or $(A^n)(A^m)=(D)(C)$; therefore, $A^{n+m}=D\cdot C$.
Converting to logarithmic form: $\log_A\left(\frac{D}{C}\right)=n-m$	Converting to logarithmic form: $\log_A(D\cdot C)=n+m$
Substituting for *n* and *m*: $\log_A\left(\frac{D}{C}\right)=\log_A(D)-\log_A(C)$	Substituting for *n* and *m*, $\log_A(D\cdot C)=\log_A(D)+\log_A(C)$

Power law	**Power law using product law**
Let $A^m=C$. Then in logarithmic form, $\log_A(C)=m$.	$\log_A(C)=m$
Raising both sides of the equation $A^m=C$ to the power of *n* results in the equation $\left(A^m\right)^n=(C)^n$; therefore, $A^{n\cdot m}=C^n$.	Add $\log_A(C)$ to both sides of the equation so that $\log_A(C)+\log_A(C)=m+\log_A(C)$.
Converting to logarithmic form: $\log_A\left(C^n\right)=n\cdot m$	Using product law and substitution: $\log_A(C\cdot C)=m+\log_A(C)$ or $\log_A\left(C^2\right)=\log_A(C)+\log_A(C)$ $\therefore \log_A\left(C^2\right)=2\log_A(C)$
Substituting for *m*: $\log_A\left(C^n\right)=n\cdot\log_A(C)$	This can be extended so that $(n-1)\cdot\log_A(C)$ is added to both sides of the equation $\log_A(C)=m$ and the product law applied to obtain the general power law. $\log_A\left(C^n\right)=n\cdot\log_A(C)$

Students should be exposed to questions where they are required to write a logarithmic expression as a single logarithm and simplify if necessary.

Example:

$$\log_6(4)-\left[\log_6(72)+\frac{1}{4}\log_6(16)\right]$$
$$\log_6(4)-\left[\log_6(72)+\log_6(16)^{\frac{1}{4}}\right]$$
$$\log_6\left(\frac{4}{72\times 16^{\frac{1}{4}}}\right)$$
$$\log_6\left(\frac{4}{72\times 2}\right)$$
$$\log_6\left(\frac{1}{36}\right)=-2$$

$$\log_b(2x)+3\left[\log_b(x)-\log_b(y)\right]$$
$$\log_b(2x)+3\log_b(x)-3\log_b(y)$$
$$\log_b(2x)+\log_b(x)^3-\log_b(y)^3$$
$$\log_b\left(\frac{2x}{x^3y^3}\right)=\log_b\left(\frac{2}{x^2y^3}\right)$$

Conversely, students should write a single logarithm as the sum and difference of multiple logarithms. For example, an expression such as $\log_5\left(\frac{\sqrt{5}x}{25}\right)$ can be simplified.

$$\log_5\left(\frac{\sqrt{5}x}{25}\right)=\left[\log_5\left(\sqrt{5}\right)+\log_5(x)\right]-\left[\log_5(25)\right]$$
$$\log_5\left(\frac{\sqrt{5}x}{25}\right)=\left[\log_5\left(5^{\frac{1}{2}}\right)+\log_5(x)\right]-\left[\log_5\left(5^2\right)\right]$$
$$\log_5\left(\frac{\sqrt{5}x}{25}\right)=\left[\frac{1}{2}+\log_5(x)\right]-(2)$$
$$\log_5\left(\frac{\sqrt{5}x}{25}\right)=\log_5(x)-\frac{3}{2}$$

This ability to expand or condense logarithmic expressions is useful in solving logarithmic equations.

Students are expected to simplify logarithmic expressions involving both numerical and variable arguments.

Although the term **argument** is not emphasized in *Pre-Calculus 12* (McAskill et al. 2012), it is suggested that you use this term when discussing logarithms.

For expressions with variable arguments, students should determine the restrictions on the variable(s). The arguments should be restricted to polynomials of degree 2 or less.

Example:

$\log_2(x+1)+\log_2(x)$

Since the argument of a logarithm must be positive, we know that $x > -1$.

$\log_2(x^2+20)-\log_2(x+5)$

Since the argument of a logarithm must be positive, we know that $x > -5$.

Students were introduced to factoring techniques in Mathematics 10 (AN05). They should now be exposed to problems where factoring is necessary in order to simplify the expression and determine the restrictions.

Example:

$$\log\left(x^2-6x+9\right)=\log(x-3)^2=3\log(x-3),\ x>3$$

Students were introduced to evaluating logarithmic expressions without the use of technology by using benchmarks (RF06). They will now extend this to approximating the solution, using technology. To determine the approximate value of $\log_2 9$, for example, the equation $\log_2 9 = x$ can be rewritten in exponential form:

Using common logarithms

$$\begin{aligned}
2^x &= 9\\
\log\left(2^x\right) &= \log(9)\\
x\log(2) &= \log(9)\\
x &= \frac{\log(9)}{\log(2)}\\
x &\doteq 3.17
\end{aligned}$$

Using natural logarithms

$$\begin{aligned}
2^x &= 9\\
\ln\left(2^x\right) &= \ln(9)\\
x\ln(2) &= \ln(9)\\
x &= \frac{\ln(9)}{\ln(2)}\\
x &\doteq 3.17
\end{aligned}$$

As students work through this example, they should be able to connect the solution of the problem to the original equation.

$$\log_2 9 = x \rightarrow x = \frac{\log 9}{\log 2} \text{ or } \frac{\ln 9}{\ln 2}$$

This leads to the property, referred to as the base change property, $\log_b a = \dfrac{\log a}{\log b}$ or $\dfrac{\ln a}{\ln b}$.

The logarithmic expression $\dfrac{\log a}{\log b}$ or $\dfrac{\ln a}{\ln b}$ can be evaluated using a calculator.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Simplify each of the following:
  (a) $\left(2^3\right)\left(2^4\right)$
  (b) $\frac{2^7}{2^3}$
  (c) $\left(2^3\right)^4$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Verify using the logarithm laws.
  (a) $\log_3 27 = \log_3 9 + \log_3 3$
  (b) $\log_5 25 = \log_5 125 - \log_5 5$
  (c) $\log_2 64 = 6 \log_2 2$
- A student was asked to simplify $\log_2 16 - \log_2 32 + 2 \log_2 4$ and provided the following solution:

  $\log_2 16 - \log_2 32 + \log_2 8$
  $\log_2 \frac{16}{32} + \log_2 8$
  $\log_2 \frac{1}{2} + \log_2 8$
  $\log_2 4$
  2

  State where the error first occurred and write the correct solution to the problem.
- If $P = \log_3 8$ and $Q = \log_3 6$, write $\log_3\left(8\sqrt{6}\right)$ in terms of P and Q.
- Evaluate using the laws of logarithms.
  (i) $3 \log_6 (2) + \log_6 (27)$
  (ii) $\log_5 (2.5) + 2 \log_5 (10) - \log_5 (2)$

- Use the laws of logarithms to simplify and evaluate each expression.
  - (a) $2\log_3 12 - 2\log_3 4$
  - (b) $\log_4 6 + \log_4\left(\frac{64}{3}\right) - \log_4 8$
  - (c) $\log_3(9 \cdot 27 \cdot 81)$
  - (d) $\frac{1}{2}\log_3 144 - \log_3 4 + 2\log_3 3$
  - (e) $\log_5\sqrt{175} - \log_5\sqrt{7}$
- Write each expression as a single logarithm in simplest form.
  - (a) $\log_2 a + \log_2 b - \log_2 c$
  - (b) $\log x^2 - 5\log y$
  - (c) $\log_5 A + \log_5\sqrt{B} - 3\log_5 C$
  - (d) $\log_7\sqrt[3]{x} - \log_7 y^3 + 2\log_7 y$
- If $\log(9) \doteq 0.9542$, use the laws of logarithms to find the value of $\log\left(\sqrt[5]{81}\right)$, correct to two decimal places.
- Using the laws of logarithms, show that $\log_b\left(\sqrt[n]{x^m}\right) = \frac{m\log_b(x)}{n}$.
- If log (x) = q, write $\frac{1}{3}\log(x)^5$ in terms of q.
- If log 2 = A, log 3 = B, and log 5 = C, write an algebraic expression in terms of A, B, and C for each of the following.
  - (a) $\log 6$
  - (b) $\log 2.5$
  - (c) $\log\sqrt{3}$
- Write each expression as a single logarithm statement.
  - (a) $\log_3 x + 2\log_3 7$
  - (b) $\log_5 x - \log_5 y$
  - (c) $\log_2 5x + \log_2 7x$
  - (d) $\log_3 54 - 4\log_3 2$
  - (e) $\ln 10 + \ln 6 - \ln 15$
- Use the laws of logarithms to expand each of the following:
  - (a) $\log(abc)$
  - (b) $3\log(6x)$
  - (c) $2\ln\left(\frac{a}{b}\right)$
- Write each expression as a single logarithm statement.
  - (a) $\log 3 + \log 7$
  - (b) $\log x + \log(x + 2)$
  - (c) $a\log(xz) + a\log(xy)$
  - (d) $\ln 20 - \ln 10$
  - (e) $2\ln e - 3\ln 1 + \ln e^6$

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Students should note that there is no general property of logarithms that can be used to simplify $\log_c (x + y)$.They sometimes mistakenly think that this expression is equal to $\log_c x + \log_c y$. To verify that this is not true, students could evaluate logarithmic expressions, such as

  $\log_{10} (2 + 3) = \log_{10} 5 \approx 0.699$

  $\log_{10} 2 + \log_{10} 3 \approx 0.301 + 0.477 = 0.778$
- Review the laws of exponents with students.
- Derive the three laws of logarithms with the class. Show that they are derived from the laws of exponents.
- The property $\log_b c = \frac{\log c}{\log b}$ is not developed in *Pre-Calculus 12* (McAskill et al. 2012), but is referenced in question 19 on page 402.

SUGGESTED MODELS AND MANIPULATIVES

- calculators

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- power law of logarithms
- product law of logarithms
- quotient law of logarithms

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 8.3 Laws of Logarithms
    - > Student Book: pp. 392–403

SCO RF09 Students will be expected to graph and analyze exponential and logarithmic functions. [C, CN, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF09.01 Sketch, with or without technology, a graph of an exponential function of the form $y = a^x$, $a > 0$.
RF09.02 Identify the characteristics of the graph of an exponential function of the form $y = a^x$, $a > 0$, including the domain, range, horizontal asymptote and intercepts, and explain the significance of the horizontal asymptote.
RF09.03 Sketch the graph of an exponential function by applying a set of transformations to the graph of $y = a^x$, $a > 0$, and state the characteristics of the graph.
RF09.04 Sketch, with or without technology, the graph of a logarithmic function of the form $y = \log_b x$, $b > 1$.
RF09.05 Identify the characteristics of the graph of a logarithmic function of the form $y = \log_b x$, $b > 1$, including the domain, range, vertical asymptote and intercepts, and explain the significance of the vertical asymptote.
RF09.06 Sketch the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_b x$, $b > 1$, and state the characteristics of the graph.
RF09.07 Demonstrate, graphically, that a logarithmic function and an exponential function with the same base are inverses of each other.

Scope and Sequence

Pre-calculus 11	**Pre-calculus 12**
RF11 Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).	**RF09** Students will be expected to graph and analyze exponential and logarithmic functions.

Background

In Mathematics 10, students worked with the laws of exponents, integral exponents, and rational exponents. They also applied the exponent laws to expressions with rational and variable bases, as well as integral and rational exponents (AN03). Additionally, students graphed, with and without technology, a set of data and determined the restrictions on the domain and range (RF01). In Pre-calculus 11, students graphed and analyzed reciprocal functions (limited to reciprocals of linear and quadratic functions) of the form $y = \frac{1}{f(x)}$ where they would have been introduced to the concept of asymptotes (RF11).

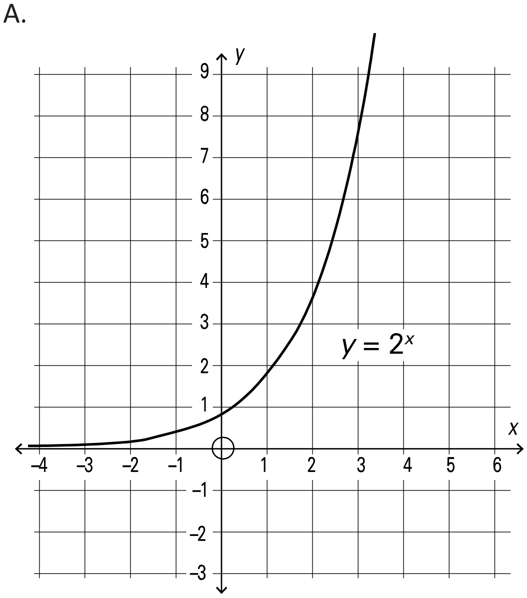
Exponential functions have the general form, $y = c^x$; $c > 0$, $c \in R$. Students will have seen exponential functions before in science and mathematics when describing situations such as population growth, compound interest, radioactive decay, and value depreciation. It is important that exponent laws be reviewed prior to working further with exponential functions.

Students will identify the characteristics of exponential graphs, which will include domain, range, horizontal asymptotes, *x*-intercept, and *y*-intercept. The importance and meaning of the horizontal asymptote for exponential graphs should be emphasized.

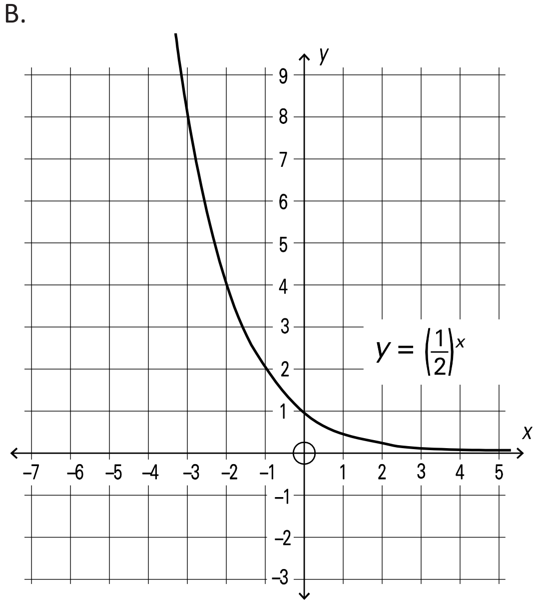
For the equation $y = c^x$, when $c > 1$, the exponential function models exponential growth (e.g., $y = 2^x$ shown on graph A below).

For the equation $y = c^x$, when $0 < c < 1$, the exponential function models exponential decay (e.g., $y = \left(\frac{1}{2}\right)^x$ shown on graph B, below).

A.



B.



For exponential functions of the form, $y = c^x$, $c > 0$, $c \neq 1$, $c \in R$

Domain: $\{x | x \in R\}$ or $(-\infty, \infty)$

Range: $\{y | y > 0, y \in R\}$ or $(0, \infty)$

Y-intercept: (0, 1)

Asymptote: *x*-axis or $y = 0$

In the Function Transformations unit of *Pre-calculus 12* (McAskill et al. 2012) students worked with horizontal and vertical translations (RF02), horizontal and vertical stretches (RF03), combinations of stretches and translations (RF04), and reflections (RF05). Mapping rules were also studied in the context of these transformations. Now, students work with exponential and logarithmic functions.

Students will explore ways that transformations affect graphs of exponential functions. They will develop the ability to sketch these graphs both with and without technology and learn to manipulate the exponential equation, $y = a(c)^{b(x-h)} + k$, to produce any given graph.

The following chart summarizes the effects of the parameters *a*, *b*, *h*, and *k*, on the graphs of exponential functions. These parameters should be discussed in relation to previous work on function transformations.

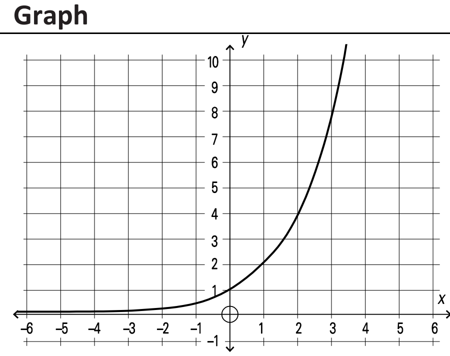
Parameter	Transformation
a	▪ Vertical stretch about the *x*-axis by a factor of $\lvert a \rvert$ ▪ For $a < 0$, reflection in the *x*-axis
b	▪ Horizontal stretch about the *y*-axis by a factor of $\frac{1}{\lvert b \rvert}$ ▪ For $b < 0$, reflection in the *y*-axis
k	▪ Vertical translation up or down by *k* units
h	▪ Horizontal translation left or right by *h* units

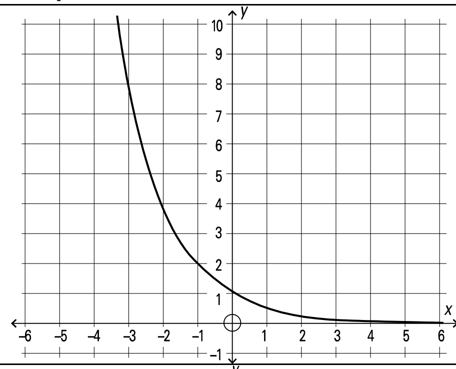
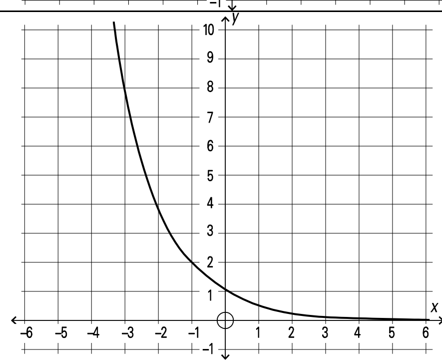
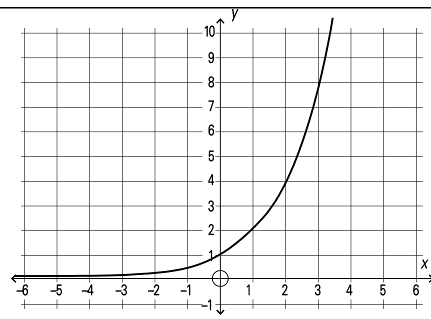
When discussing features of these graphs, the focus should be on

- *x*- and *y*-intercepts
- domain and range
- if the graph is increasing or decreasing (Note that this will depend on the values of *c* and *b*.)
- equation of the horizontal asymptote

As students think about the significance of the horizontal asymptote, they should consider in which cases the graph approaches its asymptote as *x*-values increase (tend toward positive infinity) and in which cases the graph approaches its asymptote as *x*-values decrease (tend toward negative infinity).

Students are expected to identify which functions are increasing (representing exponential growth) or decreasing (representing exponential decay) according to whether the base, *c*, is greater than 1 or between 0 and 1 and if *b* is positive or negative.

	Example	Graph	Type of function
$c > 1$, $b > 0$	$y = 2^x$		growth

	Example	Graph	Type of function
$c > 1, b < 0$	$y = 2^{-x}$		decay (Reflection in the y-axis of growth curve $y = 2^x$.) Note that the equation $y = 2^{-x}$ could be rewritten as $y = \left(\frac{1}{2}\right)^x$.
$0 < c < 1, b > 0$	$y = \left(\frac{1}{2}\right)^x$		decay
$0 < c < 1, b < 0$	$y = \left(\frac{1}{2}\right)^{-x}$		growth (Reflection in y-axis of the decay curve $y = \left(\frac{1}{2}\right)^x$.) Note that the equation $y = \left(\frac{1}{2}\right)^{-x}$ could be rewritten as $y = 2^x$.

Students are expected to identify how graphs of various exponential curves are similar and how they are different.

Students should explore graphs of other exponential functions where $c > 1$ and $0 < c < 1$. They should also graph $y = c^x$ where $c = 1$ and where $c < 0$.

When $c = 1$, a horizontal line is produced.

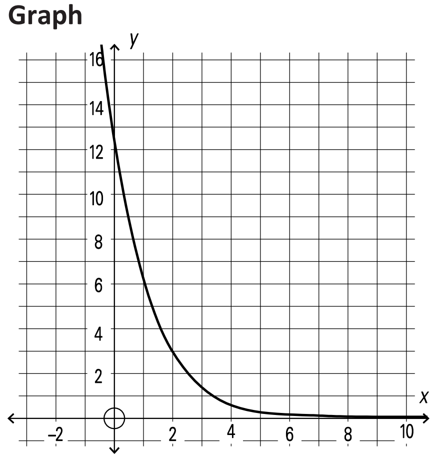
When c is negative, if integer values of x are chosen, the y-values "oscillate" between positive and negative values; for rational values of x, non-real values of y may be obtained. To see this, students could complete the table to the right for the functions $y = (-2)^x$ and $y = (1)^x$.

x	y
-2	
$-\frac{3}{2}$	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
$\frac{3}{2}$	
2	

Graphs and tables of values for exponential functions should also be analyzed by students to determine the corresponding functions.

Example:

Graph



This graph represents exponential decay. For every time x increases by 1, the value of y is cut in half. The value of y when x = 0 is 12. Therefore, the function is

$$f(x) = 12\left(\frac{1}{2}\right)^x.$$

Table of values

x	$f(x)$
0	10
1	20
2	40
3	80

This table of values represents exponential growth. For every time x increases by 1, the value of y is doubled. The value of y when x = 0 is 10. Therefore, the function is $f(x) = 10(2)^x$.

Problems involving exponential growth and decay can be introduced here in contexts such as half-lives, bacterial growth/decay, light intensity, and finance, provided these situations involve functions of the form $y = c^x$. More extensive work involving functions of the form $y = a(c)^{b(x-h)} + k$ will be done once students have applied transformations to $y = c^x$. Students should identify any necessary restrictions on the domain and range due to the context of the problem.

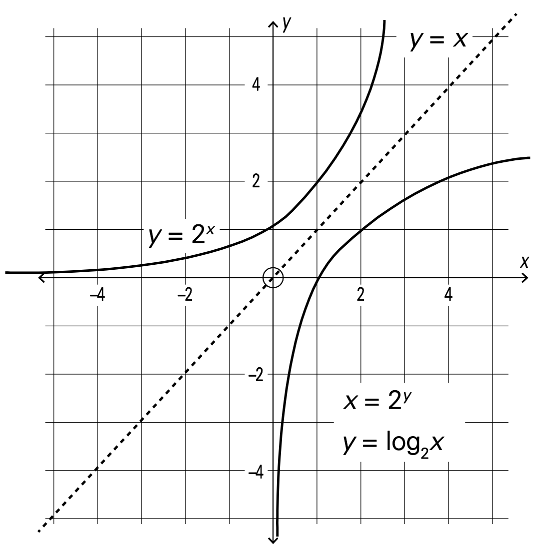
Previous work with transformations is extended to include exponential functions. Students apply reflections, stretches, and translations to exponential growth and decay curves, and then relate them to the parameters *a*, *b*, h, and *k* in a function of the form $y = a(c)^{b(x-h)} + k$, for a variety of values of *c*. Students could also write the mapping rules as they work with the transformations. It may be beneficial to first explore the types of transformations one at a time.

Once the effect of each transformation has been explored, students should work with combinations of transformations for functions of the form $y = a(c)^{b(x-h)} + k$, using a variety of values for the parameters *a*, *b*, *h*, and *k*. In the Function Transformations unit of *Pre-Calculus 12* (McAskill et al. 2012), students explored the order in which transformations should be applied. This concept should be reviewed here.

Applications of exponential growth or decay, such as cooling behaviour of a liquid, radioactive decay, medications, and light intensity, may be further addressed at this point.

It is important that students understand the inverse relationship between exponential and logarithmic functions. They can use their previous knowledge of exponential functions and inverses to explore this relation. In the Function Transformations unit of *Pre-Calculus 12* (McAskill et al. 2012), students determined inverse equations by interchanging the *x* and *y* variables and solving for *y* (RF05). They can apply this procedure to determine the inverse of an exponential function, such as $f(x) = 2^x$. Writing this function as $y = 2^x$ and switching *x* and *y* leads to solving $x = 2^y$ for *y*. This requires writing the equation in logarithmic form and results in $y = \log_2(x)$. The inverse function is $f^{-1}(x) = \log_2(x)$.

Students can use the table of values for $f(x) = 2^x$ to generate the table for $f^{-1}(x) = \log_2(x)$ by interchanging the domain and range. From this, the graphs can be created.



Point on the exponential curve $y = 2^x$	Corresponding point on the inverse function $y = \log_2 x$
$\left(-3, \frac{1}{8}\right)$	$\left(\frac{1}{8}, -3\right)$
$\left(-2, \frac{1}{4}\right)$	$\left(\frac{1}{4}, -2\right)$
$\left(-1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -1\right)$
(0, 1)	(1, 0)
(1, 2)	(2, 1)
(2, 4)	(4, 2)
(3, 8)	(8, 3)

Students are expected to observe that the graphs are reflections of each other in the line $y = x$. They are expected to identify the relationship between the domain and range for both functions. This method of graphing can be applied to any logarithmic function of the form $y = \log_c(x)$; $c > 0$, $c \neq 1$.

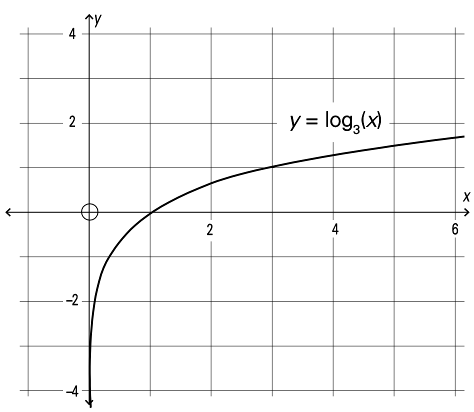
Students will explore ways that transformations affect graphs of logarithmic functions. They will develop the ability to sketch these graphs both with and without technology and learn to manipulate the logarithmic equation, $y = a \log_c [b(x - h)] + k$ to produce any given graph. Transformations, when applied to logarithmic functions, will be restricted to bases that are greater than 1.

The following chart summarizes the effects of the parameters a, b, h, and k, on the graphs of logarithmic functions. Students should understand that these effects are the same as they found for exponential functions.

Parameter	**Transformation**
a	▪ Vertical stretch about the x-axis by a factor of $\lvert a \rvert$ ▪ For $a < 0$, reflection in the x-axis
b	▪ Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ ▪ For $b < 0$, reflection in the y-axis
k	▪ Vertical translation up or down by k units
h	▪ Horizontal translation left or right by h units

Students are expected to apply their previous work with transformations to logarithmic functions and to explore the impact of the parameters a, b, h, and k on the function $y = a \log_c [b(x - h)] + k$.

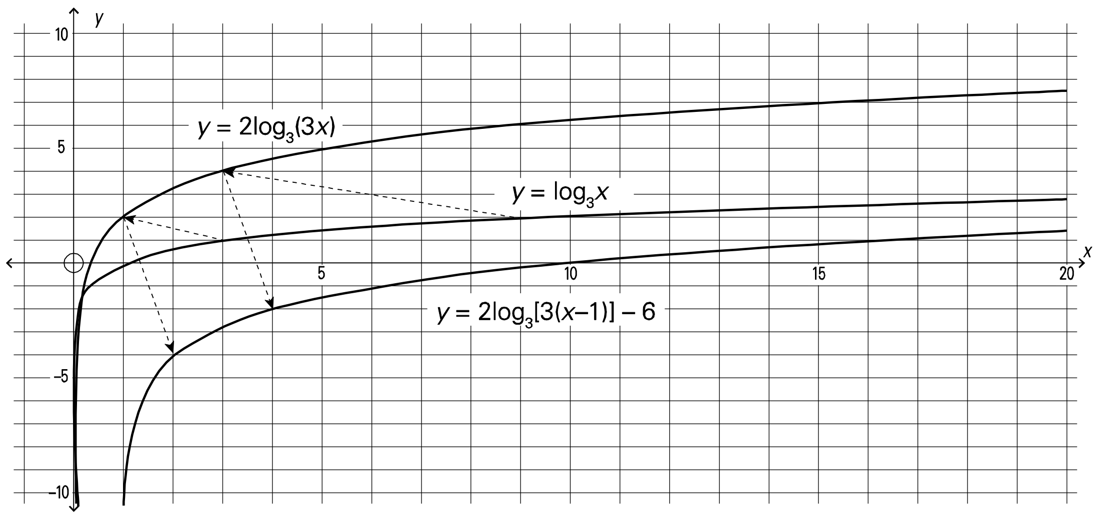
For example, students are expected to sketch a logarithmic function, such as $y = \log_3 x$, and identify the domain, range, intercepts, and vertical asymptote. They should note that the domain is restricted by the vertical asymptote.



Using this base function, students are expected to sketch a transformation, such as
$y = 2 \log_3 [3(x - 1)] - 6$.

- To graph this, students could identify the transformations.
  - The vertical stretch is 2.
  - The horizontal stretch is $\frac{1}{3}$.
  - The horizontal translation is 1.
  - The vertical translation is −6.

- Students should understand that the horizontal translation determines the vertical asymptote, which also defines the domain. Therefore, the vertical asymptote of $y = 2\log_3[3(x-1)] - 6$ will be $y = 1$. To obtain a more accurate sketch of $y = 2\log_3[3(x-1)] - 6$, students pick a point on the original, such as (3, 1), and apply the stretches to obtain $\left[\frac{1}{3}(3), 2(1)\right]$ or (1, 2) and then apply translations to that point, (1, 2), to obtain (3 + 1, 2 – 6) or (4, –4).



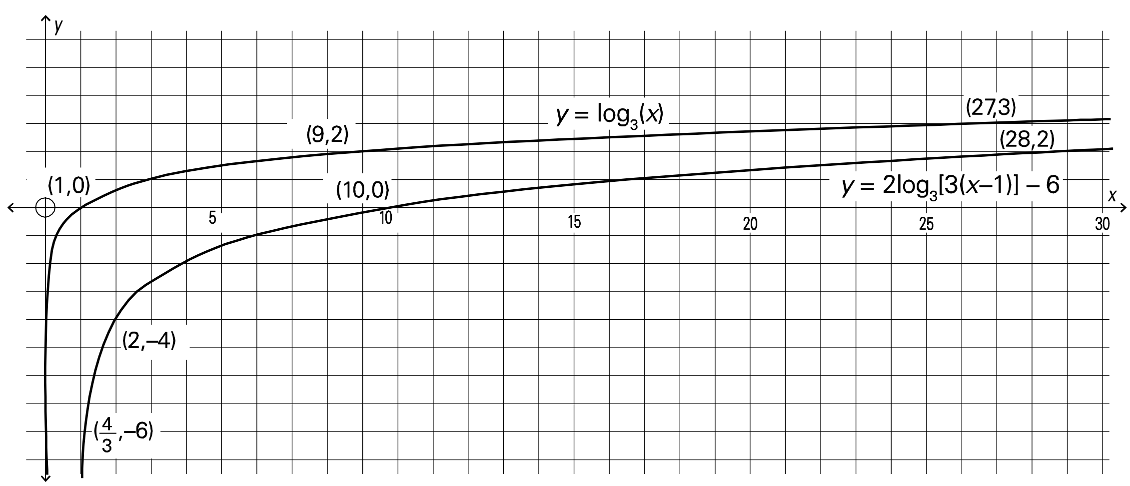
Another approach is to create the mapping rule $(x, y) \rightarrow \left(\frac{1}{3}x + 1, 2y - 6\right)$ and then produce a table of values for the transformed function.

$y = \log_3(x)$

x	***y***
1	0
9	2
27	3
81	4

$y = 2\log_3[3(x-1)] - 6$

x	***y***
$\frac{4}{3}$	–6
2	–4
10	0
28	2



The importance and meaning of the horizontal asymptote for exponential graphs, and the vertical asymptote for logarithmic graphs should be emphasized. The concept of limits can be introduced, but is not an expectation for this course.

The following chart summarizes characteristics of exponential and logarithmic functions.

	Exponential	**Logarithmic**
Function	$y = c^x$, $c > 1$, $c \neq 1$, $c \in \mathrm{R}$	$y = \log_c(x)$, $c > 0$, $c \neq 1$, $c \in \mathrm{R}$
Domain	$\{x \mid x \in \mathrm{R}\}$ or $(-\infty,\infty)$	$\{x \mid x > 0, x \in \mathrm{R}\}$ or $(0,\infty)$
Range	$\{y \mid y > 0, y \in \mathrm{R}\}$ or $(0,\infty)$	$\{y \mid y \in \mathrm{R}\}$ or $(-\infty,\infty)$
x- intercept	none	(1, 0)
y-intercept	(0, 1)	none
Increasing	When $c > 1$	When $c > 1$
Decreasing	When $0 < c < 1$	When $0 < c < 1$
Asymptote	*x*-axis or $y = 0$	*y*-axis or $x = 0$

To determine the *x*-intercept of the transformed logarithmic function, students should solve the equation for $y = 0$. They are familiar with this from work with other types of functions. They can determine the *x*-intercept of $y = 2\log_3[3(x-1)] - 6$, for example, as follows:

$0 = 2\log_3[3(x-1)] - 6$
$6 = 2\log_3(3x-3)$
$3 = \log_3(3x-3)$
$27 = 3x - 3$
$x = 10$

Similarly, when determining the *y*-intercept, students let $x = 0$. At this point, there may be cases where it is necessary to use benchmarks to approximate the intercept. Once students are familiar with the base change procedure, they will be able to determine the *y*-intercepts more accurately and quickly with the use of a calculator.

Students should also work with logarithmic functions that do not have a *y*-intercept. Evaluating $y = 2\log_3[3(x-1)] - 6$ for $x = 0$, for example, results in $y = 2\log_3(-3) - 6$. Since the domain of $y = \log_3(x)$ is $\{x \mid x > 0, x \in \mathrm{R}\}$ or $(0,\infty)$, this cannot be evaluated.

Students may also have to factor an expression in order to determine the horizontal stretch. Before $y = 3\log_4(-5x-5) - 6$ can be graphed, for example, it should be rewritten as $y = 3\log_4[-5(x+1)] - 6$.

Students should graph logarithmic equations with a reflection in the *y*-axis to see the effect on the domain. They should understand that for $y = 3\log_4[-5(x-6)] + 1$, the vertical asymptote is $x = 6$, but the domain is $\{x \mid x > 6, x \in \mathrm{R}\}$ or $(-\infty, 6)$ due to the reflection in the *y*-axis.

Students are not responsible for determining the equation of a logarithmic function, given the graph. However, given the graph of a logarithmic function, they should identify the function from a list of options.

Students should sketch the graph of a given logarithmic function, clearly showing the asymptote and intercepts, and identify the graph of a given logarithmic function from a list of options.

Assessment, Teaching, and Learning

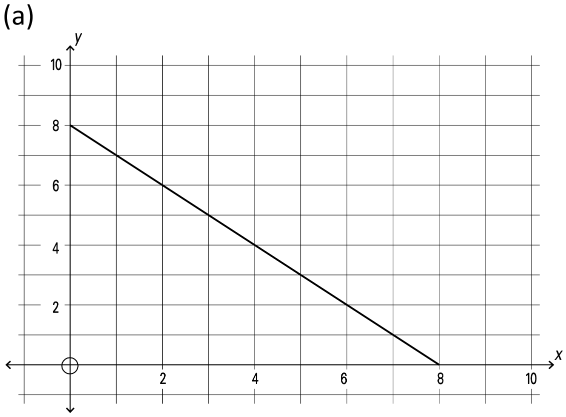
Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

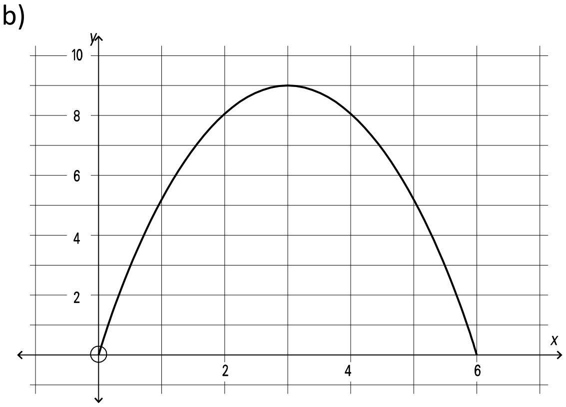
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Describe situations that might be represented by the following graphs:

(a)



(b)



- Describe situations that might be represented by the following data:

(a)

x	y
1	8
2	16
3	32
4	64
5	128

(b)

x	y
1	8
2	10
3	12
4	14
5	16

(c)

x	y
1	2
2	5
3	6
4	5
5	2

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- On the same set of axes, sketch the graphs of $y = 2^x$ and $y = x^2$. Write a sentence describing any differences you notice between the two graphs.
- For points (x, y) on the graph of the function $y = 4^x$, find the missing values.

$(-2, y)$, $(-1, y)$, $(0, y)$, $(1, y)$, $(2, y)$, $(3, y)$, $\left(x, \frac{1}{8}\right)$, $\left(x, \frac{1}{4}\right)$, $(x, 1)$, $(x, 4)$, $(x, 1024)$

- Determine whether each of the following functions is exponential.

  (a) $y = x^4$

  (b) $y = \left(\frac{1}{2}\right)^x$

  (c) $y = (0.25)^x$

- Sketch the graph of each of the following exponential functions.

  (a) $y = 5^x$

  (b) $y = \left(\frac{1}{3}\right)^x$

- For each of the following, determine whether the function is increasing or decreasing and explain why.

  (a) $y = 4^x$

  (b) $y = \left(\frac{1}{3}\right)^x$

  (c) $y = \left(\frac{5}{2}\right)^x$

- Explain why an exponential function cannot have a negative base or a base that equals 0 or 1.
- Explain why graphs of functions of the form $y = c^x$, $c > 0$, $c \neq 1$

  (a) do not have x-intercepts

  (b) always have y-intercept (0, 1)

  (c) always have the same domain and range

- Determine which situations require a value of $c > 1$ (growth) and which require a value of $0 < c < 1$ (decay). Explain your choice.

  (a) The number of neutrons present in a nuclear fission reaction triples at each stage of progression.

  (b) The volume of ice in a certain region of the arctic ice-cap is shrinking at a rate of 0.5% per year.

  (c) Lionel receives a pay increase of 2.5% per year.

- The doubling period of a bacterium is 20 min. If there are 300 bacteria initially in a culture, how many bacteria will there be after

  (a) 40 min

  (b) 2 h

- The half-life of a certain isotope is two days. How much will be left from a mass of 500 g after

  (a) six days

  (b) two weeks

- Describe how you could know, from an equation, if an exponential function would be growth or decay.

- Describe the transformations that must be applied to the graph of $y = f(x)$ to obtain the transformed function.

  (a) $f(x) = 2^x$; $g(x) = 5(2^x) + 1$

  (b) $f(x) = \left(\frac{1}{3}\right)^x$; $g(x) = \left(\frac{1}{3}\right)^{x-4} + 2$

  (c) $f(x) = 4^x$; $g(x) = \frac{1}{2}\left[4^{3(x+1)}\right] - 5$

- Graph each of the following, giving the domain, range, and equation of the horizontal asymptote.

  (a) $y = -\frac{1}{3}(2)^{x+3} - 1$

  (b) $y = 2(3)^{-\frac{1}{2}(x-2)} + 5$

- Identify the value of each parameter (a, b, h, and k) in the function $y = a(c)^{b(x-h)} + k$ and its effect on the original graph of $y = (c)^x$ for the functions.

  (a) $y = -2\left(\frac{1}{2}\right)^{3(x-1)} + 4$

  (b) $y = \frac{1}{2}(3)^{-x+2} - 4$

- Match each graph with the corresponding function.

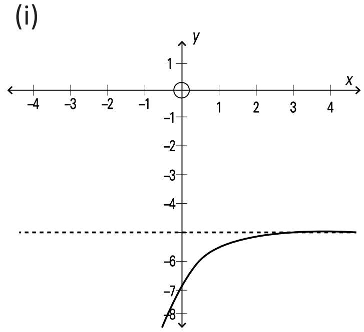
  (a) $y = 3^{2(x-1)} - 2$

  (b) $y = 2^{x-2} + 1$

  (c) $y = -4^{x+2}$

  (d) $y = -2(3)^{-x} - 5$

(i)



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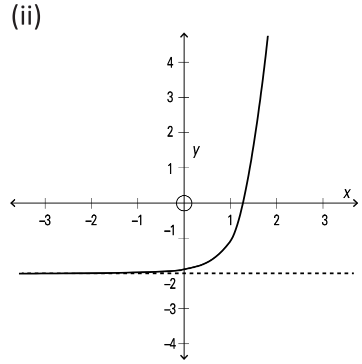

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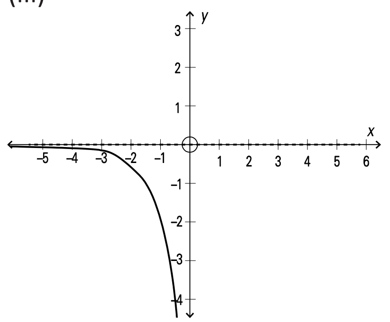

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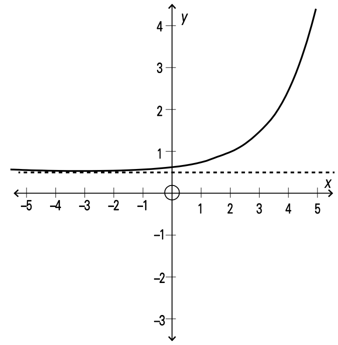


- The equation $S = (0.8)^d$ models the fraction of sunlight, S, that reaches a scuba diver under water, where d is the depth of the water, in metres. Determine what percent of sunlight reaches a diver 3 m below the surface of the water.
- Create an exponential decay function that has a horizontal asymptote at $y = -4$ and a y-intercept at (0, 2). Graph your function and explain why you selected the values of a, b, h, and k that you did.
- For points (x, y) on the graph of the function $y = \log_4(x)$, find the missing values:

  $(x, -2)$, $(x, -1)$, $(x, 0)$, $(x, 1)$, $(x, 2)$, $(x, 3)$, $\left(\frac{1}{8}, y\right)$, $\left(\frac{1}{4}, y\right)$, $(1, y)$, $(4, y)$, $(64, y)$, $(1024, y)$

- Identify the following characteristics of the graph of each function:
  - The domain and range.
  - The y-intercept (to one decimal place)
  - The equation of the asymptote
  - The x-intercept (to one decimal place)

  (a) $y = -2\log(x + 1)$

  (b) $y = \log3[2(x + 3)]$

  (c) $y = -3\log(x - 2) - 4$

  (d) $y = \log(x + 3) + 1$

  (e) $y = 2(3)x - 4 + 1$

  (f) $y = 3\left(\frac{1}{2}\right)^{x+1} - 5$

  (g) $y = \ln(x - 4) - 2$

  (h) $y = -\ln(x + 2) + 3$

- Sketch the graph of $y = 2^x$ and $y = \log_2(x)$ on the same coordinate grid. How are these functions related?
- Describe the transformations that must be applied to the graph of $y = \ln(x)$ to obtain the transformed function.

  (a) $y = \ln(x + 3) + 1$

  (b) $y = \frac{1}{3}\ln(5x)$

  (c) $y = -\ln\left[\frac{1}{2}(x+1)\right] - 5$

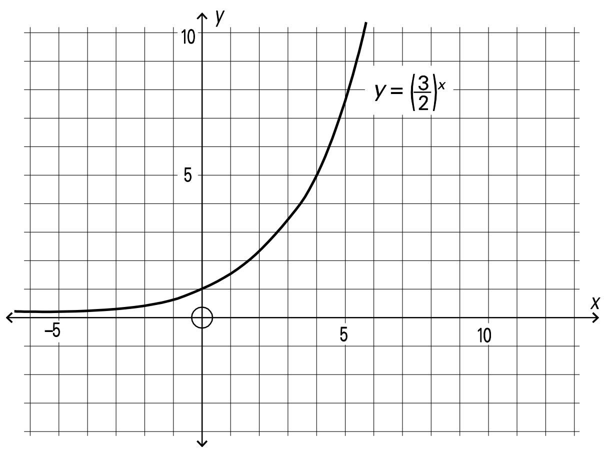
- Graph each of the following using transformations.

  (a) $y = 2\log_2(-x + 1) - 6$

  (b) $y = -\frac{1}{2}\log_3[2(x + 3)] + 1$

- Identify the intercepts, vertical asymptote, and domain of $y = 2\log_6[3(x + 4)] - 5$.

- Apply the mapping rule $(x,y)\rightarrow\left(-\frac{1}{2}x+1,3y-12\right)$ to $y = \log_4 x$. Write the resulting function and identify the domain, range, intercepts, and vertical asymptote.
- Why do all functions of the form $y = \log_c x$ intersect at (1, 0)?
- Determine the x-intercept for the graph of $y=a\log_c\left(bx\right)$.
- The point $\left(\frac{1}{36},-2\right)$ is on the graph of $y = \log_c x$. If the point (k, 216) is on the graph of the inverse, determine the values of c and k.
- The equivalent amount of energy, E, in kilowatt-hours, released from an earthquake with a Richter magnitude of R is determined by the function $R = 0.67 \log(0.36E) + 1.46$. How much energy would be released by a small earthquake measuring 3.5 on the Richter scale? Round off the answer to the nearest kilowatt-hour.
- Given the graph and equation of the exponential function $f\left(x\right)=\left(\frac{3}{2}\right)^x$, determine the equation and the domain and range of the inverse relation.



- Explain why you cannot evaluate $\log_c(-3)$ and $\log_c(0)$.

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- As an introduction to exponential functions, students should create tables of values and their corresponding graphs for exponential functions of the form $y = c^x$, where $c > 1$ and $0 < c < 1$. They could create tables for functions such as the following:

(a) $y = 2^x$

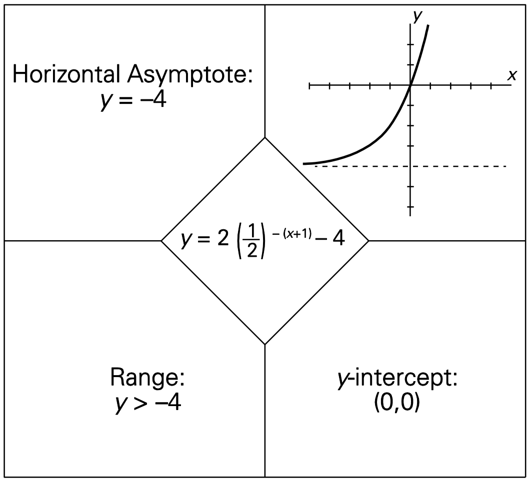
(b) $y=\left(\frac{3}{2}\right)^x$

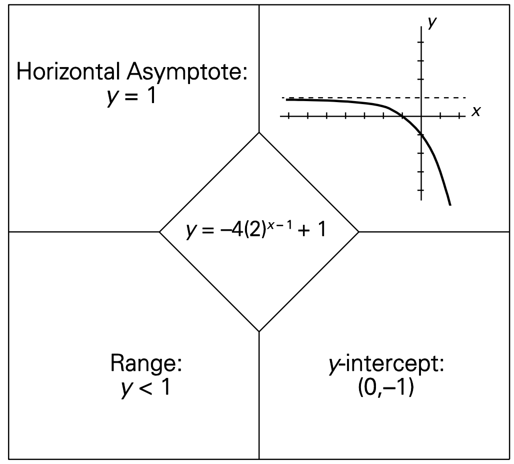
(c) $y = 2.8^x$

(d) $y=\left(\frac{2}{3}\right)^x$

(e) $y = 0.16^x$

- For exponential functions, have students explore real-life examples, such as the exponential decrease of concentration of medication in the blood stream.
- In *Pre-calculus 12* (McAskill et al. 2012), there is an excellent investigation on page 346, "Investigate Transforming an Exponential Function."
- It may be beneficial to first explore the types of transformations one at a time.
  - Translations
    - > Students could create tables of values and the graphs for functions of the form $f(x) = c^x + k$. Using functions such as $f(x) = 2^x$, $f(x) = 2^x + 3$, and $f(x) = 2^x - 4$, they should explore the connection between the value of k and the vertical translation.
    - > Students should then explore the effects of h for functions of the form $f(x) = c^{x-h}$, such as $f(x) = 2^{x+3}$ and $f(x) = 2^{x-4}$.
    - > Further exploration with other bases could be done with the aid of graphing technology. Students could also write the mapping rules relating the graph of $y = c^x$ to the transformed graphs.
  - Stretches
    - > Vertical stretches should be explored with functions of the form $y = a(c)^x$ by creating tables of values for a set of functions such as $y = 2(5)^x$, $y = 4(5)^x$, and $y = \frac{1}{3}(5)^x$.
    - > Horizontal stretches can be explored using a set of functions of the form $y = c^{bx}$, such as $y = (5)^{2x}$, $y = (5)^{\frac{x}{3}}$, and $y = (5)^{0.2x}$.
  - Reflections
    - > Reflections in the x-axis should be explored for functions of the form $y = -(c)^x$ with tables of values for a set of functions such as $y = -(2)^x$, $y = -\left(\frac{1}{3}\right)^x$, and $y = -(4)^x$.
    - > Reflections in the y-axis for functions of the form $y = (c)^{-x}$ can be explored by creating tables of values for a set of functions such as $y = (2)^{-x}$ and $y = \left(\frac{1}{2}\right)^{-x}$.
  - Note that students should consider the relationship between $y = \left(\frac{1}{c}\right)^x$ and $y = (c)^{-x}$.
- Students can work in pairs to complete puzzles containing the characteristics and graphs of various exponential functions of the form $y = a(c)^{b(x-h)} + k$. They should work with 20 puzzle pieces (four complete puzzles) to correctly match the characteristics with each function. Sample puzzles are shown on the following page.





- Ensure that students understand that logarithms and exponents are inverses of each other. As a result, many logarithmic problems can be solved by converting them to exponential problems and exponential problems can be solved by converting them to logarithmic problems.
- Starting with a logarithmic equation, $y = a \log_c[b(x - h)] + k$, have students change one factor at a time and see how each affects the graph. Students should do at least two manipulations of each (a, b, h, and k), and record their results. Students should try negative values for a, b, h, and k, to see the effect of a negative number for each of these variables.
- Give students a logarithmic graph, and the stretch of that graph, and have them determine the equation for the stretch graph (e.g., *Pre-Calculus 12,* McAskill et al. 2012., p. 330, #6).
- Students can work in pairs to complete puzzles containing the characteristics and graphs of various logarithmic functions.

Suggested Models and Manipulatives

- coins
- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- exponential decay
- exponential function
- exponential growth
- half-life
- horizontal asymptote
- logarithmic function
- vertical asymptote

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 7.1 Characteristics of Exponential Functions
    - > Student Book: pp. 334–345
  - 7.2 Transformations of Exponential Functions
    - > Student Book: pp. 346–357
  - 8.1 Understanding Logarithms
    - > Student Book: pp. 372–382
  - 8.2 Transformations of Logarithmic Functions
    - > Student Book: pp. 383–391

SCO RF10 Students will be expected to solve problems that involve exponential and logarithmic equations. [C, CN, PS, R]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF10.01 Determine the solution of an exponential equation in which the bases are powers of one another.
RF10.02 Determine the solution of an exponential equation in which the bases are not powers of one another, using a variety of strategies.
RF10.03 Determine the solution of a logarithmic equation, and verify the solution.
RF10.04 Explain why a value obtained in solving a logarithmic equation may be extraneous.
RF10.05 Solve a problem that involves exponential growth or decay.
RF10.06 Solve a problem that involves the application of exponential equations to loans, mortgages, and investments.
RF10.07 Solve a problem that involves logarithmic scales, such as the Richter scale and the pH scale.
RF10.08 Solve a problem by modelling a situation with an exponential or a logarithmic equation.

Scope and Sequence

Pre-calculus 11	**Pre-calculus 12**
RF10 Students will be expected to analyze geometric sequences and series to solve problems.	**RF10** Students will be expected to solve problems that involve exponential and logarithmic equations.

Background

In Mathematics 9, students rewrote numbers as powers and solved problems involving the laws of exponents for whole number exponents (N01, N02). In Mathematics 10, they worked with negative exponents, rational exponents, the exponent laws, radicals, and problems involving laws of exponents (AN03). They also worked with variable bases.

Students now solve exponential equations with variable exponents where the bases can both be expressed as rational powers of the same base, including radical bases. They will work with equations such as $(25)^x = \left(\frac{1}{125}\right)^3$. In this case, both bases can be expressed as integer powers of 5.

Students should develop estimation skills for the solutions of exponential equations with variable exponents, including those where the bases cannot both be expressed as rational powers of the same base.

When solving equations using logarithms, they will be better able to determine the reasonableness of solutions. Suggested strategies include systematic trial and graphing technology.

Systematic Trial

Students could solve $2^x = 10$, correct to two decimal places, using the process outlined here. Since 10 is closer to $2^3 = 8$ then to $2^4 = 16$, students might begin with $x \doteq 3.3$.

Test value for *x*	**Power**	**Approximate value**
3.3	$2^{3.3}$	9.849
3.4	$2^{3.4}$	10.556
The value obtained for $2^{3.3}$ is closer to 10, so the next estimate should be closer to 3.3 than to 3.4.		
3.31	$2^{3.31}$	9.918
3.32	$2^{3.32}$	9.987
3.33	$2^{3.33}$	10.056

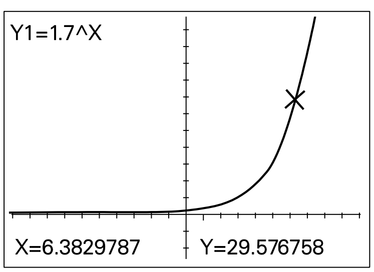
Students should reason that the best estimate is $x \doteq 3.32$ because 9.987 is closer to 10 than 10.056 is.

Students should use systematic trial to solve exponential equations with a variety of bases, including rational bases. Although the precision of estimates can vary, a minimum of one decimal place is required.

Graphing Technology

Students could also use graphing technology to determine the solution for an exponential equation. With graphing technology, the solution can be found using the graph or a table of values.

To determine the solution to the equation $1.7^x = 30$, for example, the graph of $y = 1.7^x$ can be used to determine the value of x that makes the value of the function approximately 30.

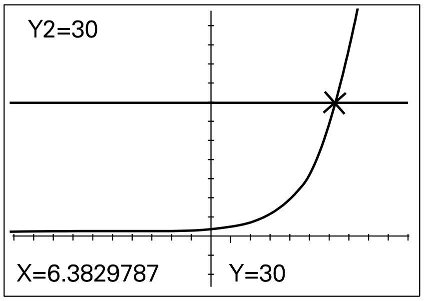


Students could also generate the table of values associated with the equation to find the value of x that makes the value of the function approximately 30.

x	*y*
6	24.138
6.1	25.453
6.2	26.84
6.3	28.303
6.4	29.845
6.5	31.472
6.6	33.187

$x = 6.4$

Alternatively, students can determine the intersection of the graphs of $y = 1.7^x$ and $y = 30$ to get the approximate solution.



Students should work with problems that can be modelled by exponential functions or equations, such as modelling the cooling behaviour of a liquid, radioactive decay, medications, half-lives, doubling time, bacterial growth/decay, light intensity, and finance. They should solve problems in situations where

- an exponential function or equation is given
- the graph of an exponential function is given
- a situation is given and they have to create an exponential model to find the solution

Students should answer questions such as the following:

- Shelly initially invests \$500, and the value of the investment increases by 4% annually.
  - Create a function to model the situation.
  - How much money is in Shelly's investment after 30 years?
  - What amount of time will it take for the investment to double?

  Students should solve this problem using systematic trial, a table of values, or graphing technology.
- The half-life of Radon 222 is 92 hours. From an initial sample of 48 g, how long would it take to decay to 6 g?

  Students could use either of the two methods below to solve this problem algebraically:

$A = 48\left(\frac{1}{2}\right)^n$

n is the number of 92-hour increments

$6 = 48\left(\frac{1}{2}\right)^n$

$\frac{1}{8} = \left(\frac{1}{2}\right)^n$

$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^n$

$3 = n$

$\therefore$ it would take 276 hours

$A = 48\left(\frac{1}{2}\right)^{\frac{1}{92}}$

t is the number of hours

$6 = 48\left(\frac{1}{2}\right)^{\frac{1}{92}}$

$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{1}{92}}$

$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{1}{92}}$

$3 = \frac{t}{92}$

$276 = t$

Students are also required to model a situation, such as the following, using an exponential function, when given specific parameters.

- A cup of hot chocolate is served at an initial temperature of 80°C and then allowed to cool in a stadium with an air temperature of 5°C. The difference between the hot chocolate temperature and the temperature of the stadium will decrease by 30% every six minutes. If T represents the temperature of the hot chocolate in degrees Celsius, measured as a function of time, t, in minutes, students can answer the following questions:
  - What is the transformed exponential function in the form $T = a(c)^{b(t-h)}\ k$?

    Students should understand that
    - > this situation would be a decay curve since the temperature is decreasing over time ($c < 1$)
    - > the horizontal asymptote would be $y = 5$ ($k = 5$)
    - > the base would be 100% – 30% or 70% ($c = 0.7$)
    - > the horizontal stretch factor would be 6 ($b = \frac{1}{6}$)
    - > the vertical stretch factor can be determined algebraically

      $$T = a(c)^{b(t-h)} + k$$
      $$T = a(0.7)^{\frac{1}{6}(t)} + 5$$

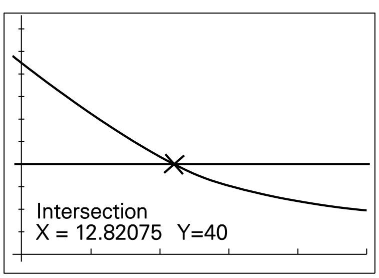
      Using the point (0, 80),

      $$80 = a(0.7)^{\frac{1}{6}(0)} + 5$$
      $$80 = a + 5$$
      $$75 = a$$

      Therefore, the equation that describes this situation is $T = 75(0.7)^{\frac{1}{6}(t)} + 5$.
  - What is the temperature at time $t = 11$ minutes?

    $$T = 75(0.7)^{\frac{1}{6}(11)} + 5$$
    $$T = 44°\mathrm{C}$$
  - How long does it take the hot chocolate to cool to a temperature of 40°C?

    As a possible method, a graphical solution is shown.



Students are required to model a situation with an exponential function from a given graph, a table of values, or a description when the equation of the horizontal asymptote is $y = 0$.

When the asymptote is $y = 0$, students are expected to determine all parameters from the given information.

When the asymptote is not $y = 0$, as in the above example, students are not required to determine the common ratio from a table of values or graph. They are required to determine the common ratio from the problem description.

To solve problems involving finance, students must become familiar with how annual interest rates are applied as well as common compounding periods. Students should understand that compounding semi-annually, for example, means interest is calculated twice a year.

Until logarithms have been introduced and explored, students solve exponential equations where the powers could both be expressed as rational powers of the same base. In cases where the bases were not the same, graphing technology and systematic trial were used to estimate the value of the variable (RF09).

Once students are familiar with logarithms, they will use them to solve these exponential equations as well as solve logarithmic equations.

They should be able to determine solutions to exponential equations as both approximate and exact values. When solving $3^{x+1} = 5^{4x-3}$, students take the common logarithms or natural logarithm of both sides of the equation, or change from exponential to logarithmic form.

Common log of both sides

$$\begin{aligned}
&3^{x+1} = 5^{4x-3}\\
&\log\left(3^{x+1}\right) = \log\left(5^{4x-3}\right)\\
&(x+1)\log(3) = (4x-3)\log(5)\\
&x\log(3) + \log(3) = 4x\log(5) - 3\log(5)\\
&x\log(3) - 4x\log(5) = -3\log(5) - \log(3)\\
&x\left[\log(3) - 4\log(5)\right] = -3\log(5) - \log(3)\\
&x = \frac{-3\log(5) - \log(3)}{\log(3) - 4\log(5)} \doteq 1.11
\end{aligned}$$

Natural log of both sides

$$\begin{aligned}
&3^{x+1} = 5^{4x-3}\\
&\ln\left(3^{x+1}\right) = \ln\left(5^{4x-3}\right)\\
&(x+1)\ln(3) = (4x-3)\ln(5)\\
&x\ln(3) + \ln(3) = 4x\ln(5) - 3\ln(5)\\
&x\ln(3) - 4x\ln(5) = -3\ln(5) - \ln(3)\\
&x\left[\ln(3) - 4\ln(5)\right] = -3\ln(5) - \ln(3)\\
&x = \frac{-3\ln(5) - \ln(3)}{\ln(3) - 4\ln(5)} \doteq 1.11
\end{aligned}$$

Changing forms

$$\begin{aligned}
&3^{x+1} = 5^{4x-3}\\
&\log_3\left(5^{4x+3}\right) = x+1\\
&(4x-3)\log_3(5) = x+1\\
&4x\log_3(5) - 3\log_3(5) = x+1\\
&4x\log_3(5) - x = 1 + 3\log_3(5)\\
&x\left[4\log_3(5) - 1\right] = 1 + 3\log_3(5)\\
&x = \frac{1 + 3\log_3(5)}{4\log_3(5) - 1} = \frac{5.395}{4.86} = 1.11
\end{aligned}$$

Students are expected to verify the solution(s) obtained. Students should be encouraged to use graphing technology or systematic trial to check the reasonableness of their solution. Verification can also be done by substituting the solution back into the original equation.

When logarithmic expressions have the same base, the **arguments** $(2x + 5)$ and (11), as in the following example, are equal to one another.

$\log_6(2x + 5) = \log_6 11$
$2x + 5 = 11$
$x = 3$

When solving a simple logarithmic equation of the form $\log_c(x) = n$ algebraically, there are two approaches. The logarithmic equation can be changed to exponential form or it can be rewritten so that the augments can be equated.

For example, to solve $\log_3(2x + 5)= 4$:

Changing to exponential form

$$\log_3(2x+5)=4$$
$$3^4=(2x+5)$$
$$81=2x+5$$
$$76=2x$$
$$x=38$$

Rewriting so arguments can be equated.

$$\log_3(2x+5)=4$$
$$\log_3(2x+5)=\log_3(3^4)$$
$$\left(2x+5\right)=\left(3^4\right)$$
$$2x+5=81$$
$$2x=76$$
$$x=38$$

Students are expected to solve logarithmic equations that require them to find solutions for both linear and quadratic equations.

Example:

$$\log_5\left(x+1\right)+\log_5\left(x-2\right)=\log_5\left(4\right)$$
$$\log_5\left[\left(x+1\right)\left(x-2\right)\right]=\log_5\left(4\right)$$

Equating the arguments we note that

$$(x+1)(x-2)=4$$
$$x^2-x-2=4$$
$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$x=3;\ x=-2$$

$x = -2$ has to be eliminated since it produces a negative argument. Therefore, the answer is $x = 3$.

$$\log_2\left(4x+1\right)-\log_2\left(2x-1\right)=3$$
$$\log_2\left(\frac{4x+1}{2x-1}\right)=3$$

Changing to exponential form:

$$\log_2\left(\frac{4x+1}{2x-1}\right)=3$$
$$2^3=\frac{4x+1}{2x-1}$$
$$8(2x-1)=4x+1$$
$$16x-8=4x+1$$
$$12x=9\ ;\quad x=\frac{3}{4}$$

Changing to logs of same base:

$$\log_2\left(\frac{4x+1}{2x-1}\right)=\log_2\left(8\right)$$
$$\frac{4x+1}{2x-1}=8$$
$$4x+1=8(2x-1)$$
$$4x+1=16x-8$$
$$12x=9\ ;\quad x=\frac{3}{4}$$

When solving equations, non-permissible values of the variable must be considered and solutions must be checked to ensure that they do not create negative arguments.

Example:

$\log_2(x+1)+\log_2(x)=1$

$\log_2[(x+1)(x)]=1$

$(x+1)(x)=2^1$

$x^2+x=2$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2; x=1$

Since the argument of a logarithm must be positive, we know that $x>-1$; therefore, the answer to this equation is $x = 1$.

$\log_2(x^2+20)-\log_2(x+5)=3$

$\log_2\left(\frac{x^2+20}{x+5}\right)=3$

$\frac{x^2+20}{x+5}=2^3$

$x^2+20=8(x+5)$

$x^2+20=8x+40$

$x^2-8x-20=0$

$(x-10)(x+2)=0$

$x=10; x=-2$

We know that $x>-5$; therefore, both $x = 10$ and $x = -2$ are correct.

Students are expected to combine their factoring skills with their knowledge of logarithms and exponents to solve equations.

Example:

$2^{2x+1}-9(2^x)+4=0$

$(2^1)(2^{2x})-9(2^x)+4=0$

$2(2^x)^2-9(2^x)+4=0$

$[2(2^x)-1][(2^x)-4]=0$

$2^x=\frac{1}{2}; 2^x=4$

$x=-1; x=2$

$2[\ln(x)]^3-8\ln(x)=0$

$2\ln(x)\left\{[\ln(x)]^2-4\right\}=0$

$2\ln(x)[\ln(x)+2][\ln(x)-2]=0$

$\ln(x)=0;\ \ln(x)=-2;\ \ln(x)=2$

$x=e^0=1; x=e^{-2}=\frac{1}{e^2}; x=e^2$

Students may find it helpful to understand that when logarithmic expressions are in the form $x=a^{\log_a(m)}$, the base will be eliminated from the final answer.

For example, to solve $x=5^{\log_5(12)}$ or $x=e^{\ln(12)}$, begin by expressing in logarithmic form.

$x=5^{\log_5(12)}$

$\log_5(x)=\log_5(12)$

Equating the arguments we can state that $x = 12$.

$x=e^{\ln(12)}$

$\log_e(x)=\ln(12)$

$\ln(x)=\ln(12)$

Equating the arguments we can state that $x = 12$.

Students are expected to algebraically solve problems where

- the logarithmic or exponential equation is given
- the graph of an exponential function is given
- a situation is given that can be modelled by an exponential or logarithmic equation

Both logarithmic and exponential equations can also be solved graphically, either by graphing as a single function and finding the *x*-intercept, or by graphing the functions that correspond to each side of the equal sign, and then identifying at the point of intersection of the two graphs.

Students will solve problems involving logarithmic scales such as the Richter scale (used to measure the magnitude of an earthquake), the pH scale (used to measure the acidity of a solution), and decibel scale (used to measure sound level). When dealing with Richter scale, pH scale, or decibel scale problems, students are not expected to develop formulas but should be given the formula when it is required.

Students may be familiar with pH scales from work in science courses. The pH scale of a solution is determined using the equation $y = -\log x$, where x is the concentration of hydrogen ions in moles per litre (mol/L). The scale ranges from 0 to 14 with the lower numbers being acidic and the higher numbers being basic. A value of pH = 7 is considered neutral. The scale is a logarithmic scale with one unit of increase in pH resulting in a ten-fold decrease in acidity. Another way to look at this would be that a one-unit increase in pH results in a ten-fold increase in basicity.

The magnitude of an earthquake, y, can be determined using $y = \log x$, where x is the amplitude of the vibrations measured using a seismograph. An increase in one unit in magnitude results in a ten-fold increase in the amplitude.

Sound levels are measured in decibels using $\beta = 10\lfloor \log(I) + 12 \rfloor$, where β is the sound level in decibels (dB) and I is the sound intensity measured in watts per metre squared (w/m^2). This would be a good opportunity for students to measure audio volumes in the environment around them. They could use a smartphone application, for example, to show the approximate decibel level of their location. Although quite accurate, the application is mainly a tool for detecting noise level in casual settings.

The formula $A = A_0(1+r)^n$ can be used for finance calculations, where A_0 represents the initial value, r represents the interest rate per compounding period and n represents the number of compounding periods. For example, if a $1000 deposit is made at a bank that pays 12% interest compounded monthly, students should be able to determine, using logarithms, how long it will take for the investment to reach $2000. Students should also be exposed to situations where it is necessary to determine the initial value, the interest rate, or the number of compounding periods.

$A = A_0(1+r)^n$ $2000 = 1000(1+0.01)^n$ $2 = (1.01)^n$	
$\log(2) = \log(1.01)^n$ $\log(2) = n\log(1.01)$ $\dfrac{\log(2)}{\log(1.01)} = n$ $n = 69.66$	$2 = (1.01)^n$ $\log_{1.01}(2) = n$ $n = \dfrac{\log(2)}{\log(1.01)}$ $n = 69.66$

Therefore, after 70 months (5 years, 10 months) the amount of money would have doubled.

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- What is the value of x?

  (a) $2^x = 8$

  (b) $3^x = 1$

  (c) $5^x = \dfrac{1}{25}$

  (d) $6^x = \sqrt{6}$

  (e) $5\left(2^x\right) = 10$

  (f) $2\left(3^x\right) - 20 = 34$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A start has been made to solving the equation shown below. Complete the solution. Solve using a different approach, and explain which approach you prefer and why.

  $16\left(2^x\right) = 1024$

  $2^x = 64$

  $\log\left(2^x\right) = \log 64$

- For which of the following equations are the y-values equivalent?

  (a) $\log_{32} y = \dfrac{2}{5}$

  (b) $\log_{16} 2 = y$

  (c) $\log_y 81 = \dfrac{4}{3}$

  (d) $\log_8 y = \dfrac{2}{3}$

- The population of a small town is 125. If it is projected to double every 50 years, how long will it take for the town to reach a population of 8000?
- A cup of coffee contains approximately 100 mg of caffeine. When you drink the coffee, the caffeine is absorbed into the bloodstream, and is eventually metabolized by the body. Every five hours the amount of caffeine in the bloodstream is reduced by 50%. How many hours does it take for the caffeine to be reduced to 15 mg?
- Solve for *x* algebraically.

  $\log_2 x + \log_2 (x - 7) = 3$
- Solve each of the following to determine an exact value for *x*, then approximate *x* to the nearest hundredth.

  (a) $e^x = 16$
  (b) $e^{3x} = 27$
  (c) $e^{2x} = 5$
  (d) $e^{5x} = 9$
  (e) $4e^{2x} = 5$
  (f) $e^{5-3x} = 2$
  (g) $e^{x+1} = 7$
  (h) $e^{2x-1} = 3$
  (i) $4 - 2e^x = -22$
- Solve each of the following to determine an exact value for *x*, then approximate *x* to the nearest hundredth.

  (a) $\ln x = -1$
  (b) $\ln x = \frac{1}{3}$
  (c) $\ln x = 5$
  (d) $\ln(2x - 1) = 3$
  (e) $\ln(x + 2) = 4$
  (f) $\ln\left(\frac{1}{x}\right) = 2$
  (g) $\ln x = \ln 5 + \ln 8$
  (h) $5 + 2\ln x = 6$
  (i) $\ln x^2 = 2\ln 4 - 4\ln 2$
  (j) $-5 + 2\ln 3x = 5$
- Solve for *x*.

  (a) $e^{\ln(x+1)} = 5$
  (b) $e^{\ln 5x} = 10$
  (c) $e^{\ln(4x+2)} = 14$
  (d) $e^{\ln x^2 + \ln x} = 8$
  (e) $e^{\ln 2x} = 6$
  (f) $\log_{10} e^x = 1$
  (g) $\log_5 e^{2x} = 3$
  (h) $\ln(e^{2x-1}) = 5$
  (i) $\ln(e^{5x+2}) = 22$
- Chemists define the acidity or alkalinity of a substance according to the formula $pH = -\log\left\lfloor H^+ \right\rfloor$ where $\left\lfloor H^+ \right\rfloor$ is the hydrogen ion concentration, measured in moles per litre. Solutions with a value of less than 7 are acidic, greater than 7 are basic, and equal to 7 (water) are neutral.

  (a) Apple juice has a hydrogen ion concentration of $\left\lfloor H^+ \right\rfloor = 0.0003$. Find the pH value and determine whether the juice is basic or acidic.

  (b) A test of ammonia shows the hydrogen ion concentration to be $\left\lfloor H^+ \right\rfloor = 1.3 \times 10^{-9}$. Find the pH value and determine whether the ammonia is basic or acidic.
- Loudness is measured in decibels (dB) calculated using the formula $dB = 10\log\left(\frac{I}{I_0}\right)$, where I_0 is the intensity of the threshold sound, or sound that can barely be perceived, and *I* is the intensity in terms of multiples of the intensity of threshold sound.
  - Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss, and a power mower has an intensity of about $I = \left(5.0 \times 10^{10}\right)I_0$. Calculate the decibels and determine if it makes sense to wear ear protection when mowing the lawn.

- Earthquake intensity is measured on the Richter scale, calculated as $R=\log\left(\frac{I}{I_0}\right)$, where I_0 is the threshold quake, or movement that can barely be detected, and the I is the intensity in terms of multiples of that threshold intensity.
  - A seismograph set up at home, indicates that there was an event while you were out that measured an intensity of $I = 989$. Given that a heavy truck rumbling by can give a Richter rating of 3 or 3.5, and moderate quakes have a Richter rating of 4 or more, what was likely the event that occurred while you were out?
- The temperature T, in °C of a cup of hot chocolate, t minutes after it is made, is given by the equation $T = 92e^{-0.06t}$.
  (a) Find the temperature of the hot chocolate 8 minutes after it is made.
  (b) How long will it take the hot chocolate to cool to 50°C?
- At the initial count, a bacterial culture contained 1250 bacteria. Another count, 1.5 hours later, revealed 80 000 bacteria. What is the doubling period for this bacterium?
- The half-life of Sodium-24 is 14.9 hours. A hospital buys a 40-mg sample of Sodium-24.
  (a) How many milligrams, to the nearest tenth, will remain after 48 hours?
  (b) After how long will only 2.5 mg remain?
- The growth of a culture of bacteria can be modelled by the equation, $N(t)=N_0e^{0.105t}$, where $N(t)$ is the number of bacteria after t hours and N_0 is the initial number of bacteria.
  (a) If the culture has 300 bacteria initially, what is the estimated population in 12 hours?
  (b) How much time will be required for it to double in population?
- A radioactive element decays exponentially according to the formula $A=A_0\, e^{-0.04463t}$, where A is the amount present after t days and A_0 is the initial amount. If the initial amount is 80 grams,
  (a) find the amount remaining after 45 days
  (b) after how much time will the radioactive element decay to 20% of the initial amount
- Determine the solution for equations, such as the following:
  (a) $9^{2x+1} = 81$
  (b) $16^{2x+1}=\left(\frac{1}{2}\right)^{x-3}$
  (c) $40 = 4^{2x+3}$
- Algebraically determine the solution for the following equations:
  (a) $\left(\frac{1}{3}\right)^{2x-1}=(81)^{3-x}$
  (b) $5\left(\frac{1}{4}\right)^{x}=80$
  (c) $\sqrt{5}=25^{x-1}$
  (d) $27^{2x-1}=\sqrt[3]{3}$
  (e) $\sqrt[5]{8^{x-1}}=\sqrt[3]{16^{x}}$
  (f) $\sqrt{3^{x}}=9^{2x+1}$
- Is it best to estimate the solution to $(5)^{3x-1}=(12)^{2x+1}$ using systematic trial or graphing technology? Defend your choice.

- When can you solve an exponential equation algebraically, and when must you use an estimation method (graphing or systematic trial)?
- The half-life of a certain radioactive isotope is 30 hours. Algebraically determine the amount of time it takes for a sample of 1792 mg to decay to 56 mg.
- If a new car, purchased for $20,000, depreciates at a rate of 28% every two years,
  (a) what will be the value of this car after six years
  (b) what amount of time will it take for the car to lose half its value
- Solve the following equations:
  (a) $\ln^3 x - 20 = 7$
  (b) $4e^{2x} - 60 = 4$
  (c) $4e^{2x} - 5e^x - 6 = 0$
  (d) $\ln^2 x - 9 \ln x + 20 = 0$
- Choose between two investment options and justify your choice: earning 12% interest per year compounded annually or 12% interest per year compounded monthly.
- Create a logarithmic equation where the only solution is $x = 4$.
- Explain why the equation $\log_4(1 - x) + \log_4(x - 3) = 1$ has no roots.
- Why must the solution to $\log_4(x + 2) + \log_4(x - 4) = 2$ must be in the domain $\{x > 4, x \in R\}$.
- Solve each of the following logarithmic equations.
  (a) $\log_2 x = \log_2 5 + \log_2 3$
  (b) $\log x = 1 + \log 2$
  (c) $4 \log_5 x = \log_5 625$
  (d) $\log_2 (x - 2) + \log_2 x = \log_2 3$
  (e) $\log_5 (3x\ 1) + \log_5 (x - 3) = 3$
  (f) $\log_3 (x - 2) + \log_3 10 - \log_3 (x^2 + 3x - 10) = 0$
- Identify the error in the given solution and explain why it is incorrect. Write the correct solution.

$$10^x + 5 = 60$$
$$\log\left(10^x + 5\right) = \log 60$$
$$\log 10^x + \log 5 = \log 60$$
$$x \log 10 = \log 60 - \log 5$$
$$x = \frac{\log 60 - \log 5}{\log 10}$$

- The magnitude of an earthquake, M, as measured on the Richter scale, is given by $M = \log(I)$, where I is the intensity of the earthquake (measured in micrometres from the maximum amplitude of the wave produced on a seismograph). A town experiences an earthquake with a magnitude of 4.2 on the Richter scale. Four years later, the same town experiences an earthquake that is five times as intense as the first earthquake. What is the magnitude of the second earthquake?
- After taking a cough suppressant, the amount, A, in mg, remaining in the body is given by $A = 10(0.85)^t$, where t is given in hours.
  (a) What was the initial amount taken?
  (b) What percent of the drug leaves the body each hour?
  (c) How much of the drug is left in the body six hours after the dose is administered?
  (d) How long is it until only 1 mg of the drug remains in the body?

- Answer the following questions using the table below:

Location and Date	Magnitude
Chernobyl: 1987	4
Haiti: January 12, 2012	7
Northern Italy: May 20, 2012	6

(a) How many times as intense was the earthquake in Haiti compared to the one in Chernobyl?

(b) How many times as intense was the earthquake in Haiti compared to the one in Northern Italy?

(c) How many times as intense was the earthquake in Northern Italy compared to the one in Chernobyl?

(d) If a recent earthquake was half as intense as the one in Haiti, what would be the approximate magnitude?

- How long will it take $1000 to double if it is invested at 3%/year, compounded annually? Round up the answer to the nearest year.
- How long will it take 20 mg of Iodine-131 to decay to 16.85 mg if the half-life of Iodine-131 is 8.1 days? Round off the answer to one decimal place.
- In 2011, the population of Canada was 33.5 million, and was increasing at a rate of 3.7%/year.
- Assuming that the population continues to increase at that rate, in what year will the population of Canada reach 50 million?

Planning for Instruction

Consider the following sample instructional strategies when planning lessons.

- Using a Quiz-Quiz-Trade activity, students can solve a variety of exponential equations for which both sides can be written as rational powers of the same base.

(a) Prepare a set of cards that at least matches the number of students in the class. One side of the card contains the question and the other side contains the answer.

(b) Give each student a card. Allow students a minute or two to become familiar with the question on the card. Then they find a partner to whom they will ask the question.

(c) If the partner is not able to answer the question, they should coach them first before providing the answer.

(d) After both partners are finished asking and answering the questions, they switch cards and find a new partner.

- Have students practise different strategies to solve a given problem. For example, to solve $3\log_8 x+\log_8 5=\log_8 625$:

Strategy 1

$$3\log_8 x+\log_8 5=\log_8 625$$
$$\log_8 x^3+\log_8 5=\log_8 625$$
$$\log_8 5x^3=\log_8 625$$
$$5x^3=625$$
$$x^3=125$$
$$x=5$$

Strategy 2

$$3\log_8 x+\log_8 5=\log_8 625$$
$$\log_8 x^3+\log_8 5=\log_8 625$$
$$\log_8 5x^3=\log_8 625$$
$$\log_8 5x^3-\log_8 625=0$$
$$\log_8\left(\frac{5x^3}{625}\right)=0$$
$$\left(\frac{5x^3}{625}\right)=8^0=1$$
$$5x^3=625$$
$$x^3=125$$
$$x=5$$

- Have students discuss the differences in the solution methods for $\log_2 x+\log_2 3=\log_2 9$ and $\log_2 x+\log_2 3=9$.
- A common error that students may make is not using the distributive property correctly when multiplying the logarithm and the variable expression. Encourage them to put brackets around the exponent portion of the equation when moving it to the front of the logarithm. For example, $\log(3)^{x+2}=(x+2)\log(3)$ rather than $\log(3)^{x+2}=x+2\log(3)$.
- A common student error occurs when students solve questions similar to $7(2^{3x}) = 21$. Discuss with students why $7(2^{3x})$ cannot be written as 14^{3x}. Students should divide both sides of the equation by 7 in order to isolate the power ($2^{3x} = 3$).
- Students could work in small groups to put together a jigsaw puzzle where the expressions on the adjacent sides of the puzzle pieces have to be equivalent. This activity provides students with the opportunity to practice work with logarithms, make mathematical arguments about whether or not pieces fit together, and check and revise their work.
- Ask students to create a flowchart of the steps needed to solve exponential equations and a flowchart of the steps needed to solve logarithmic equations. Then have them compare the flowcharts for similarities. Ask them to consider if one flowchart can be designed that handles both types of equations.
- Students can work in groups of two for the activity Pass the Problem. Each pair gets a problem that involves a situation to be modelled with an exponential or a logarithmic equation. Ask one student to write the first line of the solution and then pass it to the second student. The second student verifies the workings and checks for errors. If there is an error, students should discuss what the error is and why it occurred. The student then writes the second line of the solution and passes it to their partner. This process continues until the solution is complete.
- Provide opportunities for students to practise solving real-world problems that involve exponential functions. Examples include problems involving depreciation, pH levels, the Richter scale, half-life of radioactive elements, and compound interest.
- Students may find the following mnemonic helpful:

  $B^e=n \quad \log_B n=e$ as "Ben the log bunny"

SUGGESTED MODELS AND MANIPULATIVES

- jigsaw puzzles
- exponential equation cards

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- equating arguments
- equating exponents
- exponential equation
- logarithmic equation

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 7.3 Solving Exponential Equations
    - > Student Book: pp. 358–365.
  - 8.4 Logarithmic and Exponential Equations
    - > Student Book: pp. 404–415.

SCO RF11 Students will be expected to demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). [C, CN, ME]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF11.01 Explain how long division of a polynomial expression by a binomial expression of the form $x-a,\ a\in Z$ is related to synthetic division.
RF11.02 Divide a polynomial expression by a binomial expression of the form $x-a,\ a\in Z$ using long division or synthetic division.
RF11.03 Explain the relationship between the linear factors of a polynomial expression and the zeros of the corresponding polynomial function.
RF11.04 Explain the relationship between the remainder when a polynomial expression is divided by $x-a,\ a\in Z$ and the value of the polynomial expression at $x = a$ (remainder theorem).
RF11.05 Explain and apply the factor theorem to express a polynomial expression as a product of factors.

Scope and Sequence

Pre-calculus 11	Pre-calculus12
RF01 Students will be expected to factor polynomial expressions of the following form where *a*, *b,* and *c* are rational numbers. ▪ $ax^2 + bx + c,\ a\neq 0$ ▪ $a^2x^2 - b^2y^2,\ a\neq 0,\ b\neq 0$ ▪ $a[f(x)]^2 + b[f(x)] + c,\ a\neq 0$ ▪ $a2[f(x)]^2 - b^2[g(y)]^2,\ a\neq 0,\ b\neq 0$ **RF05** Students will be expected to solve problems that involve quadratic equations.	**RF11** Students will be expected to demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients).

Background

Linear functions (M10RF05) and quadratic functions (M11RF02) are examples of polynomial functions that students have already studied. They will now extend their study of polynomials to include cubic, quartic, and quintic functions.

A **polynomial function** is a function consisting of two or more terms with only one variable. Within each term the variable is raised to a whole number power and is multiplied by a constant. In general, a polynomial with real coefficients can be represented as $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$.

- a_nx^n is the **leading term**

- a_n is the **leading coefficient**
- n is the **degree of the polynomial**

Students were introduced to factoring techniques for quadratic functions in Mathematics 10 (AN05). They used common factors and trinomial factoring to express quadratics in factored form. In Pre-calculus 11, these factoring techniques were extended (RF01) to factor expressions with rational coefficients of the following forms:

- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$
- $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$
- $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$

In this unit, these techniques are used to factor cubic, quartic, and quintic polynomials.

To factor higher-order polynomials, long division or synthetic division is used in combination with factoring techniques.

Students should be introduced to synthetic division in terms of its connection to long division. They should observe that long division with polynomials is similar to division with numerical expressions.

To determine the quotient for the expression $(6x^2 + 21x + 9) \div (x - 2)$, for example, either long division or synthetic division could be used. Students should see the connection between the divisor and dividend in long division and the root and coefficients in synthetic division.

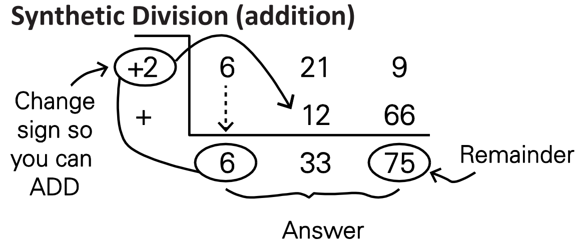
Long Division

$$
\begin{array}{r}
6x + 33 \\
x - 2\overline{)\,6x^2 + 21x + 9} \\
\underline{-6x^2 - 12x} \\
33x + 9 \\
\underline{-33x - 66} \\
75
\end{array}
$$

$$\frac{6x^2 + 21x + 9}{x - 2} =$$

$$6x + 33 + \frac{75}{x - 2}$$

Synthetic Division (addition)



$$\frac{6x^2 + 21x + 9}{x - 2} =$$

$$6x + 33 + \frac{75}{x - 2}$$

Students should see that there may be a remainder at the end of the process. In the example above, $(6x^2 + 21x + 9) \div (x - 2) = (6x + 33) + \dfrac{75}{x - 2}$. Students should also be encouraged to include any restrictions that may exist. In this particular case, the restriction is $x \neq 2$.

Synthetic division can be regarded as a more efficient method than doing long division of two polynomials when the divisor is a linear function of the form $(x - a)$.

When the divisor is not in the form of ($x - a$), then synthetic division cannot be done.

Example:

$\dfrac{x^3+2x^2-6x+1}{x^2-4x+3}$ can be divided using long division, as shown here, to obtain $(x+6)+\dfrac{15x-17}{x^2-4x+3}$.

$$
\begin{array}{r}
x+6 \\
x^2-4x+3\overline{\smash{\big)}\,x^3+2x^2-6x+1} \\
\underline{-x^3-4x^2+3x} \\
6x^2-9x+1 \\
\underline{-6x^2-24x+18} \\
15x-17
\end{array}
$$

Sometimes a quotient can be rewritten to allow for synthetic division,
For example, $\dfrac{4x^3-8x^2+12x-24}{2x-8}$ can be rewritten as $\dfrac{2(2x^3-4x^2+6x-12)}{2(x-4)}=\dfrac{2x^3-4x^2+6x-12}{x-4}$.

Then synthetic division could be used to obtain $\left(2x^2+4x+22\right)+\dfrac{76}{x-4}$.

$$
\begin{array}{r|rrrr}
4 & 2 & -4 & 6 & -12 \\
+ & & 8 & 16 & 88 \\
\hline
& 2 & 4 & 22 & 76
\end{array}
$$

If students had used long division, for this example, they would have obtained $\left(2x^2+4x+22\right)+\dfrac{152}{2x-8}$,

which is equivalent to $\left(2x^2+4x+22\right)+\dfrac{76}{x-4}$.

Although synthetic division can be completed using either the addition or subtraction operation, it is recommended that the addition operation be used. Students make far fewer errors when adding than when subtracting. The use of addition also helps make a connection between synthetic division and the remainder theorem.

Students are expected to carefully examine the polynomial when setting up synthetic division. A zero must be included for each missing term in the dividend. A common error occurs in questions such as $\left(x^4-10x^2+2x+3\right)\div(x-3)$ when students do not include zero for the missing cubic term. Students should check the result by multiplying the quotient and the divisor. This can help inform them if the division is done correctly.

Students are expected to understand that the **remainder theorem** states that if ($x - a$) is a linear divisor of a polynomial function $P(x)$, then $P(a)$ is the remainder. Initially, students may not see a need for this theorem since they can obtain remainders by using synthetic division. Exposure to polynomials such as $\left(x^9-8x^2\right)$ or $\left(x^{100}+1\right)$ should help them see that the remainder theorem is more efficient in some cases.

The **factor theorem**, which states that a polynomial, $P(x)$, has a factor $(x - a)$ if and only if $P(a) = 0$, can be used to determine the factors of a polynomial expression. For example, for $P(x) = x^5 - x^4 + x^3 - 2x + 1$, since $P(1) = 0$, we know that $(x - 1)$ is a factor of $P(x)$.

To find the roots of a polynomial, *Pre-Calculus 12* (McAskill et al. 2012) uses the **integral zero theorem** to relate the factors of a polynomial and the constant term of the polynomial. The integral zero theorem is a special case of the **rational root theorem**.

For the integral zero theorem, the factors of the constant term indicate possible factors of the polynomial. They then verify using the factor theorem. For the polynomial $f(x) = x^3 - 3x^2 - 4x + 12$, for example, the possible integral zeros are the factors of 12. When applying the integral zero theorem, students are expected to test both the positive and negative factors. Since $f(2) = 0$, $f(-2) = 0$, and $f(3) = 0$, the factors of the polynomial are $(x - 2)$, $(x + 2)$ and $(x - 3)$. This method, where only the factors of the constant term are considered, is restricted to polynomial functions with a leading coefficient of 1 having distinct integral zeros only.

$P(x) = x^3 - 2x^2 + 3x - 6$ is a polynomial where the integral zero theorem would apply since its leading coefficient is 1. The integral zero theorem states that the possible roots, are $\pm 1, \pm 2, \pm 3, \pm 6$. The value of r that $P(r) = 0$ is +2, so $(x - 2)$ is a factor of $P(x)$. If the quotient can be factored, the remaining factor(s) can be determined by using long division or synthetic division or other factoring methods.

Students should work with polynomial functions that have leading coefficients other than one and that may have non-integer zeros. The **rational root theorem** can also be used in conjunction with synthetic division to factor a polynomial. When the leading coefficient of a polynomial does not equal 1, the **rational root theorem** states that the possible test values for a rational root, a, are the factors of the constant divided by the factors of the leading coefficient.

Students should use the rational root theorem to determine possible factors and verify one of the factors using the factor theorem. Synthetic division is then applied, resulting in a polynomial to be factored further. For the polynomial $f(x) = 4x^3 - 12x^2 + 5x + 6$, the rational root theorem would indicate that the possible rational roots for $f(x)$ are all the factors of $6\left(\pm\{1, 2, 3, 6\}\right)$ divided by all the factors of $4\left(\pm\{1, 2, 4\}\right)$. Thus, the possible rational roots are $\pm\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{6}{1}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{6}{2}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{6}{4}\right\}$ or $\pm\left\{1, 2, 3, 4, \frac{1}{2}, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}\right\}$.

$f(x) = 4x^3 - 12x^2 + 5x + 6, f(2) = 0$, using synthetic division gives

$$
\begin{array}{r|rrrr}
2 & 4 & -12 & 5 & 6 \\
 & & 8 & -8 & -6 \\
\hline
 & 4 & -4 & -3 & \boxed{0}
\end{array}
$$

This results in $(x - 2)(4x^2 - 4x - 3)$, which can be further factored to $(x - 2)(2x - 3)(2x + 1)$.

Factoring a polynomial using the rational root theorem may require some adjustment to the final answer if the polynomial has integer coefficients and some of the roots are not integral.

Example:

For $P(x)=2x^4+5x^3-5x-2$, the possible roots are: $\pm1, \pm\frac{1}{2}$, and ±2. The values of *r* that make $P(r)=0$ are –1, +1, $-\frac{1}{2}$, and –2, so factors of *P*(*x*) are $(x+1)$, $(x-1)$, $\left(x+\frac{1}{2}\right)$, and $(x-2)$.

$\therefore P(x)=k(x-1)(x+1)(x-\frac{1}{2})(x-2)$

Since the leading coefficient of *P*(*x*) is 2, we know that *k* = 2.

$\therefore P(x)=2\,(x-1)(x+1)(x-\frac{1}{2})(x-2)$

In order to write *P*(*x*) with integral coefficients,

$P(x)=(x-1)(x+1)(2)(x-\frac{1}{2})(x-2)$

$P(x)=(x-1)(x+1)(2x-1)(x-2)$

In addition to the rational root theorem, the factor theorem, and synthetic division, students should also be able to use alternative methods to factor polynomials, such as grouping. They should have been introduced to grouping when they factored quadratics using decomposition. This can be extended to polynomials, such as $f(x)=x^3-3x^2-4x+12$.

$$f(x)=x^3-3x^2-4x+12$$
$$f(x)=x^2(x-3)-4(x-3)$$
$$f(x)=(x-3)(x^2-4)$$
$$f(x)=(x-3)(x+2)(x-2)$$

Some specialized factoring techniques students are expected to use are

- sum and difference of cubes

  $(a^3x^3+d^3)=(ax+d)(a^2x^2-adx+d^2)$

  $(a^3x^3-d^3)=(ax-d)(a^2x^2+adx+d^2)$
- factoring by grouping

  $x^3-2x^2-16x+32=x^2(x-2)-16(x-2)=(x-2)(x^2-16)=(x-2)(x-4)(x+4)$
- factoring quartics as trinomials

  $x^4-5x^2-4=(x^2)^2-5x^2-4$

Once a polynomial has been factored, students apply the **zero product property** to determine the zeros. This is a natural extension of the work done with solving quadratic equations in Pre-calculus 11 (RF05).

Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Explain why $(x-5)(2x+3)=0$ has solutions of $+5$, $-\frac{3}{2}$.
- Expand:
  (a) $(x-3)(x+1)$
  (b) $(x+1)(x-1)(2x-4)$
- Factor:
  (a) x^2-9
  (b) x^2-5x-6
  (c) $2x^2-7x-15$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine whether or not each of the following is a polynomial function and explain your reasoning.
  (a) $f(x)=3-2x+\frac{1}{2}x^2$
  (b) $f(x)=\frac{1}{x^2-4x+4}$
  (c) $f(x)=(x-1)^3$
  (d) $f(x)=6-\sqrt{x}$
  (e) $f(x)=x^{-2}+5x^{-1}+6$
- Determine the degree, type, leading coefficient, and constant term of each polynomial function.
  (a) $f(x)=x^3+2x^2-3x-4$
  (b) $f(x)=2x^4-5x^2-2x+2$
  (c) $f(x)=3-2x-x^2$
- Explain the connection between long division and synthetic division, using the example $(2x^3-x^2-13x-6)\div(x+2)$.

- Determine each of the following quotients using long division.

  (a) $\dfrac{x^4-2x^3+x^2+12x-6}{x+2}$

  (b) $\dfrac{x^2-5x-3}{x-2}$

  (c) $\dfrac{2x^3+12x-6}{2x-6}$

- Determine each of the following quotients using synthetic division.

  (a) $\dfrac{x^3-1}{x-1}$

  (b) $\dfrac{x^4-2x^3+x^2+12x-6}{x+2}$

  (c) $\dfrac{2x^3+12x-6}{2x-6}$

- Use the remainder theorem to determine the remainder when each polynomial is divided by $(x-1)$.

  (a) $2x^2+5x+7$

  (b) x^3+2x^2-4x+1

- Find the value of k in each dividend such that the remainder is 3 after completing the division.

  (a) $\left(kx^2+3x+1\right)\div\left(x+2\right)$

  (b) $\left(x^3+x^2+kx-12\right)\div\left(x-2\right)$

- Determine the remainder for the function $f(x) = x^9 + 3x^4 - 5x + 1$ if it is divided by $(x-3)$.
- The volume of a rectangular prism is given by $V = x^3 + 18x^2 + 80x + 96$. Determine the missing dimension(s) if two of the dimensions are $(x + 2)$ and $(x + 12)$. Identify any restrictions on the variable x.
- When x^2+5x+7 is divided by $(x+k)$, the remainder is 3. Find the possible value(s) for k.
- When ka^3-3a^2+5a-8 is divided by $(a-2)$, the remainder is 22. What is the remainder when ka^3-3a^2+5a-8 is divided by $(a+1)$?
- Explain why the remainder theorem is an efficient way to find the remainder when a polynomial expression is divided by $(x-a)$.
- Explain how to determine the remainder when $(10x^4 - 11x^3 - 8x^2 + 7x + 9)$ is divided by $(2x - 3)$.
- Use both long division and synthetic division to determine the result when $\left(15x^3-45x^2+30x+12\right)$ is divided by $(3x-6)$.
- Explain and demonstrate, using the integral zero theorem, the factor theorem, and synthetic division, how you would determine the values of k that make $(x-k)$ a factor of $f(x) = x^3 - 4x^2 - 11x + 30$.
- Determine the corresponding binomial factor of each polynomial, $P(x)$, given the value of the zero.

  (a) $P(-2) = 0$

  (b) $P(0) = 0$

- State whether each polynomial has a factor of $(x - 4)$.

  (a) $3x^2 - 7x - 21$

  (b) $x^3 - 5x^2 + 3x + 4$

  (c) $-x^4 + 3x^3 + 5x^2 - 16$

- Determine the possible integral zeros of each polynomial.

  (a) $P(x) = x^3 + 5x^2 - x - 5$

  (b) $P(x) = x^4 - 17x^2 + 16$

- Factor the following using specialized techniques.

  (a) $8x^3 - 125$

  (b) $x^3 + 2x^2 - x - 2$

  (c) $2x^4 + 5x^2 - 3$

  (d) $x^3 + 2x^2 + x + 2$

  (e) $64x^3 + 125$

  (f) $3x^4 - 192x$

  (g) $4x^4 - 37x^2 + 9$

  (h) $x^5 - 5x^4 - 10x^3 + 50x^2 + 9x - 45$

- Factor completely each of the following polynomials.

  (a) $x^3 + 8x^2 + 19x + 12$

  (b) $x^3 + 4x^2 + 2x - 3$

  (c) $x^3 - 27$

  (d) $x^4 + 3x^3 - 14x^2 - 48x - 32$

  (e) $x^5 - 5x^3 + 4x$

- List the possible rational zeros of each function.

  (a) $f(x) = 4x^3 + 12x^2 + x + 3$

  (b) $y = 3x^3 + 10x^2 + 4x - 8$

  (c) $k(x) = 30x^3 - x^2 - 6x + 1$

- Is $(x - 1)$ a factor of $(x^{100} - 1)$? Is $(x + 1)$? Explain.

- If $(x - 1)$ is a factor of $ax^3 + bx^2 + cx + d$, what is the value of $a + b\ c + d$? Explain. Use this result to determine if $(x - 1)$ is a factor of the following polynomials.

  (a) $3x^3 + 5x^2 - 6x - 2$

  (b) $2x^3 - 9x^2 - x - 8$

  (c) $-5x^3 + 4x + 1$

- Create a polynomial of degree ≥ 3 using linear factors. Explain the relationship between linear factors and the corresponding zeros of the function.

- Find the values of a and b if $(ax^3 + bx^2 + 3x - 4)$ has a remainder of -2 when divided by $(x - 1)$ and a remainder of 2 when divided by $(x - 2)$.

- When a polynomial $P(x)$ is divided by $(2x - 1)$ the quotient is $x^2 - 2x - 1$ and the remainder is -4. Find $P(x)$.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Review the concept of a polynomial with the class. Classify and give examples of polynomials that are constant, linear, and quadratic.
- Many students may not recall the algorithm of long division. This will need to be reviewed.
- Complete long division with numbers and then with polynomials side by side and compare the process.

Long division with numbers

$$\frac{523681}{4}$$

$$=130920\frac{1}{4}$$

$$\begin{array}{r} 130920 \\ 4\overline{)523681} \\ -\underline{4} \\ 12 \\ -\underline{12} \\ 036 \\ -\underline{36} \\ 08 \\ -\underline{8} \\ 01 \end{array}$$

Long division with polynomials

$$\frac{2x^3+6x^2+12x+20}{x+3}$$

$$2x^2+12-\frac{16}{x+3}$$

$$\begin{array}{r} 2x^2+12 \\ x+3\overline{)2x^3+6x^2+12x+20} \\ -\underline{2x^3+6x^2} \\ 0+12x+20 \\ -\underline{12x+36} \\ -16 \end{array}$$

- Take time to explain how the process of synthetic division is a specialized form of long division where some assumptions are made. For example, to determine the result of $(6x^2+21x+9)\div(x-2)$:

<table>
<tr><th>Long division</th><th></th><th>Synthetic division (subtraction)</th><th>Synthetic division (addition)</th></tr>
<tr><td>$x-1\overline{)6x^2+21x+9}$</td><td>Rather than writing in the x and x^2, we can agree that we will know they are there by the location of the coefficients.</td><td>$\begin{array}{r|rrr} -2 & 6 & 21 & 9 \\ - & & & \\ \hline \end{array}$</td><td>$\begin{array}{r|rrr} 2 & 6 & 21 & 9 \\ + & & & \\ \hline \end{array}$</td></tr>
<tr><td>$\begin{array}{r} 6x \\ x-1\overline{)6x^2+21x+9} \\ -\underline{6x^2-12x} \end{array}$</td><td>We know that the first number in the answer will be the same as the leading coefficient since it is being divided by 1x.</td><td>$\begin{array}{r|rrr} -2 & 6 & 21 & 9 \\ - & & -12 & \\ \hline & 6 & & \end{array}$</td><td>$\begin{array}{r|rrr} 2 & 6 & 21 & 9 \\ + & & 12 & \\ \hline & 6 & & \end{array}$</td></tr>
<tr><td>$\begin{array}{r} 6x+33 \\ x-1\overline{)6x^2+21x+9} \\ -\underline{6x^2-12x} \\ 33x+9 \end{array}$</td><td>Subtract or add the negative.</td><td>$\begin{array}{r|rrr} -2 & 6 & 21 & 9 \\ - & & -12 & \\ \hline & 6 & 33 & \end{array}$</td><td>$\begin{array}{r|rrr} 2 & 6 & 21 & 9 \\ + & & 12 & \\ \hline & 6 & 33 & \end{array}$</td></tr>
<tr><td>$\begin{array}{r} 6x+33 \\ x-1\overline{)6x^2+21x+9} \\ -\underline{6x^2-12x} \\ 33x+9 \\ -\underline{33x-66} \\ 75 \end{array}$</td><td>Finish the synthetic division to determine the result of the division.</td><td>$\begin{array}{r|rrr} -2 & 6 & 21 & 9 \\ - & & -12 & -66 \\ \hline & 6 & 33 & 75 \end{array}$</td><td>$\begin{array}{r|rrr} 2 & 6 & 21 & 9 \\ + & & 12 & 66 \\ \hline & 6 & 33 & 75 \end{array}$</td></tr>
</table>

Long division		**Synthetic division (subtraction)**	**Synthetic division (addition)**
Answer: $6x+33+\frac{75}{x-2}$	State the answer.	Answer: $6x+33+\frac{75}{x-2}$	

- To assist students in understanding the remainder and factor theorems, make the link between the remainder when $P(x)$ is divided by $(x - r)$ and the point $[r, P(r)]$ on $P(x)$.

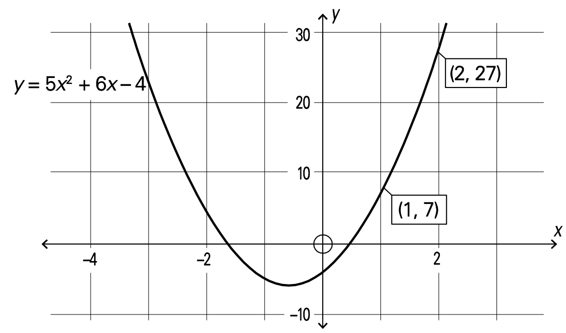
  For example, have students divide a polynomial such as $P(x)=5x^2+6x-4$ by several linear equations such as $(x - 2)$ and $(x - 1)$. Then ask students to graph $P(x)$ and observe the points (2, 28) and (1, 7).

$$\frac{5x^2+6x-4}{x-2}$$

$$\begin{array}{r|rrr} 2 & 5 & 6 & -4 \\ + & & 10 & 32 \\ \hline & 5 & 16 & 28 \end{array}$$

$$\frac{5x^2+6x-4}{x-1}$$

$$\begin{array}{r|rrr} 1 & 5 & 6 & -4 \\ + & & 5 & 11 \\ \hline & 5 & 11 & 7 \end{array}$$



- *Pre-Calculus 12* (McAskill et al. 2012) uses subtraction when showing synthetic division. This is fine, but many students make fewer mistakes when adding than when subtracting and may find using synthetic division with addition a quicker process.
- *Pre-Calculus 12* (McAskill et al. 2012) does not mention the rational root theorem. The integral zero theorem is a special case of the rational root theorem where the polynomial has a leading coefficient of 1. The resource will need to be supplemented in this area. Use questions such as, Use the rational root theorem, factor theorem, and synthetic division to factor the following:

  (a) $y=x^3+x^2-9x+7$

  (b) $y=x^4-x^3-4x^2+2x+4$

  (c) $y=2x^5-x^4-26x^3+13x^2+72x-36$

  (d) $y=12x^2-28x+15$

  (e) $y=21x^2-11x-40$

  (f) $y=2x^4+5x^3-x^2+5x-3$

- Make a set of cards with factorable polynomials and a set with possible factors. Hand each student a card when they enter the classroom. Have them find their partner by matching the factor cards with the appropriate polynomial card. This will be their seat partner for the day. This could also be played as a memory game or with a set of cards with possible zeros.

Suggested Models and Manipulatives

- calculator

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- difference and sum of cubes
- factor by grouping
- factor theorem
- integral zero theorem
- long division
- polynomial function
- rational root theorem
- remainder theorem
- synthetic division

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 3.1 Characteristics of Polynomial Functions
    - Student Book: pp. 106–117
  - 3.2 The Remainder Theorem
    - Student Book: pp. 118–125
  - 3.3 The Factor Theorem
    - Student Book: pp. 126–135

SCO RF12 Students will be expected to graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). [C, CN, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF12.01 Identify the polynomial functions in a set of functions, and explain the reasoning.
RF12.02 Explain the role of the constant term and leading coefficient in the equation of a polynomial function with respect to the graph of the function.
RF12.03 Generalize rules for graphing polynomial functions of odd or even degree.
RF12.04 Explain the relationship among the zeros of a polynomial function, the roots of the corresponding polynomial equation, and the x-intercepts of the graph of the polynomial function.
RF12.05 Explain how the multiplicity of a zero of a polynomial function affects the graph.
RF12.06 Sketch, with or without technology, the graph of a polynomial function.
RF12.07 Solve a problem by modelling a given situation with a polynomial function and analyzing the graph of the function.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. (PC11)** **RF05** Students will be expected to solve problems that involve quadratic equations. (PC11)**	**RF12** Students will be expected to graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).

* M11—Mathematics 11
** PC11—Pre-calculus 11

Background

In Mathematics 10, students related linear relations expressed in slope-intercept form, general form, and slope-point form to their graphs (RF06). In Mathematics 11, they analyzed quadratic functions to identify the characteristics of the corresponding graph (RF02). In this unit, students graph and analyze polynomial functions of degree 5 or less.

Students will explore how a polynomial function's degree, x-intercepts, y-intercept, and the sign of its leading coefficient affect the shape of its graph. They will also look at how the **multiplicity of zeros**, which is the number of times a zero (x-intercept) occurs, impacts on the graph. Students should be able to sketch the basic shape of a graph, showing information learned from the sign of a_n, the degree of the polynomial, and known rational roots. This is just a sketch as local maximum and minimum points are not known. Students should also understand that there may be other irrational, real roots.

Students will be expected to solve contextual problems involving polynomial functions using two different methods. One method is to graph the equation that is created and to estimate the solution from this graph. The other method is to create an equation and to solve it algebraically using factoring methods.

Students have previously sketched the graph of polynomial functions of degree 0, 1, and 2. They should understand the following:

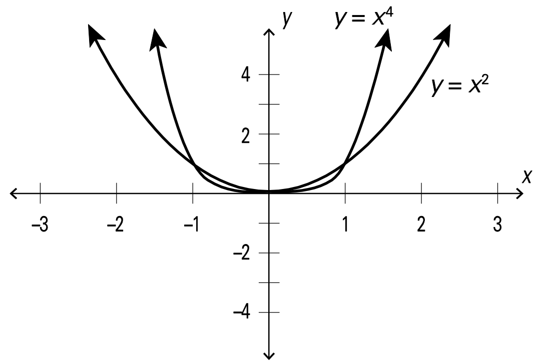
Function	**Degree**	**Type of Function**	**Graph**
$f(x)=a$	0	constant	horizontal line
$f(x)=ax+b$	1	linear	line with slope a
$f(x)=ax^2+bx+c$	2	quadratic	parabola

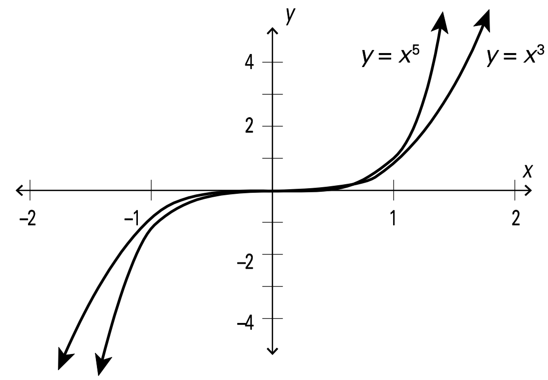
They will now be introduced to some of the basic features of the graphs of polynomial functions of degree greater than 2:

- The graph of a polynomial function is continuous.
- The graph of a polynomial function has only smooth turns. A function of degree n has at most $(n-1)$ turns.
- If the leading coefficient of the polynomial function is positive, then as $x \to +\infty$, $y \to +\infty$. If the leading coefficient is negative, then as $x \to +\infty$, $y \to -\infty$.
- The constant term is the y-intercept of the graph.

The intent at this point is that students learn to recognize these basic features. Later in the unit, they will use these features, point-plotting, and intercepts to make reasonably accurate sketches.
To examine the basic features, students could graph polynomials with technology, such as graphing calculators or other graphing software. Graphing software apps available for students' mobile devices could also be used.

Students should explore the graphs of various polynomials with even degree and odd degree. The polynomial functions that have the simplest graphs are the monomial functions $f(x) = a_n x^n$. When n is even, the graph is similar to the graph of $f(x) = x^2$. When n is odd, the graph is similar to the graph of $f(x) = x^3$. The greater the value of n, the flatter the graph of a monomial is on the interval $-1 \le x \le 1$.





Through exploration, students should see that if the degree of a polynomial function is even, then its graph has the same behaviour to the left and right. For the graph of $f(x) = x^4$, for example, as $x \to +\infty, y \to +\infty$ and as $x \to -\infty, y \to +\infty$.

If the degree is odd, the graph has opposite behaviours to the right and left. For the graph of $f(x) = -x^3$, for example, $x \to +\infty, y \to +\infty$ and $x \to -\infty, y \to -\infty$.

Students could use graphing technology to determine any similarities and differences between polynomials such as the following:

- $f(x) = 2x + 1$
- $f(x) = x^2 + 2x - 3$
- $f(x) = x^3 + 2x^2 - x - 2$
- $f(x) = x^4 + 5x^3 + 5x^2 - 5x - 6$
- $f(x) = 0.2x^5 - x^4 - 2x^3 + 10x^2 + 1.4x - 9$

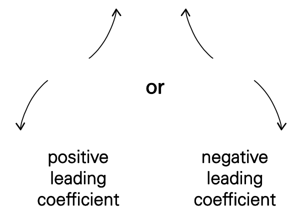
Students could then graph each of these with a negative leading coefficient. From this, they should identify a pattern in the graphs of odd and even degree functions. Students should note patterns in the end behaviour, the constant term, the number of vertices, and the number of real x-intercepts. Examples should be limited to polynomials with real x-intercepts to allow students to easily identify the patterns.

Students should be expected to make observations such as the following:

***x*-intercepts**

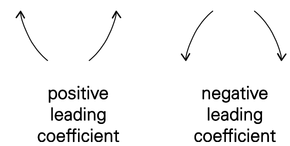
When the degree is odd:

- there is at least one *x*-intercept
- the maximum number of *x*-intercepts equals the degree



When the degree is even:

- there could be no *x*-intercepts
- the maximum number of *x*-intercepts equals the degree



Vertices

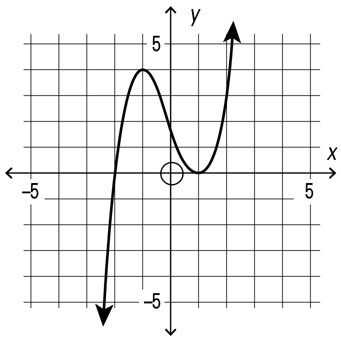
- Maximum number will be one less than the degree
- If degree is even: minimum of one vertex
- If degree is odd: minimum of zero vertices
- Even degree: must have odd number of vertices
- Odd degree: must have an even number of vertices (or no vertices)

Students were introduced to the basic features of the graphs of polynomial functions of degree greater than 2 at the beginning of this unit. They will now use these features, along with the intercepts, to graph polynomials of degree 5 or less.

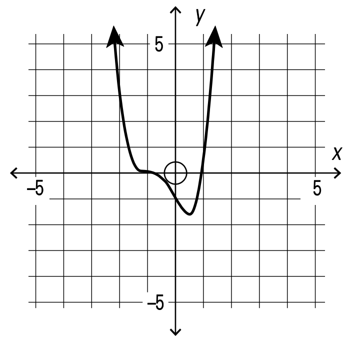
Students were introduced to the zeros of a quadratic function, the roots of the quadratic equation, and the *x*-intercepts of the graph in Mathematics 11 (RF02). It is important that they distinguish between the terms **zeros**, **roots**, and ***x*-intercepts**, and use the correct terms in a given situation. Students could be asked to find the roots of the equation $3x^3 - 10x^2 - 23x - 10 = 0$, find the zeros of the function $f(x) = 3x^3 - 10x^2 - 23x - 10$, or determine the *x*-intercepts of the graph of $f(x) = 3x^3 - 10x^2 - 23x - 10$. In each case, students are identifying the factors of the polynomial and solving to arrive at the solution $x = -1, x = -\frac{2}{3}, x = 5$.

Students should understand that the degree of a polynomial indicates the maximum number of *x*-intercepts for its graph. For each real *x*-intercept, there is a linear factor and a zero for the polynomial function.

Students have solved polynomials to find distinct, real zeros that correspond to distinct, real x-intercepts. They need to be aware that the zeros will not always be distinct. Some polynomial functions may have a multiplicity of a zero (i.e., double, triple), also referred to as the order of a zero. Students should graph polynomials such as $f(x) = (x - 1)(x - 1)(x + 2)$ or $f(x) = (x + 1)(x + 1)(x + 1)(x - 1)$ to see the effect of multiplicity of roots.



$f(x) = (x-1)(x-1)(x+2)$



$f(x) = (x+1)(x+1)(x+1)(x-1)$

Ask students questions, such as

- What would happen to the graph if there was a zero of multiplicity 4?
- What would be the effect on the graph if there are two double roots?
- What effect does a triple root have on the graph? A double root? A single root?

Students should be encouraged to check other possibilities for multiplicity of zeros for polynomials of degree ≤ 5 and the effect on their respective graphs. They would be expected to make observations such as the following:

- $(x - r)^{\text{odd}} \rightarrow$ passes through x-axis at $x = r$.
- $(x - r)^{\text{even}} \rightarrow$ vertex touching x-axis at $x = r$.

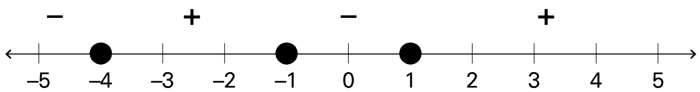
Students should understand that in order to sketch a graph of any polynomial without graphing technology, they have to identify points such as the x- and y-intercepts. They use the value of the leading coefficient to determine the end behaviour and consider how multiplicity of zeros affects the graph of the function.

Students should also be able to identify when the function is positive, $f(x) > 0$, and when it is negative, $f(x) < 0$. A table of intervals or a sign diagram consisting of a number line, roots and test points can help with this. Students should understand that the function is neither positive nor negative at the x-intercepts. The intervals should be expressed as set or interval notation. Students should be familiar with both types of notation from Mathematics 10 (RF01).

Ask students to determine the intervals where the graph represented by the function $f(x) = (x + 4)(x + 1)(x - 1)$ is positive or negative. They could use a table to determine the intervals:

Interval	$x < -4$	$-4 < x < -1$	$-1 < x < 1$	$x > 1$
Sign	–	+	–	+

Sign lines were introduced in Pre-calculus 11 when students graphed absolute value functions (RF02). Relating the sign diagram to the *x*-axis of the graph, students can substitute an *x*-value from each interval into the function to determine where the function is positive or negative.



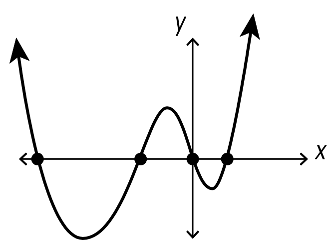
Remind students of the effect of a multiplicity of a zero on the graph of a polynomial function. Discuss how this would appear on a sign diagram. Students should understand that for a zero of odd multiplicity (e.g., a single root or a triple root), the sign of the function changes. If a function has a zero of even multiplicity (e.g., a double root), the sign does not change.

Encourage students to link the positive and negative intervals to their understanding of the graph of a polynomial.

Example:

$x(x+9)(x+3)(x-2)<0$

$f(x)<0$ for intervals
$x\in(-9,-3)\cup(0,2)$

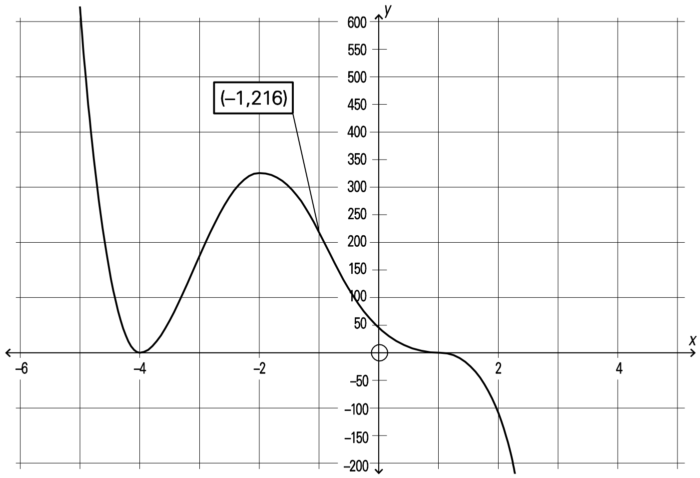


In Pre-calculus 11, students solved problems by determining and analyzing a quadratic equation (RF05). This will now be extended to polynomial functions. When solving a problem, it is often necessary to simplify the problem and express the problem with mathematical language or symbols to make a mathematical model. It is an expectation that students will apply skills developed in this unit to solve problems in various contexts.

For example, they should answer questions based on area, volume, and numbers, such as the following:

- Three consecutive integers have a product of −720. What are the integers?
  - Students could model this situation with a polynomial function such as $P(x)=(x)(x+1)(x+2)$ or $P(x)=(x-1)(x)(x+1)$ and solve the equation $P(x)=-720$ to determine the integers.
- An open box is to be made from a 10 in. × 12 in. piece of cardboard by cutting *x*-in. squares from each corner and folding up the sides. If the volume of the box is 72 in.^{3}, what are the dimensions?
  - It is necessary here to place restrictions on the independent variable. Ask students why, in this case, the value of *x* is restricted to $0<x<5$. It is important for students to consider the possibility of inadmissible roots in the context of the problem. They should understand that time, length, width, and height, for example, cannot be negative values.

Students should also be able to analyze a graph and create a polynomial equation that models the graph. They should be given a variety of graphs that have both distinct roots and multiplicity of roots and asked to determine the equation of the polynomial function that represents the graph. For example, the polynomial shown below has a double root at $x=-4$ and a triple root at $x=1$.



$P(x) = a(x+4)^2(x-1)^3, a < 0$

Using the point (–1, 216),

$216 = a(-1+4)^2(-1-1)^3, a < 0$

$216 = a(3)^2(-2)^3$

$216 = -72a$

$-3 = a$

$P(x) = -3(x+4)^2(x+1)^3, a < 0$

In cases where a graph may possibly represent a polynomial of either degree 3 or 5, or of either degree 2 or 4, students would need to be provided with the degree of the resulting equation.

Assessment, Teaching, and Learning

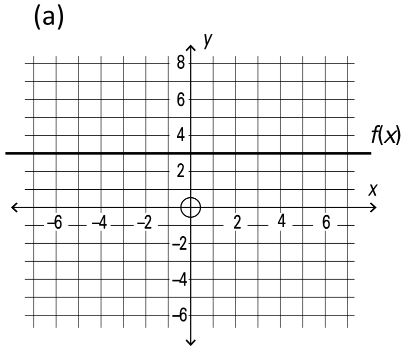
Assessment Strategies

Assessing Prior Knowledge

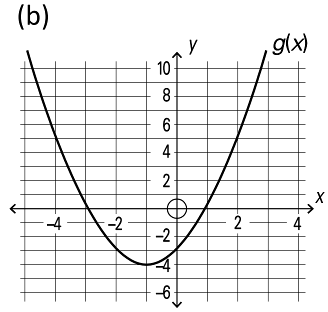
Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- State the equation for each of the following:

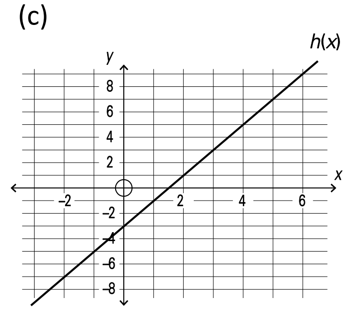
(a)



(b)



(c)

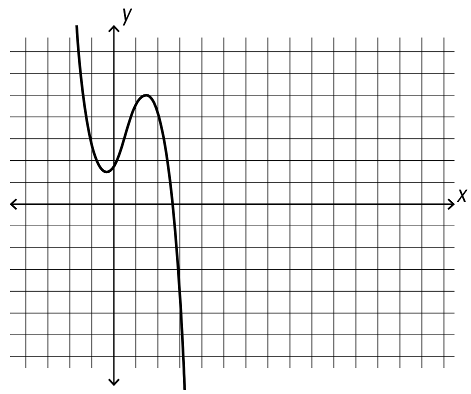


Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- For the graph of the given function,
  (a) determine whether the graph represents an odd-degree or an even-degree function

(b) determine whether the leading coefficient is positive or negative



- Describe the end behaviour of the corresponding graphs of each of the following functions. State the possible number of x-intercepts and y-intercepts.

  (a) $f(x) = x^2 - 3x + 2$

  (b) $f(x) = 4 - 3x^2 - x^3$

  (c) $f(x) = -x^4 - 4x^3 + x^2 + 6x$

- Graph each of the following polynomial functions, clearly indicating x-intercepts, y-intercept, and the end behaviours.

  (a) $y = (x-2)(x+3)(x-1)$

  (b) $y = (x+1)(x-4)(2x+5)(x-3)$

  (c) $f(x) = -x(x-1)(x+3)$

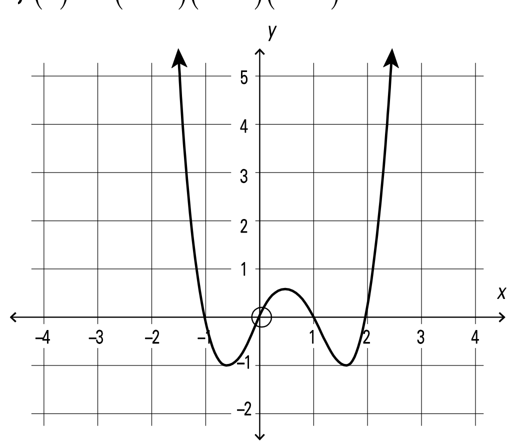
  (d) $g(x) = \frac{1}{3}(x^2 - 16)(x^2 - 6x)$

  (e) $y = -\frac{1}{3}(x-2)^2(x+2)^3$

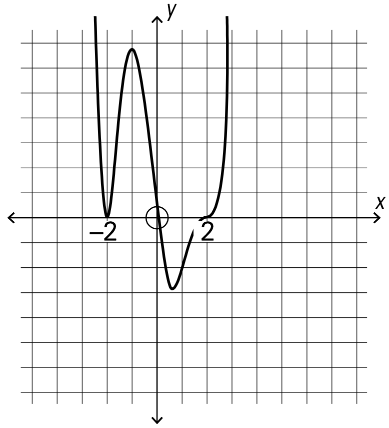
  (f) $y = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$

- Identify the features of the graph related to the function $f(x) = -3x^2 + 9x + x^5$.
- How many vertices can the graph of a polynomial function of degree 5 have? Explain.
- Describe the characteristics of the graphs of cubic and quartic functions with the largest possible number of terms.
- Given the equation and the graph below, identify the zeros, roots, and x-intercepts and explain how they are related.

  $f(x) = x(x-1)(x+1)(x-2)$

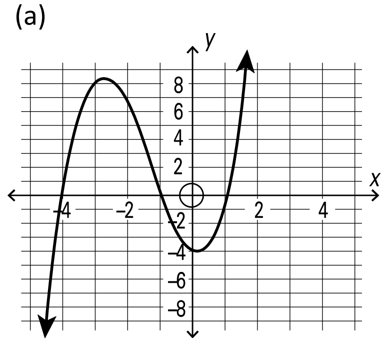


- For the polynomial $f(x) = x^3 - 4x^2 + x + 6$,
  (a) algebraically or graphically determine the linear factors of $f(x)$
  (b) explain the relationship between the linear factors and any x-intercepts
- Explain, using an example, how a zero of multiplicity 5 affects the graph of a polynomial.
- For the given graph, determine the multiplicity of the zeros and explain your reasoning.

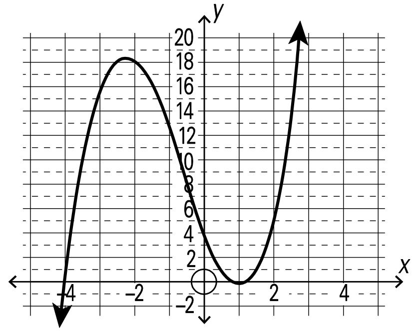


- Examine the following graphs. Identify their zeros and their order of multiplicity.

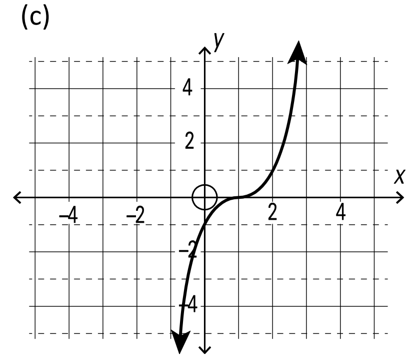
(a)



(b)



(c)



- Solve each of the following equations.
  (a) $x(x+3)(x-8)=0$
  (b) $x^3+x^2-6x=0$
  (c) $x^3+3x^2-x-3=0$
  (d) $x^3-19x=30$
  (e) $x^4+2x^3-7x^2-8x+12=0$
- Without using graphing technology, sketch the graph of the polynomial $f(x) = x^3 - 2x^2 - 4x + 8$.
- Graph each of the following functions and determine
  – the domain and range
  – any zero(s) and their multiplicity
  – the y-intercept
  – the interval(s) where the function is positive and negative
  – the end behaviour

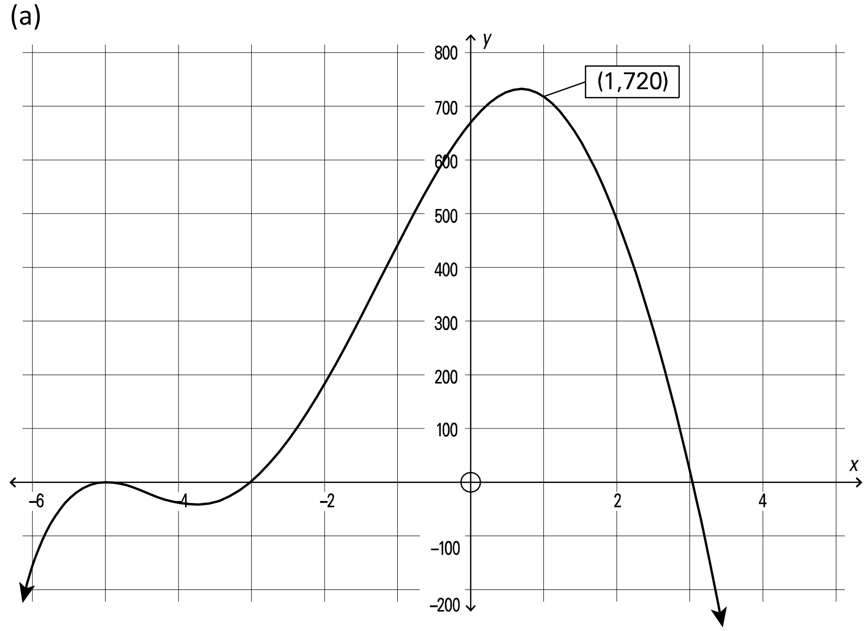
  (a) $f(x)=x^2(3-x)$

(b) $f(x) = x^3 - 2x^2 + x$

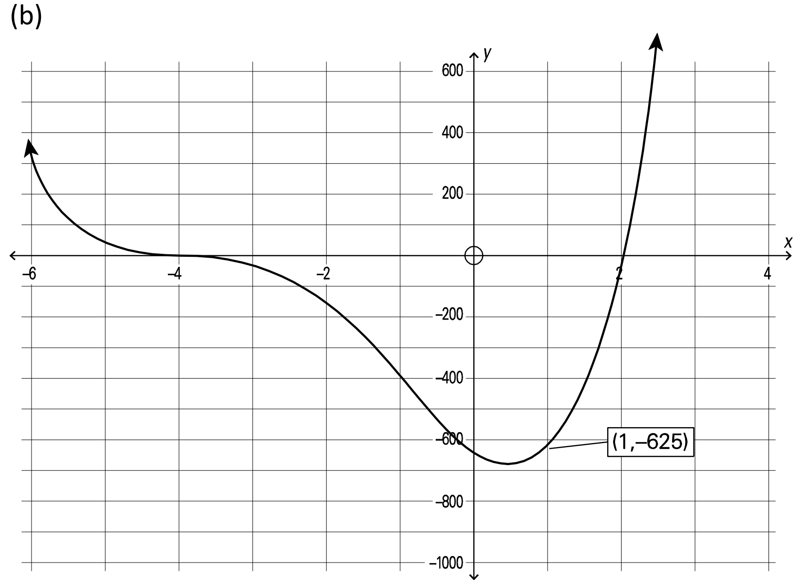
(c) $f(x) = x^4 - 4x^2$

- Sketch the graph of $f(x) = -(x+2)^3(x-4)$ and verify using graphing technology. Explain what characteristics of the polynomial (e.g., intercepts, multiplicity of zeros, leading coefficient, positive or negative intervals) helped you sketch the graph.
- State the equation that corresponds to each of the following graphs:

(a)



(b)

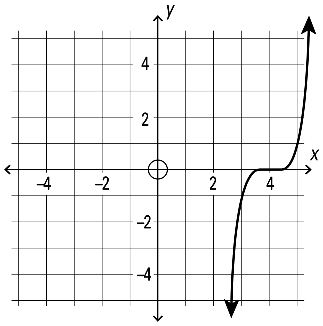


- An open-topped box with a volume of 900 cm^3 is made from a rectangular piece of cardboard by cutting equal squares from four corners and folding up the sides.

(a) If the original dimensions of the cardboard are 30 cm × 40 cm, find the side length of the square that is cut from each corner.

(b) Calculate the surface area.

- The actual and projected number, C (in millions), of computers sold for a region between 2010 and 2020 can be modelled by $C = 0.0092(t^3 + 8t^2 + 40t + 400)$ where $t = 0$ represents 2010. During which year are 8.51 million computers projected to be sold?
- A rectangular shipping container has a volume of 2500 cm^3. The container is four times as wide as it is deep, and 5 cm taller than it is wide. What are the dimensions of the container?
- Explain how the number of zeros affect the graph of a polynomial function.
- Model the polynomial function represented by the graph as both a cubic function and a quintic function.



Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Students can work in groups to match the equations with the appropriate graph. They should explain the features that guided the selection of the appropriate graph.

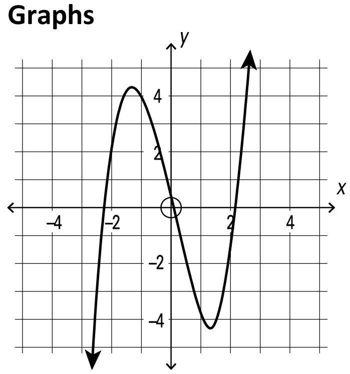
Functions

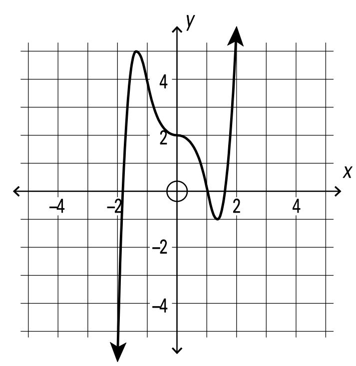
$f(x) = x^3 - 5x$

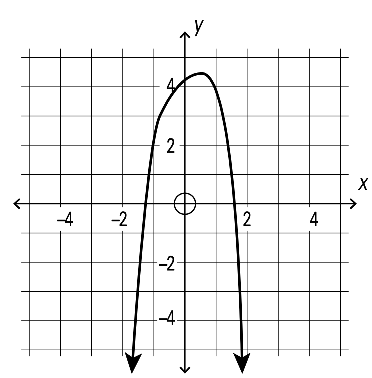
$f(x) = x^5 - 3x^3 + 2$

$f(x) = -x^4 + x + 4$

Graphs





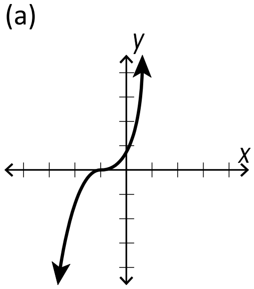


- Ask students to create a foldable or a graphic organizer to summarize the rules for graphing polynomial functions of odd or even degree.
- The activity Commit and Toss gives students an opportunity to anonymously commit to an answer and provide a justification for the answer they selected. Provide students with a selected response question, as shown below. Students write their answer, crumble their solutions into a ball, and toss the papers into a basket. Once all papers are in the basket, ask students to reach in and take out one.

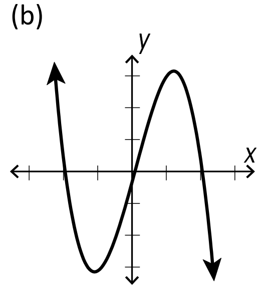
  They then move to the corner of the room designated to match the selected response on the paper they have taken. In their respective corners, they should discuss the similarities and differences in the explanation provided and report back to the class.

  Which of the following is the graph of an even degree function? Explain your reasoning.

(a)



(b)



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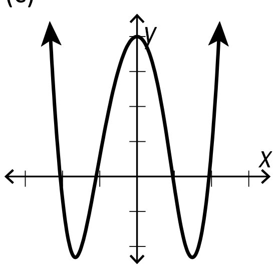
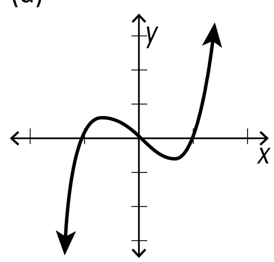
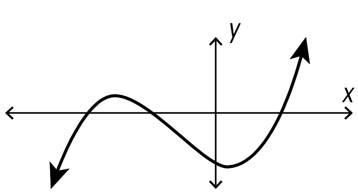
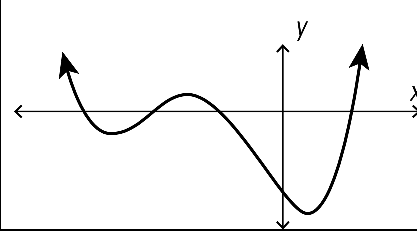
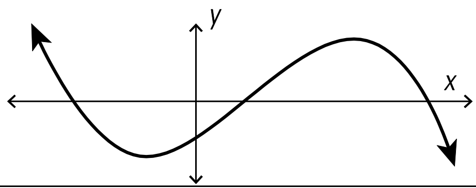
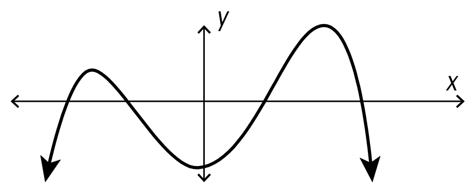


(d)



- Using Think-Pair-Share, give individual students time to think about the similarities and differences among the graphs with respect to multiplicity of zeros. Students then pair up with a partner to discuss their ideas. After pairs discuss, students share their ideas in a small group or whole-class discussion.
- Students are not expected to use transformations to graph polynomial functions.
- Students can develop their own chart or graphic organizer, like that shown below, to summarize the characteristics of odd- and even-degree polynomial functions.

Odd-Degree Polynomial Functions	**Even-Degree Polynomial Functions**
When a_n is positive, the graph starts in the third quadrant and ends in the first quadrant. 	When a_n is positive, the graph starts in the second quadrant and ends in the first quadrant. 

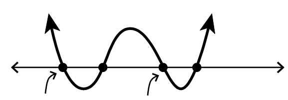
Odd-Degree Polynomial Functions	**Even-Degree Polynomial Functions**
When a_n is negative, the graph starts in the second quadrant and ends in the fourth quadrant. 	When a_n is negative, the graph starts in the third quadrant and ends in the fourth quadrant. 

- Give Me Five is an activity that provides students with an opportunity to individually and publicly reflect on their learning. Ask students, What was the most significant thing you learned about graphing polynomials? Give time for students to quietly reflect before asking for five volunteers to share their reflections. A show of hands can also indicate how many students had a similar thought each time a student shares his or her reflection.
- Students can work in groups of two for the activity Pass the Problem. Each pair gets a problem that involves a situation to be modelled with a polynomial function. Ask one student to write the first line of the solution and then pass it to the second student. The second student verifies the workings and checks for errors. If there is an error, students should discuss what the error is and why it occurred.
- The student then writes the second line of the solution and passes it to the partner. This process continues until the solution is complete.

  Example:

  The length, width, and height of a rectangular box are x cm, $(x - 4)$ cm, and $(x + 5)$ cm, respectively. Find the dimensions of the box if the volume is 132 cm^3.
- Students should assemble a polynomial toolkit, with fairly accurate sketches of specific polynomials that illustrate each of these cases with examples, such as
  - $P(x) = x^3 + 2x^2 + x + 2$, which has only one real root
  - $P(x) = 2x^3 + x^2 - 6x - 3$, which has three real roots, one of which is rational

- As enrichment, ask them to solve polynomial inequalities using the graph of the corresponding function. For example, when is $P(x)\le 0$ for $P(x)=x^4+x^3-2x^2-x+1$?

$x^4+x^3-2x^2-x+1\le 0$ **Step 1:** Solve the related equation. $\underbrace{x^4+x^3-2x^2-x+1}_{P(x)}=0$ Rational roots: $\pm\left\{\frac{1}{1}\right\}$ $\quad P(1)=0 \quad \therefore (x-1)$ is a factor $\quad P(-1)=0 \quad \therefore (x+1)$ is a factor	**Using synthetic division:** $(x-1)\ \begin{array}{r\|rrrrr} 1 & 1 & 1 & -2 & -1 & 1 \\ + & & 1 & 2 & 0 & -1 \\ \hline & \multicolumn{4}{l}{\underbrace{1 \quad 2 \quad 0 \quad -1}_{\text{cubic}}} & 0 \end{array}\ (x-1)\left(x^3+2x^2+0x-1\right)=0$ $(x+1)\ \begin{array}{r\|rrrr} -1 & 1 & 2 & 0 & -1 \\ + & & -1 & -1 & 1 \\ \hline & \multicolumn{3}{l}{\underbrace{1 \quad 1 \quad -1}_{\textit{quadratic}}} & 0 \end{array}\ (x-1)(x+1)\left(x^2+x-1\right)=0$
$(x-1)(x+1)\left(x^2+x-1\right)=0$ $x-1=0;\ x+1=0;\ x^2+x-1=0$ $\boxed{x=1}$ $\quad \boxed{x=-1}$ $\quad x=\frac{-6\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-1\pm\sqrt{(1)^2-4(1)(-1)}}{2(1)}$ $x\doteq 0.62$ $\boxed{x=\frac{-1\pm\sqrt{5}}{2}}$ $x\doteq -1.62$	**Step 2** Number line and answer:  $\frac{-1-\sqrt{5}}{2}$ $\quad \frac{-1+\sqrt{5}}{2}$ When is $P(x)\le 0$? Answer: $\left[\frac{-1-\sqrt{5}}{2},-1\right]\cup\left[\frac{-1+\sqrt{3}}{2},1\right]$

Suggested Models and Manipulatives

- grid paper

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- double root
- end behaviour
- even-degree polynomial
- multiplicity of a zero
- odd-degree polynomial
- order of a zero
- triple root

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 3.1 Characteristics of Polynomial Functions
    - Student Book: pp. 106–117
  - 3.4 Equations and Graphs of Polynomial Functions
    - Student Book: pp. 136–152

SCO RF13 Students will be expected to graph and analyze radical functions (limited to functions involving one radical). [CN, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF13.01 Sketch the graph of the function $y = \sqrt{x}$, using a table of values, and state the domain and range.

RF13.02 Sketch the graph of the function $y - k = a\sqrt{b(x-h)}$ by applying transformations to the graph of the function $y = \sqrt{x}$, and state the domain and range.

RF13.03 Sketch the graph of the function $y = \sqrt{f(x)}$, given the graph of the function $y = f(x)$, and explain the strategies used.

RF13.04 Compare the domain and range of the function $y = \sqrt{x}$ to the domain and range of the function $y = f(x)$, and explain why the domains and ranges may differ.

RF13.05 Describe the relationship between the roots of a radical equation and the x-intercepts of the graph of the corresponding radical function.

RF13.06 Determine, graphically, an approximate solution of a radical equation.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)* **AN02** Students will be expected to solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. (PC11)** **AN03** Students will be expected to solve problems that involve radical equations (limited to square roots). (PC11)** **RF03** Students will be expected to analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, and x- and y-intercepts. (PC11)**	**RF13** Students will be expected to graph and analyze radical functions (limited to functions involving one radical).

Mathematics 11 / Pre-calculus 11 (continued)	Pre-calculus 12 (continued)
RF05 Students will be expected to solve problems that involve quadratic equations. (PC11)**	

* M11—Mathematics 11
** PC11—Pre-calculus 11

Background

Solving radical equations algebraically is a review from Pre-calculus 11. It is not the focus of this outcome. The intent here is to solve equations graphically.

In Pre-calculus 11, students worked with radicals and radical expressions with numerical and variable radicands (AN02) and solved equations and problems dealing with radical equations (AN03). Students now use their knowledge of transformations from the previous unit (RF04) to graph radical functions. They also determine the domain and range of radical functions.

While students have been introduced to radical equations (AN03), they may or may not have been exposed to what the graphs of radical functions look like. They should be introduced to the graphs by creating the graph of $y=\sqrt{x}$ using a table of values. Once the graph is produced, the domain and range of $y=\sqrt{x}$ should be discussed.

Students should note that because it is not possible to take the square root of negative numbers in the real number system, the radicand must be greater than or equal to zero, resulting in a domain for $y=\sqrt{x}$ of $[0, \infty)$. Technology could also be used to create the table of values and the graph.

Once students are familiar with the characteristics of the graph of $y=\sqrt{x}$, the characteristics of a transformed radical function of the form $y-k=a\sqrt{b(x-h)}$ or $y=a\sqrt{b(x-h)}+k$ should be examined. The graphs can be produced by either applying the transformations or creating a mapping rule, deriving a table of values, and plotting the points. Students are expected to be able to use both of these methods for graphing a function such as $f(x)=-3\sqrt{\frac{1}{2}(x+6)}+1$.

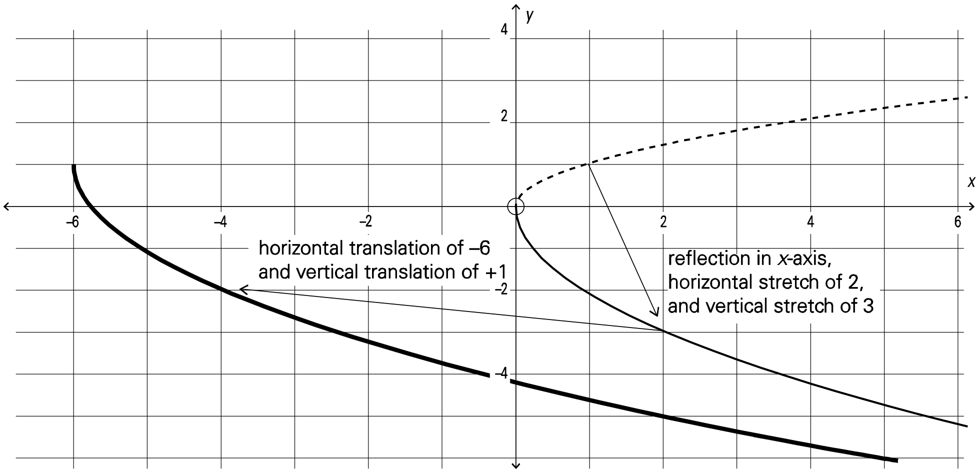
Method 1: Applying Transformations
Students have already been exposed to these graphing techniques using general functions (RF04). They should now apply these techniques to the specific function $y=\sqrt{x}$.

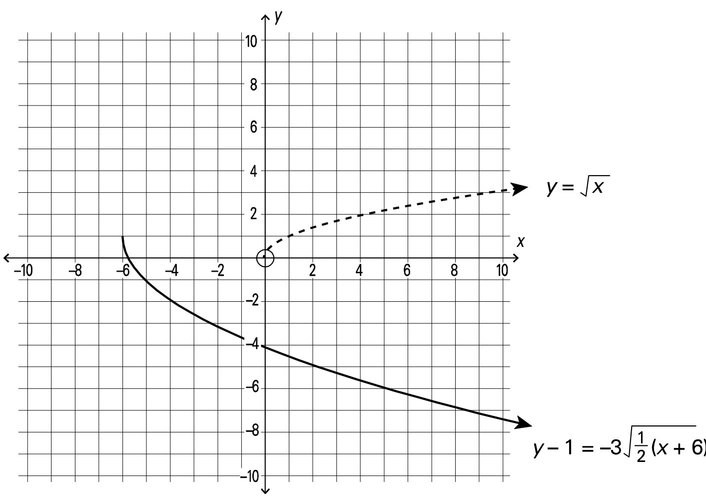
Students could explore a radical function such as $y=-3\sqrt{\frac{1}{2}(x+6)}+1$.

Using transformations, they should see that

- $a = -3$, so all points have a vertical stretch of 3
- since $a < 0$, all points are reflected in the x-axis
- $b = 2$, so all points have a horizontal stretch of 2
- $h = -6$, so all points have a horizontal translation of 6 units left
- $k = 1$, so all points have a vertical translation of 1 unit up

When students create the graph by applying transformations, remind them that they must apply stretches and reflections first, and translations last.





Method 2: Creating a Mapping Rule, Deriving a Table of Values, and Plotting the Points

Students could also write the mapping rule for the function, transform the points of $y=\sqrt{x}$, and plot these points to create the graph.

Points for $y=\sqrt{x}$

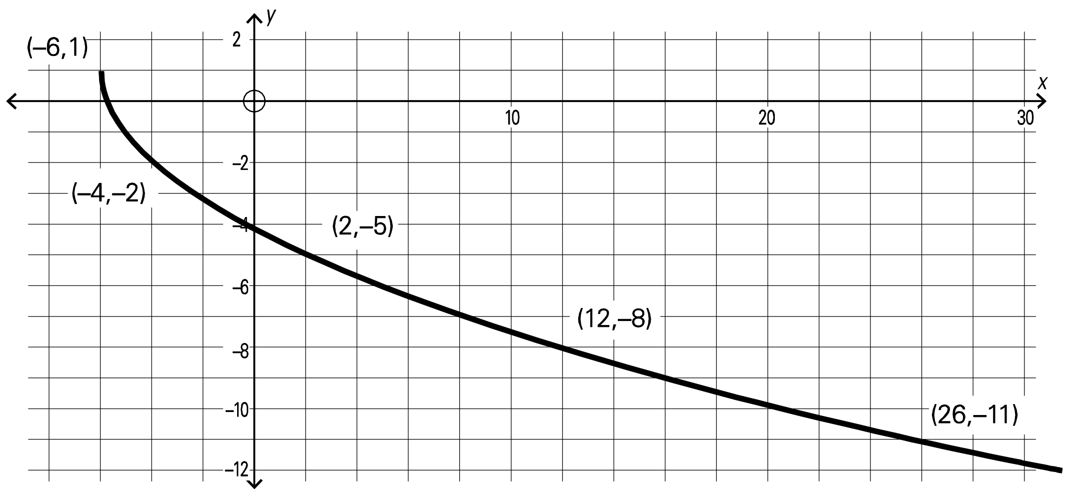
x	***y***
0	0
1	1
4	2
9	3
16	4

Mapping rule:
$(x, y) \rightarrow (2x-6, -3y+1)$

2*x* – 6	**–3*y* + 1**
2(0) – 6	–3(0) + 1
2(1) – 6	–3(1) + 1
2(4) – 6	–3(2) + 1
2(9) – 6	–3(3) + 1
2(16) – 6	–3(4) + 1

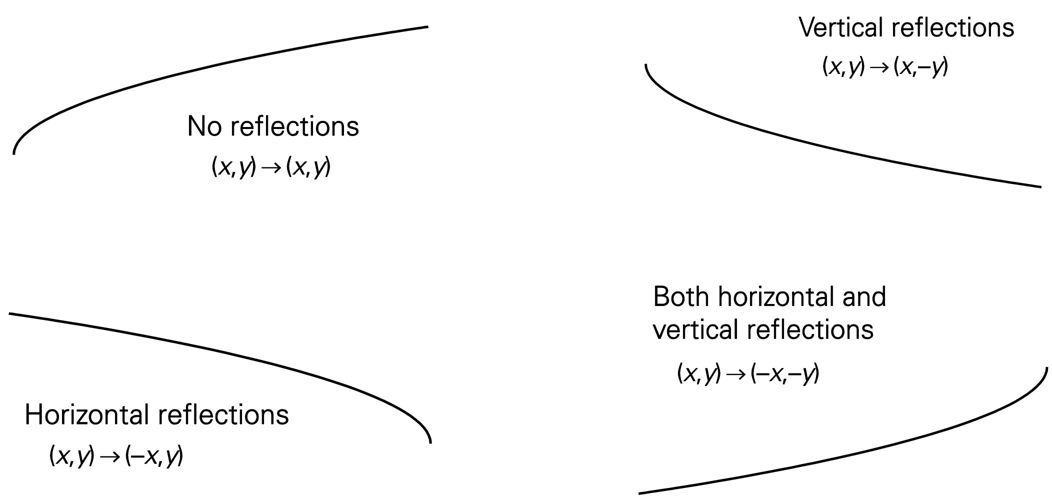
Points for
$y-1=-3\sqrt{\frac{1}{2}(x+6)}$

x	***y***
–6	1
–4	–2
2	–5
12	–8
26	–11

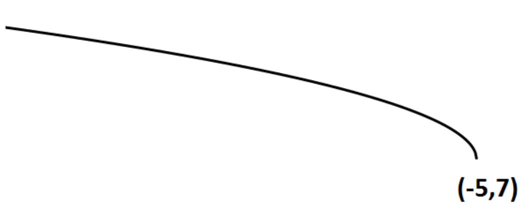


Students should determine the domain and range of the function from the graph. After they have worked through a number of examples, they should see a pattern that allows them to determine the domain and range of a radical expression such as $y = 4\sqrt{-2(x+5)} + 7$ without creating an accurate graph. This can be done by examining the reflections and translations. Students should understand that stretches do not affect the domain and range of radical functions.

Specifically, students would expect the following general shapes:



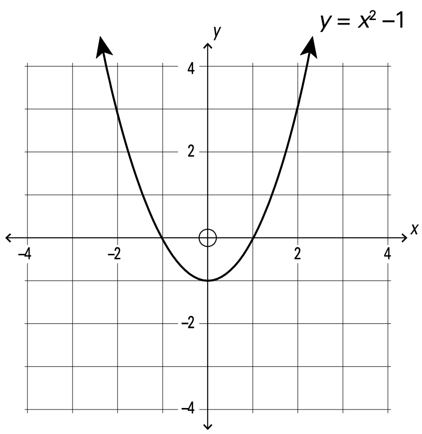
Therefore, $y = 4\sqrt{-2(x+5)} + 7$ would have a horizontal reflection only and would look like this:



Therefore, students would conclude that $y = 4\sqrt{-2(x+5)} + 7$ has a domain of $(-\infty, -5]$ and a range of $[7, +\infty)$.

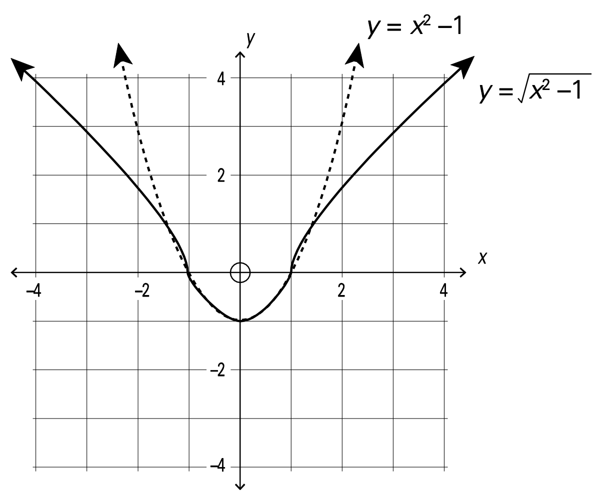
It is important to note that students are not expected to determine the equation of a radical function given its graph or points on the graph.

Given the graph of $y = f(x)$, students are expected to graph $y = \sqrt{f(x)}$.
For example, the graph of $y = x^2 - 1$.



One method to produce the graph of $y = g(x) = \sqrt{x^2 - 1}$ is to first generate a table of values for $y = f(x) = x^2 - 1$. Then, to graph $g(x) = \sqrt{f(x)} = \sqrt{x^2 - 1}$, students take the square root of the *y*-values of function $y = f(x)$.

x	$f(x)$	$\sqrt{f(x)}$
−2	3	$\sqrt{3}$
−1	0	0
0	−1	undefined
1	0	0
2	3	$\sqrt{3}$



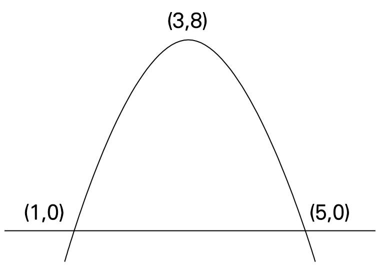
Students are expected to understand that $y = \sqrt{f(x)}$ is undefined where $f(x) < 0$, that the invariant points occur where $f(x) = 0$ or $f(x) = 1$, and that $f(x)$ is above the graph of $y = \sqrt{f(x)}$ where $0 < f(x) < 1$. These results, along with other key points, can then be used to help create the graph of $y = \sqrt{f(x)}$ when $y = f(x)$ is given.

Students could also be given the equation $y = f(x)$ and use it to generate a table of values. From this, they can then graph $y = \sqrt{f(x)}$. The graph and table of values of $y = f(x)$ can also be generated with the use of technology.

The graphs of $y = f(x)$ are limited to linear and quadratic functions. Students can use the graphs of $y = f(x)$ and $y = \sqrt{f(x)}$ to determine the domain and range of both functions. When equations are given for the functions, they can first graph each function and then determine the domain and range.

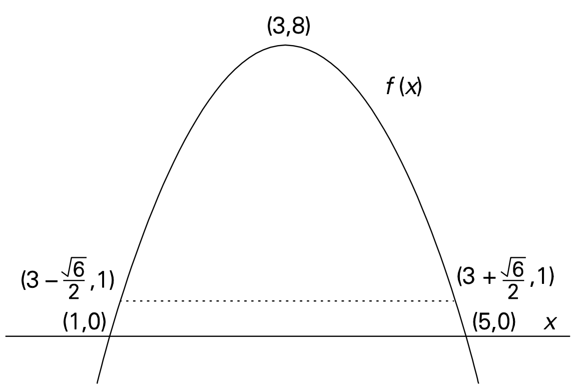
Example:

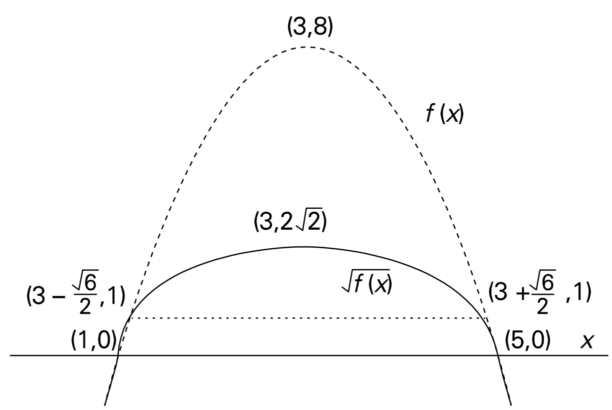
$y = \sqrt{-2(x-3)^2 + 8}$ can be sketched by sketching $y = -2(x-3)^2 + 8$.
From working with quadratics in Mathematics 11 (RF02), students know that this quadratic opens down, it has a vertex at (3, 8). The *x*-intercepts of $y = -2(x-3)^2 + 8$ can be calculated to be *x* = 5 and *x* = 1.



From this rough sketch, it is evident that $y = \sqrt{-2(x-3)^2 + 8}$ has a domain of [1, 5] and a range of [0, 8].

To sketch a more accurate graph of $y = \sqrt{-2(x-3)^2 + 8}$, students need to also determine the values where $-2(x-3)^2 + 8 = 1$ and plot those invariant points as well as the *x*-intercepts and vertex. With this information, it is possible to sketch *f*(*x*) and then $\sqrt{f(x)}$ as shown below.





An alternative method to determine domain and range of $y = f(x)$ and $y = \sqrt{f(x)}$ involves analyzing key points. Given $y = -x^2 + 6x - 5$, for example, students can use knowledge of quadratic functions from Mathematics 11 to determine the *x*- and *y*-intercepts of the parabola and the vertex of the parabola (RF02). This information can then be used to identify key points on the graph of $y = \sqrt{-x^2 + 6x - 5}$.

Function	$\boldsymbol{y=-x^2+6x-5}$	$\boldsymbol{y=\sqrt{-x^2+6x-5}}$
x-intercepts	1 and 5	1 and 5
y-intercepts	–5	none
maximum value	(3, 4)	(3, 2)
minimum value	none	0

From the above information, students should determine that the domain of $y=\sqrt{-x^2+6x-5}$ is [1, 5] and the range is [0, 2].

This should lead to the generalization that the domain of $y=\sqrt{f(x)}$ consists of all values where $f(x) \geq 0$, and the range consists of the square roots of all of the values in the range of $f(x)$ for which $f(x)$ is defined.

The relationship between the roots of a general function and the *x*-intercepts of the corresponding graph has been established earlier in this course (RF12). This is now extended to radical functions.

Initially, students should be provided with a graph of a radical function, or they should use graphing technology to create the graph, in order to approximate the solution. Students should only be expected to graph without technology radical functions that include simple transformations, such as $f(x)=\sqrt{(x-1)}+3,$ $g(x)=-\sqrt{(x-1)}+2,$ and $h(x)=\sqrt{(5-x)}-3.$

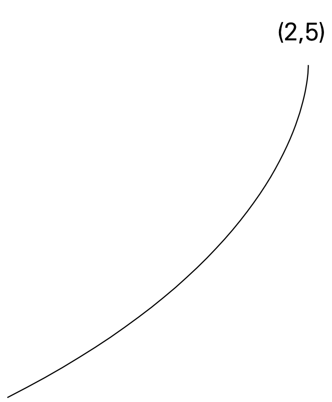
In Pre-calculus 11, students solved radical equations algebraically (AN03).

Example:

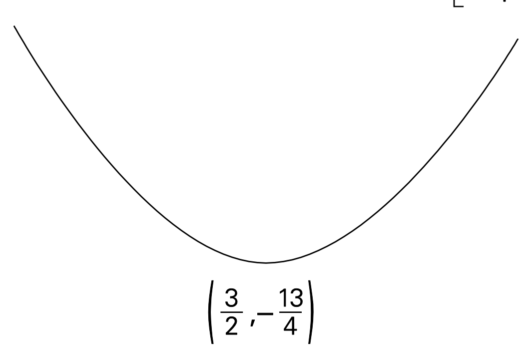
Students would be expected to approximate the solution for the equation $-3\sqrt{(4-2x)}+5=x^2-3x-1$ graphically.

They should anticipate the following:

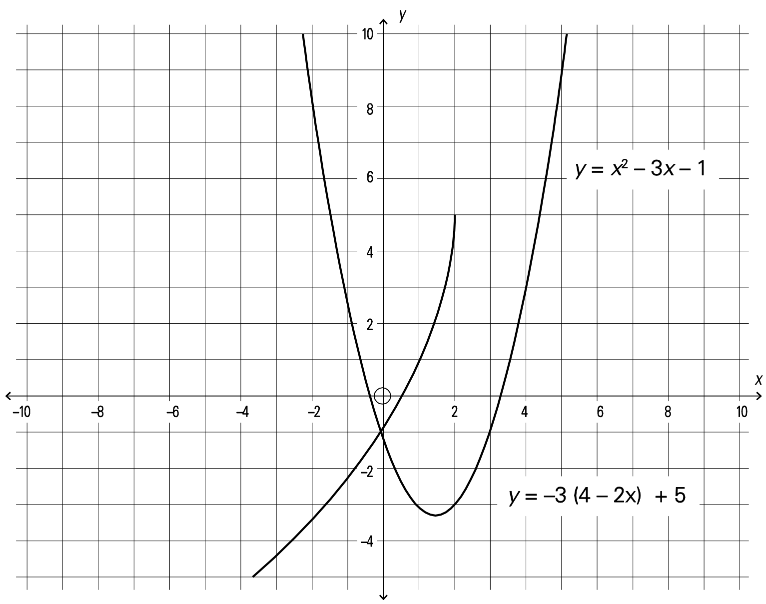
The graph of $y=-3\sqrt{(4-2x)}+5$ is a square root function that has both horizontal and vertical reflections, and since it can be rewritten as $y=-3\sqrt{-2(x-2)}+5,$ the domain would be $(-\infty,2]$ and the range would be $(-\infty,5]$.



The graph of $y=x^2-3x-1$ is a quadratic that opens up and has a vertex at $\left(\frac{3}{2},-\frac{13}{4}\right)$ or (1.5, –3.25), thus its range would be $\left[-\frac{13}{4},+\infty\right)$.



Making window adjustments, if necessary, students are expected to graph both equations and determine their intersection point(s) as shown below.



Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- State any restrictions for the value of x in each of the following expressions:
  (a) $\sqrt{x}$
  (b) $\sqrt{x-3}$
- Determine the x-intercepts, direction of opening, and the vertices for each of the following:
  (a) $y=x^2-3x-6$
  (b) $y=4\left(x-3\right)^2-9$
  (c) $y=8x-x^2$
- Use graphing technology to solve the equation $2(x-1)^2+5=7-x$.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Explain how to transform the graph of $y=\sqrt{x}$ to obtain the graph of each function.

  (a) $y=\sqrt{x}-4$

  (b) $y=\frac{1}{2}\sqrt{-x}+2$

  (c) $y=-5\sqrt{9-x}+1$

- Describe the similarities and differences between the graphs of $y=\sqrt{x}$ and $y=\frac{1}{3}\sqrt{-2(x)+1}-4$.

  Discuss the domain and range of each function.

- Using the graph of $y=\sqrt{x}$, create the graph of $y=-\sqrt{\left[\frac{1}{2}(x-6)\right]}+3$, describing all transformations and stating the domain and range.

- Without the use of a graphing utility, sketch the graph of each of the following radical functions. Then, identify the domain and the range.

  (a) $y=-\sqrt{x-4}$

  (b) $y=\sqrt{-(x+3)}+2$

  (c) $y=2\sqrt{(x-4)}$

  (d) $y=-2\sqrt{-(x-3)}+4$

- Write the equation of the radical function that results by applying each set of transformations to the graph of $y=\sqrt{x}$.

  (a) A horizontal stretch by a factor of four, then a horizontal translation of six units right.

  (b) A vertical reflection in the *x*-axis, then a horizontal translation four units left, and a vertical translation down 11 units.

  (c) A horizontal reflection in the *y*-axis, then a horizontal stretch by a factor of $\frac{1}{5}$ and a vertical stretch by a factor of three.

- Sketch the graphs of $f(x)=\sqrt{x-3}$ and $g(x)=-\sqrt{x-3}$ on the same coordinate plane. Explain why a quadratic equation cannot be used to define *y* as a function of *x* for the resulting graph.

- For each point on the graph of $y=f(x)$, does a corresponding point on the graph of $y=\sqrt{f(x)}$ exist? If so, state the coordinates of the image point. Round off answers to two decimal places, when necessary.

  (a) (1, –9)

  (b) (–2, 6)

  (c) (–1, 0)

  (d) (17, 5)

- Given $f(x)=9-x^2$, sketch the graph of $y=\sqrt{f(x)}$. Identify the domain and the range of both functions.

- Without graphing, determine the domain and range of each function.

  (a) $y=\sqrt{x^2+4x-5}$

  (b) $y=\sqrt{-\frac{1}{2}x^2+x+12}$

  (c) $y=\sqrt{2(x-3)^2-5}$

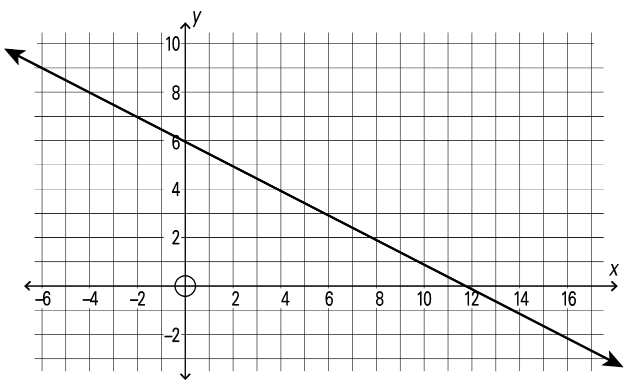
  (d) $y=\sqrt{25x-x^2}$

  (e) $y=\sqrt{2x+5}$

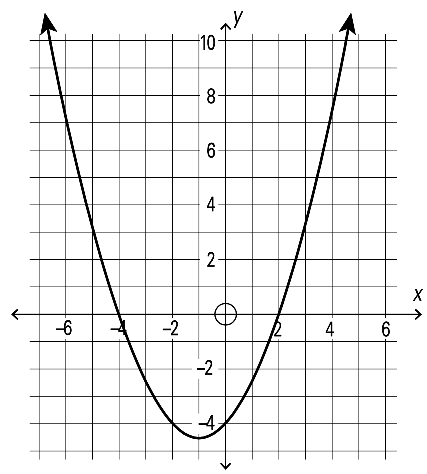
  (f) $y=\sqrt{(x-2)(x+1)}$

- Given the graph of $y=f(x)$ below, create the graph of $y=\sqrt{f(x)}$ by examining the intercepts and the invariant points:

(a)



(b)



- Explain why the domain of $y=\sqrt{x^2+4}$ is $(-\infty,\infty)$ but the domain of $y=\sqrt{x^2-4}$ is $(-\infty,-2]\cup[2,\infty)$.
- In general, in what ways does the graph of $y=f(x)$ resemble that of $y=\sqrt{f(x)}$? In what ways does it differ?
- Explain why the function $y=\sqrt{2x+5}$ has a restricted domain while the function $y=\sqrt{2x^2+5}$ has no restrictions.
- Graph the following. Label all vertices and invariant points.

  (a) $y=\sqrt{x^2-9}$

  (b) $y=\sqrt{16-x^2}$

  (c) $y=\sqrt{x^2-x-12}$

  (d) $y=\sqrt{15-2x-x^2}$

- Solve each of the following equations using graphing technology.

  (a) $3-\sqrt{x+1}=1$

  (b) $x-3=\sqrt{2x+5}$

  (c) $\sqrt{x^2-25}-3=x+5$

- On grid paper, sketch the graphs of the radical functions below and approximate their solutions graphically.

  (a) $\sqrt{x+5}=4$

  (b) $\sqrt{x^2-1}=x+3$

  (c) $\sqrt{-(x-4)}=x-4$

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- Students should be reminded that all of the transformation rules apply to radical functions.
- Review with the class the methods for solving equations using graphing technology.
- Create several pairs of cards where one card contains the equation of a radical function, and the second card contains the range. Distribute the cards among the students and have them find their partner by matching the radical function with its range. Once students have found their partners, they should create the graph of the radical function and find its domain.
- When exploring the graph of $\sqrt{f(x)}$, where $f(x)=x^2-1$, ask the following questions:

  (a) Why is the graph undefined for values of *x* between –1 and 1?

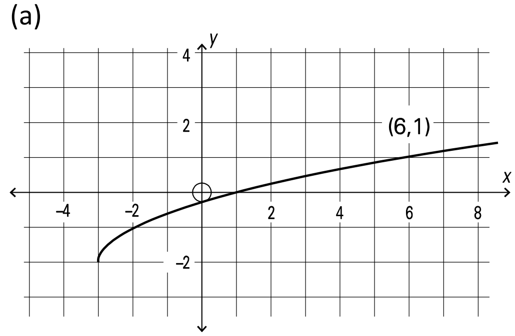
  (b) Are there any invariant points? If so, what are they?

  (c) Where is the graph of $y=\sqrt{f(x)}$ above $y=f(x)$? Below $y=f(x)$?

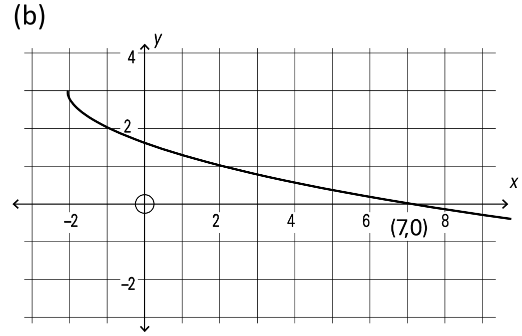
- For enrichment, this outcome can be extended to writing the equation of a radical function from a graph or a description of a graph.

Example:

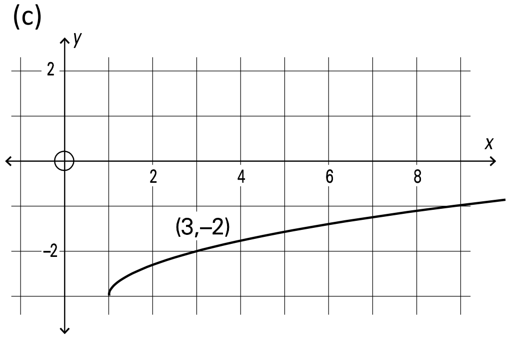
(a)



(b)

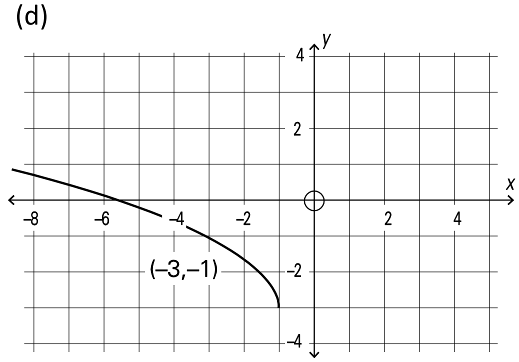


(c)



$f(x)$ has a domain of $[-3,+\infty)$ and a range of $[2,+\infty)$. It passes through the point (1, 4).

(d)



$g(x)$ has a domain of $(-\infty,4)$ and a range of $[-2,+\infty)$. It passes through the point (2, –1).

- Have students use pipe cleaners or straws to model transformations of functions.

Suggested Models and Manipulatives

- grid paper
- ruler

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- radical function
- square root of a function

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 2.1 Radical Functions and Transformations
    - Student Book: pp. 62–77
  - 2.2 Square Root of a Function
    - Student Book: pp. 78–89
  - 2.3 Solving Radical Equations Graphically
    - Student Book: pp. 90–98

SCO RF14 Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials). [CN, R, T, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

RF14.01 Graph, with or without technology, a rational function.
RF14.02 Analyze the graphs of a set of rational functions to identify common characteristics.
RF14.03 Explain the behaviour of the graph of a rational function for values of the variable near a non-permissible value.
RF14.04 Determine if the graph of a rational function will have an asymptote or a hole for a non-permissible value.
RF14.05 Match a set of rational functions to their graphs, and explain the reasoning.
RF14.06 Describe the relationship between the roots of a rational equation and the *x*-intercepts of the graph of the corresponding rational function.
RF14.07 Determine, graphically, an approximate solution of a rational equation.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry. (M11)* **AN04** Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). (PC11)** **AN05** Students will be expected to perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). (PC11)** **AN06** Students will be expected to solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials, or trinomials). (PC11)**	**RF14** Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials).

Mathematics 11 / Pre-calculus 11 (continued)	Pre-calculus 12 (continued)
RF01 Students will be expected to factor polynomial expressions of the following form where *a*, *b,* and *c* are rational numbers. ▪ $ax^2 + bx + c, a \neq 0$ ▪ $a^2x^2 - b^2y^2,\ a \neq 0,\ \ b \neq 0$ ▪ $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$ ▪ $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0$ (PC11)** **RF03** Students will be expected to analyze quadratic functions of the form $y = a\left(x - p\right)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, and *x*- and *y*-intercepts. (PC11)** **RF11** Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). (PC11)**	

*   M11—Mathematics 11
** PC11—Pre-calculus 11

Background

In Mathematics 11, students were introduced to quadratic functions (RF02). In Pre-calculus 11 students developed this understanding further and would have also worked with rational expressions and equations (A04, A05, A06). In Pre-calculus 11 students have experience graphing the reciprocal of linear and quadratic functions (RF11).

Earlier in Pre-calculus 12, students explored in detail the effect of transformations on graphs of functions and their related equations.

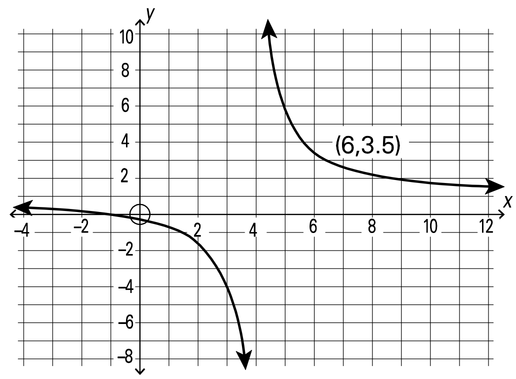
This outcome will build on this previous work, as well as build on previous work in which non-permissible values were explored. The focus now shifts to rational functions.

Students will be introduced to characteristics for rational functions including non-permissible values (**vertical asymptotes** or **points of discontinuity**); zeroes and behaviour near these values; and **end behaviour**, which may include **horizontal** or **slant (oblique) asymptotes**. Students will learn to graph these functions using a table of values, technology, and/or vertical and horizontal transformations.

Students are expected to identify, from the equation of a rational function, zeros, points of discontinuity, and vertical asymptotes. If a function has one or more vertical asymptotes they are expected to understand the behaviour of the function around the vertical asymptote. It is not expected that the end behaviour of a rational function will be explored using limits but instead discussed in an informal way.

Example:

$y=f(x)=\frac{x^2-5x-6}{x^2-10x+24}$, when factored, is $y=f(x)=\frac{(x-6)(x+1)}{(x-6)(x-4)}$ or $y=f(x)=\frac{(x+1)}{(x-4)}, x\neq 6.$

<table>
<tr><td>Point of discontinuity</td><td>Where x = 6, point of discontinuity is $\left(6,\frac{7}{2}\right)$.</td><td rowspan="4">+

</td></tr>
<tr><td>Zero</td><td>(−1, 0)</td></tr>
<tr><td>Vertical asymptote</td><td>x = 4
▪ As x gets close to 4 from below 4, the function gets large, but is negative.
▪ As x gets close to 4 from above 4, the function gets large, and is positive.</td></tr>
<tr><td>sEnd behaviour</td><td>As x gets larger, the function
$f(x)=\frac{(x+1)}{(x-4)}$, can be rewritten as
$f(x)=\frac{x+3-3-1}{x+3}$
$f(x)=\frac{(x+3)}{(x+3)}+\frac{-4}{x+3}$
$f(x)=1-\frac{4}{x+3}$
Therefore, $f(x)$ gets closer to 1 as x gets larger and the end behaviour of $f(x)$ is the horizontal asymptote y = 1.</td></tr>
</table>

Graphs may intersect the horizontal or slant asymptote but will not intersect the vertical asymptote(s). The horizontal or slant asymptote will reflect the end behaviour of the graph, whereas the vertical asymptotes will indicate a non-permissible value for the given function.

Students are expected to understand and compare rational functions presented in different forms—equation, graph, or word problem—and for each be able to determine where the non-permissible values are located and which type they represent, either points of discontinuity or vertical asymptotes.

Using technology would be helpful here to allow students to analyze more functions in a short period of time.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- State the restrictions for x.

  (a) $y = \dfrac{3}{x+1}$

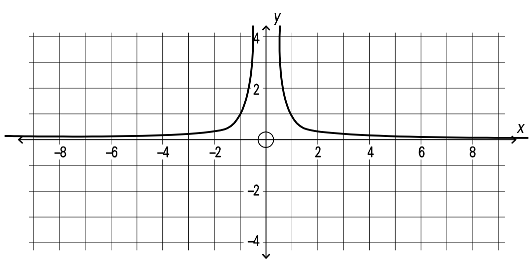
  (b) $y = \dfrac{2x+1}{x-1} + 2$

  (c) $y = \dfrac{6-3x}{x+5} - \dfrac{2}{x}$

- Match the following graphs and equations.

  (a) $y = \dfrac{1}{x}$ (b) $y = \dfrac{1}{x^2}$ (c) $y = \dfrac{1}{x^2+1}$ (d) $y = \dfrac{1}{x^2-1}$

  (i)



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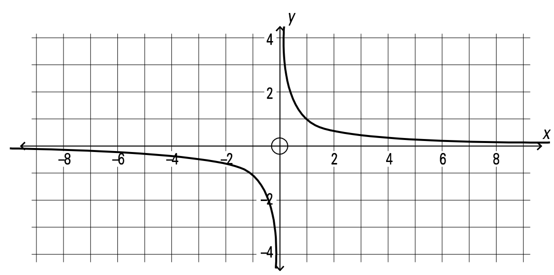

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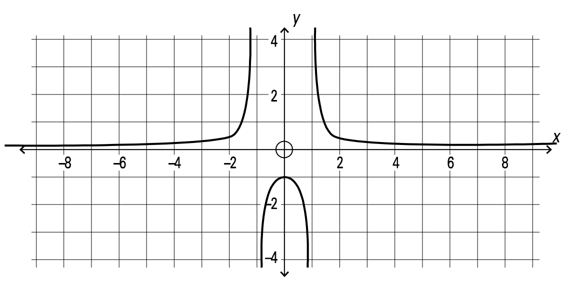

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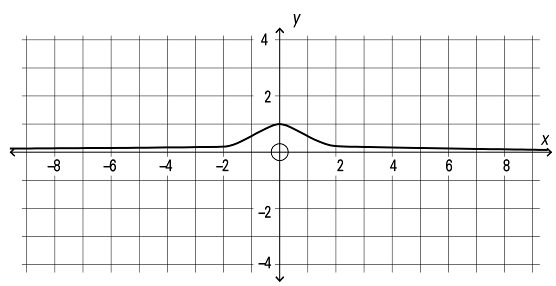


- Use synthetic division to determine the result of the division $\left(2x^2 - 5x + 6\right) \div \left(x - 3\right)$.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Describe the transformations that must be applied to the graph of $y=\frac{1}{x}$ to obtain the transformed function.

  (a) $y=\frac{1}{x-3}+1$

  (b) $y=\frac{5}{x+2}$

  (c) $y=-\frac{1}{x}-4$

- Sketch the graph of each of the following functions. Identify the domain, range, intercepts, and asymptotes.

  (a) $y=\frac{2}{x+3}$

  (b) $y=\frac{5}{x-2}-1$

  (c) $y=\frac{-1}{x+3}+4$

  (d) $y=\frac{2x-1}{x+3}$

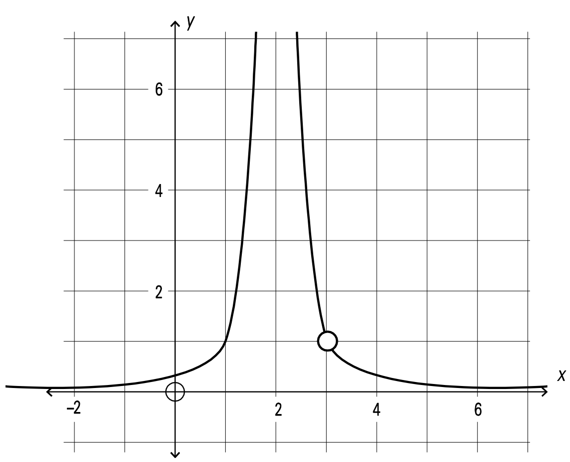
- Rewrite each of the following functions in the form $y=\frac{a}{x-h}+k$.

  (a) $f(x)=\frac{3-2x}{x}$

  (b) $f(x)=\frac{3x}{x-1}$

- Sketch the graph of the function $y=\frac{1}{x+4}$, showing all important characteristics.

- Given the graph of a rational function, $f(x)$, shown below, complete the characteristics chart that follows.



Characteristics	$\boldsymbol{y=\dfrac{1}{x^2-4x+4}}$
Non-permissible value(s)	
Feature at the non-permissible value(s) (point of discontinuity or vertical asymptote)	
Behaviour near non-permissible value(s)	
End behaviour	
Domain	
Range	
Intercepts (state as points)	
Equation of vertical asymptote	
Equation of horizontal asymptote	

- Given the function, $y=\dfrac{x^2-4}{x^2-x-2}$, draw a labelled graph and give a chart of its characteristics.

- State any non-permissible values for each of the functions. Determine whether each of the following rational functions has a point of discontinuity or a vertical asymptote at its non-permissible value.

  (a) $y=\dfrac{x^2-2x}{x}$

  (b) $y=\dfrac{x^2+5x+6}{x-3}$

  (c) $y=\dfrac{x^2-4}{x+2}$

- What is the end behaviour of each of the following rational functions?

  (a) $f(x)=\dfrac{2x-3}{4x+3}$

  (b) $f(x)=\dfrac{x^2-3x+2}{x-1}$

  (c) $f(x)=\dfrac{2x^2-3}{4x+3}$

  (d) $f(x)=\dfrac{2x-3}{4x+3}$

- Sketch the graph of each of the following rational functions.

  (a) $y=\dfrac{4-x^2}{x-2}$

  (b) $y=\dfrac{x^2-x-6}{x-3}$

  (c) $y=\dfrac{x^2+1}{x+1}$

- Write a possible equation of a rational function with a vertical asymptote at $x = 2$, a point of discontinuity at $\left(-2, \frac{1}{2}\right)$.
- Approximate the solution for each of the following graphically.

  (a) $\frac{4}{x} - 1 = \frac{3}{x+1}$

  (b) $\frac{10}{2x-1} - 1 = \frac{x}{3}$

- Match the graphs shown to their equations. Explain your reasoning.

  (a) $y = \frac{x+1}{x-2}$

  (b) $y = \frac{x-2}{x+1}$

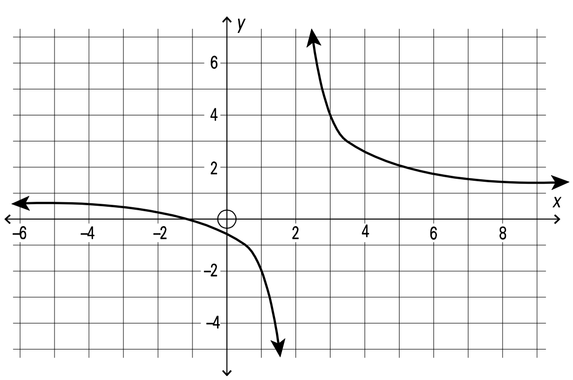
  (c) $y = \frac{3}{x-2}$

  (d) $y = \frac{3(x-2)}{x^2-4}$

  (e) $y = \frac{x-1}{x^2-1}$

  (f) $y = \frac{x^2-1}{x-1}$

(i)



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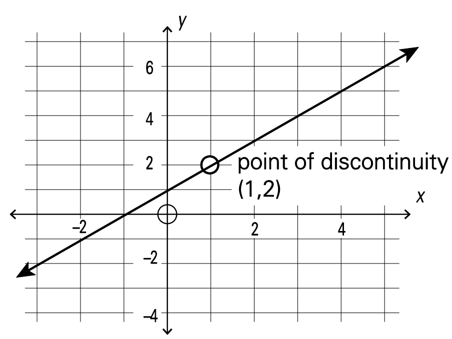

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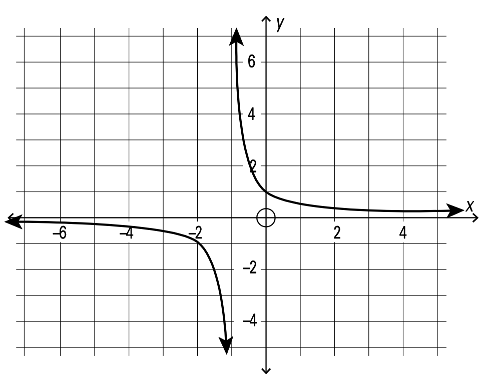

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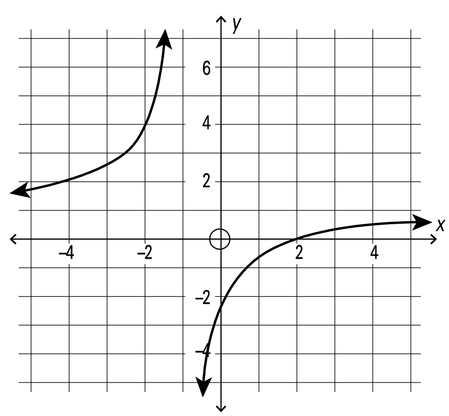

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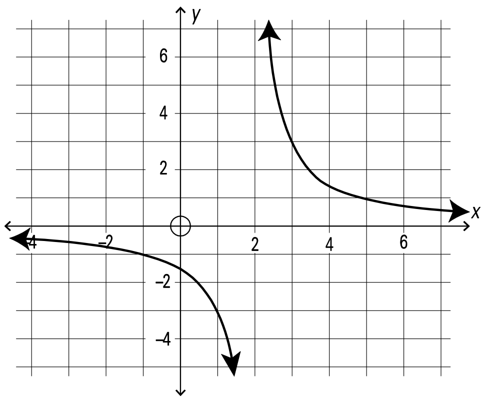

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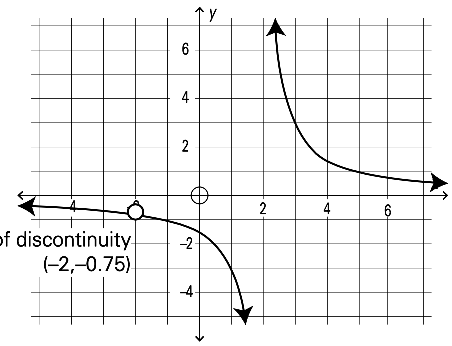


- Dividing 20 by a number gives the same result as dividing 12 by 2 less than the number. Find the number.
- Find the exact positive value of x in the numbers (x), (x + 1), and (x + 2) such that the reciprocal of the smallest number equals the sum of the reciprocals of the other two numbers.
- Jason ran 4 km/h faster than Ming walked. Jason ran 15 km in the time it took Ming to walk 9 km. What were their speeds?

Planning for Instruction

Consider the following sample instructional strategies when planning lessons.

- To develop students' understanding, remind students of transformations of linear, quadratic, radical, trigonometric, and absolute value functions that they have already studied.
- Have students compare how the values of a, h, and k, in the quadratic $y = a\left(x-h\right)^2 + k$, affect the original graph of $y = x^2$ and how these values in $y = \frac{a}{\left(x-h\right)} + k$ affect the original graph of $y = \frac{1}{x}$ (including the transformation of the asymptotes). Students should then be ready to try a linear divided by a linear example and be able to rearrange into the above form to aid them in graphing.

Example:

$y = \frac{2x-3}{x+1}$ could be rewritten as $y = 2 - \frac{5}{x+1}$ or $y = \frac{-5}{x+1} + 2$ using the method in *Pre-Calculus 12* (McAskill et al. 2012) or using synthetic division.

Method in *Pre-Calculus 12* (McAskill et al. 2012)

$$y=\frac{2x+2-2-3}{x+1}$$

$$y=\frac{(2x+2)-2-3}{x+1}$$

$$y=\frac{2(x+1)}{x+1}+\frac{-5}{x+1}$$

or $y=2-\dfrac{5}{x+1}$ or $y=\dfrac{-5}{x+1}+2$

Synthetic division

$$\begin{array}{r|rr} -1 & 2 & -3 \\ + & & -2 \\ \hline & 2 & -5 \end{array}$$

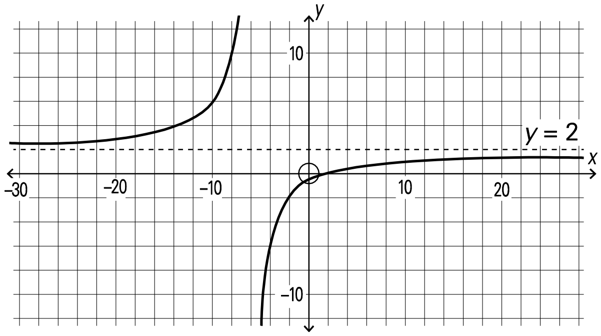
Therefore, $y=2-\dfrac{5}{x+1}$ or $y=\dfrac{-5}{x+1}+2$.

- Students should add to their function tool kit a collection of carefully drawn graphs for functions with specific values of *a*, *h*, and *k*.
- Ensure that students are able to distinguish between a rational function that has a vertical asymptote and one that has a point of discontinuity.
- Ensure that students understand the connection between the roots of a rational equation and the *x*-intercepts of the related rational function.
- Synthetic division can be used to rewrite a function and clarify the end behaviour of a function.

Example:

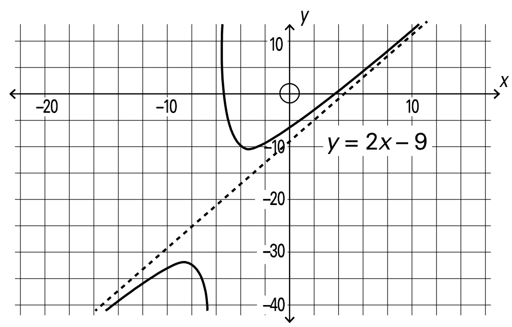
$y=f(x)=\dfrac{2x-3}{x+6}$ can be rewritten, using division, as $y=f(x)=2-\dfrac{15}{x+6}$.

As *x* gets really large, the fraction $\dfrac{15}{x+6}$ gets really close to zero, so the function $y=f(x)=2-\dfrac{15}{x+6}$ gets really close to y = 2. Thus, the end behaviour of the function $y=f(x)=\dfrac{2x-3}{x+6}$ is the horizontal asymptote y = 2.



$y=g(x)=\dfrac{2x^2+3x-40}{x+6}$ can be rewritten, using division, as $y=g(x)=(2x-9)+\dfrac{14}{x+6}$.

As *x* gets really large, the fraction $\dfrac{14}{x+6}$ gets really close to zero, so the function $y=g(x)=(2x-9)+\dfrac{14}{x+6}$ gets really close to $y = 2x - 9$. Thus, the end behaviour of the function $y=g(x)=\dfrac{2x^2+3x-40}{x+6}$ is the slant asymptote $y = 2x - 9$.



SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- end behaviour
- horizontal asymptote
- point of discontinuity
- rational function
- vertical asymptote

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - 9.1 Exploring Rational Functions Using Transformations
    - Student Book: pp. 430–445
  - 9.2 Analysing Rational Functions
    - Student Book: pp. 446–456
  - 9.3 Connecting Graphs and Rational Equations
    - Student Book: pp. 457–467

Permutations, Combinations, and Binomial Theorem
10–15 hours

GCO: Students will be expected to develop algebraic and numeric reasoning that involves combinatorics.

Assessment Strategies

Assessment for learning can and should happen every day as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts should be used for assessing all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual student?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote learning of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SCO PCB01 Students will be expected to apply the fundamental counting principle to solve problems. [C, PS, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

PCB01.01 Count the total number of possible choices that can be made, using graphic organizers such as lists and tree diagrams.
PCB01.02 Explain, using examples, why the total number of possible choices is found by multiplying rather than adding the number of ways the individual choices can be made.
PCB01.03 Solve a simple counting problem by applying the fundamental counting principle.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
–	**PCB01** Students will be expected to apply the fundamental counting principle to solve problems.

Background

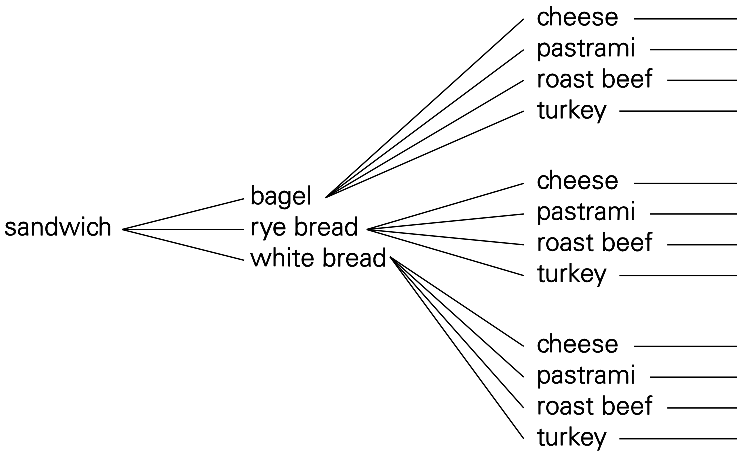
For this outcome, students will use graphic organizers, such as tree diagrams, to visualize and calculate sample space. From patterns observed, they will formulate an understanding of the **fundamental counting principle**.

This **fundamental counting principle**, also known as the **multiplication principle**, states that if one event has m possible outcomes and a second independent event has *n* possible outcomes, there will be a total of $m \times n$ possible outcomes for these two outcomes occurring together. This enables finding the number of outcomes without listing and counting each one.

The fundamental counting principle is a means of finding the number of ways of performing two or more operations together. It will be developed by solving problems through the use of graphic organizers such as lists and tree diagrams. Later, the fundamental counting principle will be applied in work with permutations and combinations.

Tree diagrams were used in Grade 7 (SP05 and SP06) and Grade 8 (SP02) to determine the number of possible outcomes in probability problems. Students now apply tree diagrams and other graphic organizers to counting problems.

For example, if a person is to choose a sandwich for lunch and had three different bread choices and four different meat choices, they would have 12 possible sandwich choices as illustrated in the tree diagram below.



Students are expected to develop an understanding of how the fundamental counting principle works, when it can be used, and the advantages and disadvantages it has over methods that involve direct counting when used to solve counting problems. In PCB02 and PCB03, the fundamental counting principle will be applied in work with permutations and combinations.

It is important for students to understand when to use the fundamental counting principle, specifically. It is also important for students to distinguish between the words **and** and **or**.

To help make this comparison, ask students to calculate the number of possibilities when choosing a sandwich and a side and a beverage compared to choosing a sandwich or a side or a beverage.

When asking students what effect the words **and** and **or** have on their answers, they are expected to conclude that

- when choosing a sandwich and a side and a beverage, the word **and** indicates these three selections (operations) are performed together, so the number of ways of doing each individual selection are multiplied
- when choosing a sandwich or a side or a beverage, the word **or** indicates these selections are being done separately, so the number of ways of doing each individual selection are added

In summary, students should understand that

- **and** implies multiplication and the use of the fundamental counting principle
- **or** implies addition

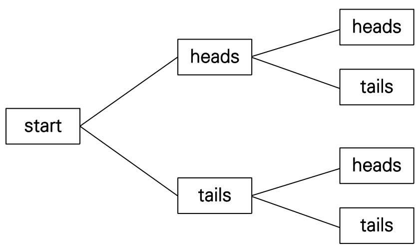
Assessment, Teaching, and Learning

Assessment Strategies

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- A small restaurant offers a lunch combo for $10 where a person can order one sandwich (chicken, turkey, grilled cheese), one side (fruit, yogurt, soup), and one drink (juice, milk). Draw a tree diagram or create a table to determine the possible lunch combos.
- Write a statement that you can conclude based on the tree diagram shown below.



- The school cafeteria advertises that it can serve up to 24 different meals consisting of one item from each of three categories:
  - *Fruit:* apples, bananas, or cantaloupe
  - *Sandwiches:* roast beef or turkey
  - *Beverages:* lemonade, milk, orange juice, or pineapple juice

  Is their advertising accurate?

	Fruit	Sandwiches	Beverages
Meals with apples	A	R	L
	A	R	M
	A	R	O
	A	R	P
	A	T	L
	A	T	M
	A	T	O
	A	T	P
Meals with bananas	B	R	L
	B	R	M
	B	R	O
	B	R	P
	B	T	L
	B	T	M
	B	T	O
	B	T	P
Meals with cantaloupe	C	R	L
	C	R	M
	C	R	O
	C	R	P
	C	T	L
	C	T	M
	C	T	O
	C	T	P

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- A restaurant offers “select-your-own” sundaes choosing one item from each of three categories:

Frozen Yogurt	Fresh Fruit	Extras
vanilla	blueberries	cherries
strawberry	strawberries	peanuts
chocolate	raspberries	

  (a) Using a tree diagram, list all possible desserts that can be ordered.

  (b) Find the number of ways to make a sundae using the fundamental counting principle and compare your answer to your answer in part a).

- A certain model car can be ordered with one of three engine sizes, with or without air conditioning, and with an automatic or a manual transmission.

  (a) Show, by means of a tree diagram, all the possible ways this model car can be ordered.

  (b) Calculate the number of different ways this model car can be ordered.

- In a restaurant there are four kinds of soup, 12 different entrees, six desserts, and three kinds of drinks. How many different four-course meals can a patron choose from?
- In Nova Scotia a license plate consists of a letter-letter-letter-digit-digit-digit arrangement, such as CXT 132.

  (a) How many license plate arrangements are possible?

  (b) How many license plate arrangements are possible if no letter or digit can be repeated?

  (c) How many license plate arrangements are possible if vowels (a, e, i, o, u) are not allowed?

- Canadian postal codes consist of a letter-digit-letter-digit-letter-digit arrangement.

  (a) How many codes are possible if all letters and numbers are used, and how does this compare with the number of license plates in Nova Scotia?

  (b) In Nova Scotia, all postal codes begin with the letter B. How many postal codes are possible?

- Research the format and restrictions on a license plate in a province or country of your choice (i.e., some letters are not used, such as I, O, and/or Q to avoid confusion with the digits 1 and 0). Determine the number of plates possible and present your findings to the class.
- How many three-digit numbers can you make using the digits 1, 2, 3, 4, and 5? Repetition of digits is not allowed.
- How does the application of the fundamental counting principle change if repetition of the digits is allowed? Determine how many three-digit numbers can be formed that include repetitions.
- How many ways can you order the letters MUSIC if it must start with a consonant and end with a vowel?
- Create and solve your own problems that require the use of the fundamental counting principle. Exchange problems and compare your solutions.

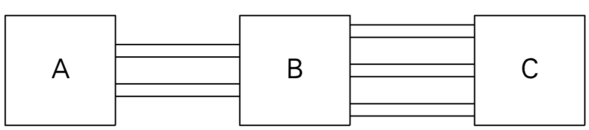
Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Begin instruction with having students construct tree diagrams to represent different outfits (tops, pants, socks, shoes) or different meal combinations served in a restaurant (appetizers, main course, desserts, beverages), and then count the total number of different arrangements/combinations in each situation.
- The following example could be used to activate students' prior knowledge.
  - The school cafeteria restaurant offers a lunch combo for $6 where a person can order one sandwich (chicken, turkey, grilled cheese), one side (fruit, yogurt, soup), and one drink (juice, milk). Ask students to draw a tree diagram or create a table to determine the possible lunch combos.

  Modifying this example to include a fourth category or adding more options in one of the categories can be used to illustrate the limitations on the practicality of using graphic organizers for counting problems and offers a good introduction to the fundamental counting principle. When the sample space is very large, the task of listing and counting all the outcomes in a given situation is unrealistic. The fundamental counting principle enables students to find the number of outcomes without listing and counting each one.
- Use examples that are relevant to the student.
- Ask students to investigate the configuration for each of the provincial license plates. Could you tell from the configuration which province(s) likely have the largest populations?
- In groups of two or three students, ask students to discuss the advantages and disadvantages of using graphic organizers to determine the sample space compared to using the fundamental counting principle. Teachers should observe the groups and ask questions such as the following:
  - Which solution do you prefer?
  - Why do you prefer that solution?
- Ask students to write a journal entry summarizing any shortcuts they observe as to how they can calculate the total number of outcomes. Verify the results obtained by applying the fundamental counting principle to each situation.
- A visual that may help students understand the meaning of **and** as well as **or** is a maze.

  Imagine beginning in Room A, move to B, **and** then to C. How many paths are there? (2 × 3 =6)

  Imagine beginning in Room B and move to A **or** to C. How many paths are there? (2 + 3 = 5)
- Provide students with examples, such as the following, to help them determine when the fundamental counting principle can be used.
  - How many possible outcomes exist if we first flip the coin and then roll the die?
  - How many possible outcomes exist if you either flip the coin or roll the die?

   - Determine the number of ways that, on a single die, the result could be odd or greater than four?
   - A buffet offers five different salads, 10 different entrees, eight different desserts, and six different beverages. In how many different ways can a customer choose a salad, an entree, a dessert, and a beverage?

  Students should understand the mathematical meaning behind the words **and** and **or**, as well as the strategy they will use to solve problems that involve these words.

- Students should be exposed to examples such as the following where restrictions exist.
   - In how many ways can a teacher seat five boys and three girls in a row of eight seats if a girl must be seated at each end of the row?

  Teachers should ask the following questions to help guide students as they work through the example:

   - Are there any restrictions for seating girls and boys?
   - Why should you fill the girls seats first?
   - How many choices are there for seat 1 if a girl must sit in that seat?
   - How many girls remain to sit in seat 8?
   - How many choices of boys and girls remain to sit in each of seats 2 through 7?
   - Which mathematical operation should you use to determine the total number of arrangements?

Students should conclude, by the fundamental counting principle, the teacher can arrange the girls and boys in $3 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 = 4320$ ways.

SUGGESTED MODELS AND MANIPULATIVES

- calculator

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- fundamental counting principle
- tree diagram

Resources

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
   - Section 11.1 Permutations
      - Student Book: pp. 516–527 (part of this section only; P01 and P02 are both addressed here)
      - Teacher Resource: pp. 280–286

SCO PCB02 Students will be expected to determine the number of permutations of n elements taken r at a time to solve problems. [C, PS, R, V]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

PCB02.01 Count, using graphic organizers such as lists and tree diagrams, the number of ways of arranging the elements of a set in a row.
PCB02.02 Determine, in factorial notation, the number of permutations of n different elements taken n at a time to solve a problem.
PCB02.03 Determine, using a variety of strategies, the number of permutations of n different elements taken r at a time to solve a problem.
PCB02.04 Explain why n must be greater than or equal to r in the notation ${}_nP_r$.
PCB02.05 Solve an equation that involves ${}_nP_r$ notation.
PCB02.06 Explain, using examples, the effect on the total number of permutations when two or more elements are identical.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
–	**PCB02** Students will be expected to determine the number of permutations of n elements taken r at a time to solve problems.

Background

This outcome should be taught in conjunction with PC03, which involves combinations.

Students will be introduced to factorial notation and how this relates to the concept of permutations. A formula will be developed and applied in problem-solving situations, including those that involve permutations with conditions. Students should first be introduced to permutations of n different elements taken n at a time. They will then move to permutations of n different elements taken r at a time.

It is expected that students will be able to explain, in practical terms, why n must be greater than or equal to r and why both n and r must be whole numbers.

As students create tree diagrams and determine the number of possible outcomes using the fundamental counting principle (PCB01), they should learn to recognize and use $n!$ (n factorial) to represent the number of ways to arrange n distinct objects.

$4!$ is the product of all positive integers less than or equal to 4 . Thus, $4! = 4 \times 3 \times 2 \times 1 = 24$. In general, $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$, where $n \in \mathrm{N}$.

It is the intention of this course for students to explore $n!$ for $n \in \mathbb{N}$. Some students, however, may discover that the calculator evaluates factorials of numbers that are not integers, such as $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$ and $\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$. These and similar calculations involve the gamma function and require a knowledge of calculus.

Students are expected to observe and apply the fact that $n! = n\,(n-1)!$. It is expected that students would use the value of 5!, for example, to determine 6!.

Students should simplify expressions where the product can be expressed as a single factorial or the expression can be written as the product of two binomials. For example, students are expected to simplify expressions similar to $\frac{8!}{6!}$ and $\frac{n!}{(n-2)!}$.

Students should also solve algebraic equations that involve factorial notation. In some cases, extraneous roots will occur due to the nature of the definition of factorial notation, when applied to counting and permutations, as $n \in \mathbb{N}$. Limit questions of this type to ones where the resulting equation is the product of two or three binomials resulting in a quadratic or a cubic equation. Students should solve and then verify their solutions. Students are expected to solve an equation, such as $\frac{n!}{(n-3)!} = 1716$, where the resulting equation $n(n-1)(n-2) = 1716$ is cubic. Depending on when this outcome is taught in this course, students may or may not be familiar with solving a cubic equation algebraically.

When order matters, factorials can be used to find the number of possible arrangements or permutations for a given number of people or objects.

For example, 3! will give the number of permutations for three people standing in a line. There are three people to choose from for the first position, two people left to choose from for the second position in the line, and only one person to choose for the end of the line, or (3)(2)(1) = 3! possible arrangements.

Providing students with several examples where they determine the number of ways to arrange *n* distinct objects makes the link to factorials clear. For example, ask students to determine the number of ways to arrange or permute a group of five people in a line by filling in the table provided.

1st person	**2nd person**	**3rd person**	**4th person**	**5th person**
5 options	4 options remaining	3 options remaining	2 options remaining	1 option remaining

By the fundamental counting principle, there are 5 × 4 × 3 × 2 × 1 or 120 ways or 5! ways. This should allow them to make the conjecture that the number of permutations is $n!$.

Students should also work with problems where only some of the objects are used in each arrangement (i.e., arranging a subset of items). Students can use the fundamental counting principle to develop an understanding of a permutation of *n* elements taken *r* at a time ${}_nP_r = \frac{n!}{(n-r)!}$, $0 \le r \le n$. This formula should not just be provided to students. It will make more sense if students can see how it is derived from the work they have already completed with factorials.

In applying the permutation formula, students will encounter 0!. It is important that 0! , including 0! = 1, be discussed at this point (see instructional strategies).

Emphasize to students that in the notation, ${}_nP_r$, n is the number of elements in a set and r is the number of elements to be selected at any given time. With this understanding well in hand, students will understand that $n \geq r, n \in \mathrm{N}$, and that $r \in \mathrm{W}$.

Rather than giving students the formula for permutations, it is expected that students will develop the formula through consideration of various situations.

For example, if first, second, and third prizes are to be awarded to a group of eight individuals, there is a choice of eight people for the first prize, seven people for the second prize, and six people for the third prize. Therefore, there are (8)(7)(6) = 336 ways to award three prizes to eight people (assuming the prizes are different), or more formally stated, there are 336 permutations of eight objects taken three at a time. This is expressed as ${}_8P_3$ = (8)(7)(6). It is equivalent to

$${}_8P_3 = \frac{(8)(7)(6)(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)} = \frac{8!}{5!} = \frac{8!}{(8-3)!}.$$

The general form is ${}_nP_r$ for n objects taken r at a time, so ${}_nP_r = \frac{n!}{(n-r)!}$.

When working with the permutation formula, it is important for students to use factorial notation and that they be able to simplify the resulting quotient of factorials. When using the formula to evaluate ${}_6P_2$, for example, encourage students to use mental mathematics to simplify ${}_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = 6 \times 5 = 30$.

When n and r are large, however, using the formula in conjunction with a calculator would be more manageable.

Another situation that should be understood is when some objects of a set are the same. In this case, there are fewer permutations because some arrangements are identical.

If a set of three marbles consists of two identical green marbles and one blue marble, students may initially write the number of arrangements as 3! They may not understand that the set {G1, G2, B} is identical to {G2, G1, B}. This configuration is counted as two different arrangements instead of one; therefore, it must be removed from the total count.

A set of n objects containing a identical objects of one kind and b identical objects of another kind can be arranged in $\frac{n!}{a!b!}$ ways.

Dividing n! by a! and b! eliminates arrangements that are the same and that would otherwise be counted multiple times.

For example, for the word BANANA there is one B, two Ns, and three As, so the total number of permutations of the letters for the word BANANA will be $P = \frac{6!}{1!2!3!} = 60$.

To be successful, it is important for students to read the problem carefully and decide whether they should apply the fundamental counting principle or use the permutation formula. Ask students to work through a mixture of problems. They should be asking themselves the following questions:

- Does the fundamental counting principle apply?
- Is it possible to use the formula ${}_nP_r$?

Permutation problems sometimes involve conditions. In certain situations, objects may be arranged in a line where two or more objects must be placed together, or certain objects(s) must be placed in certain positions. For example, How many arrangements of the word FAMILY exist if A and L must always be together? When certain items are to be kept together, students should treat the joined items as if they were only one object. Therefore, there are five groups in total, and they can be arranged in 5! ways. However, the letters AL can be arranged in 2! ways, and so the total arrangements would be
5! × 2! = 240.

Students will work through counting problems that include words such as **at most** or **at least**, such as the following:

- To open the garage door of Mary's house, she uses a keypad containing the digits 0 through 9. The password must be at least a four-digit code to a maximum of six-digits, and each digit can only be used once in the code. How many different codes are possible?

Students may find this difficult and need to be guided as they calculate the number of permutations possible for 10 numbers, where each arrangement uses four, five, or six of those objects. They should understand there are three cases to consider ${}_{10}P_4 + {}_{10}P_5 + {}_{10}P_6$.

Knowledge of permutations should be applied to solve equations of the form ${}_nP_r = k$. Students should solve equations where they simplify the expression so that it no longer contains factorials. Limit examples where they are only required to work with equations where they solve for n. They will not be expected to solve the equation for r. For example, ${}_nP_2 = 30$ can be solved as follows:

$$
\begin{aligned}
{}_nP_2 &= 30 \\
\frac{n!}{(n-2)!} &= 30 \\
(n)(n-1) &= 30
\end{aligned}
$$

Students can reason that consecutive integers with a product of 30 are needed. Since 6(6 – 1) = 30, n = 6.

$$
\begin{aligned}
\frac{n!}{(n-2)!} &= 30 \\
(n)(n-1) &= 30 \\
n^2 - n - 30 &= 0
\end{aligned}
$$

Alternatively, they can solve the quadratic equation $n^2 - n - 30 = 0$. This equation has roots 6 and –5. Since n must be a natural number, we know that n = 6.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve the following equations:

  (a) $(n)(n-1)=30$

  (b) $\dfrac{(n-1)(n-2)}{n-2}=30$

  (c) $(n)(n-1)(n-2)=60$

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Arrange the following in order of magnitude.

  (a)

  $6!$ $\quad 11!$ $\quad \dfrac{15!}{12!}$ $\quad 3! \times 2!$ $\quad 3! \times 4$

  (b)

  $9!-7!$ $\quad \dfrac{10!}{8!}$ $\quad \dfrac{9!}{3!}$ $\quad \dfrac{86!}{53!}$ $\quad 6!-5!$

  Pick three of the above expressions and create a problem in which these symbols would be used in the solution.

- Write each as a ratio of factorials.

  (a) $7 \times 6 \times 5 \times 4$

  (b) $35 \times 34 \times 33 \times 32 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

  (c) $14 \times 13 \times 8 \times 7 \times 6 \times 3 \times 2 \times 1$

  (d) $25 \times 24 \times 23 \times 5 \times 4 \times 3 \times 2 \times 1$

- Sheila is attempting to simplify $\frac{640!}{638!4!}$. She wrote the following steps:

$$\begin{aligned}&\frac{640!}{638!4!}\\&=\frac{640\times 639\times 638!}{638!4!}\\&=\frac{640\times 639}{4!}\\&=160\times 639\\&=102\ 240\end{aligned}$$

  (a) Explain the strategy that was used in the second step of the solution.

  (b) Identify and explain any errors in Shelia's work.

  (c) Write the correct solution.

- Solve the following equations for n:

  (a) $\frac{(n+2)!}{(n+1)!}=10$

  (b) $\frac{(n+5)!}{(n+3)!}=56$

  (c) $\frac{n!}{(n-2)!}=182$

  (d) $\frac{2(n+3)!}{(n+1)!}=180$

  (e) $\frac{(n+1)!}{(n-2)!}=60$

- Create a question that has an answer of 8!. Exchange your problem with a partner to discuss the similarities and differences.
- In how many different ways can a set of five distinct books be arranged on a shelf?
- In how many different orders can 15 different people stand in a line?
- Consider the word COMPUTER and the ways you can arrange its letters using each letter only once.

  (a) One possible permutation is PUTMEROC. Write five other possible permutations.

  (b) Use factorial notation to represent the total number of permutations possible. Write a written explanation to explain why your expression makes sense.

- The electronic lock to a house has six buttons. To open, a four-button combination has to be entered in sequence and can be tried only once before the lock freezes.

  (a) If none of the buttons is repeated, how many different possible lock combinations are possible?

  (b) If you are allowed to repeat buttons, how many different possible lock combinations are possible?

- Arrange a set of four different coloured blocks into as many different configurations of groups of two blocks as you can. Explicitly state any assumptions you are making and then calculate the number of configurations to verify your results.
- There are 10 movies playing at a theatre. In how many ways can you see two of them on consecutive evenings?
- A soccer league has 12 teams, and each team plays each other twice; once at home, and once away. How many games are scheduled?

- Determine the number of ways four different graduation scholarships can be awarded to 30 students under each of the following conditions:
  - (a) No student may receive more than one scholarship.
  - (b) Any student may receive any number of scholarships.
- Nine people try out for nine positions on a baseball team. Each position is filled by selecting players at random. In how many ways could the nine positions be filled
- The code for a lock consists of three numbers selected from 0, 1, 2, 3, with no repeats. For example, the code 1-2-1 would not be allowed but 3-0-2 would be allowed. Using the permutation formula, determine the number of possible codes.
- How many two-digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6 if repetition is allowed?
- How many distinct arrangements of three letters can be formed using the letters of the word LOCKERS?
- Jean, Kyle, Colin, and Lori are to be arranged in a line from left to right.
  - (a) How many ways can they be arranged?
  - (b) How many ways can they be arranged if Jean and Lori cannot be side by side?
  - (c) How many ways can they be arranged if Kyle and Colin must be side by side?
  - (d) How many ways can they be arranged if Jean must be at one end of the line?
- If there are nine different cookies (four chocolate chip, three oatmeal, and two raisin), in how many different orders can you eat all of them?
- How many ways can the letters in ACTION be ordered if all the vowels must be kept together?
- Determine the number of arrangements (of any number of letters) that can be formed from the word MASK? (Hint: arrangement can be two letters, three letters, four letters)
- Show that you can form 120 distinct five-letter arrangements from GREAT but only 60 distinct five-letter arrangements from GREET.
- How many distinct arrangements can be formed using all the letters of STATISTICS?
- Find the total number of arrangements of the word SILK and the total number of arrangements of the word SILL. How do your answers compare? Explain why this relationship exists.
- Mary has a set of posters to arrange on her bedroom wall. She can fit only two posters side by side. If there are 72 ways to choose and arrange two posters, how many posters does she have in total?
- Solve the following equations for *n*.
  - (a) ${}_{n}P_{2} = 42$
  - (b) ${}_{n+1}P_{2} = 20$
  - (c) ${}_{n-1}P_{2} = 12$
- A code consists of three letters chosen from A to Z and three digits chosen from 0 to 9, with no repetition of letters or numbers. Students should explain why the total number of possible codes can be found using the expression ${}_{26}P_{3} \times {}_{10}P_{3}$.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Many of these questions can be solved several ways: using a graphic organizer such as a tree diagram, listing all the arrangements (provided the numbers are reasonable), using the fundamental counting principle, and using a formula involving permutations. It is important for teachers to allow students the opportunity to choose their method (within reason).
- In preparation for working with formulas involving permutations, students should be given opportunities to simplify factorial expressions. Ask students to simplify an expression, such as $\frac{9!}{6!}$. Initially, they may rewrite each factorial in its expanded form and then cancel common factors to obtain the final answer. Ask them to think about a more efficient way to rewrite the numerator so they do not have to expand the denominator. Students sometimes mistakenly reduce $\frac{9!}{6!}$ to $\frac{6!}{2!}$.

  Encourage them to write out the expanded form so they can identify their mistake.
- When students solve the equation $\frac{n!}{(n-2)!}=20$, for example, ask them if both solutions $n = 5$ and $n = -4$ are correct and to explain why or why not. Since factorial notation, in the context of this course and in terms of permutations, is only valid for natural numbers, students should understand that $n = -4$ is an extraneous root.
- When working with permutations, it is important for students to make the connection to the fundamental counting principle and to factorial notation. In the previous example, there were three letters (A, B, and C) in the arrangement to consider. For the first letter, students should notice there are three options since any of the three letters can be selected. Each time a letter is placed, there is one fewer letter to choose from the group. As they write the product of the numbers ($3 \times 2 \times 1 = 6$), they should observe that this process is similar to the fundamental counting principle. Introduce this product in abbreviated form as 3!.
- Using their prior knowledge, allow students to work in groups to solve a simple problem similar to the following: Adam, Marie, and Brian line up at a banking machine. In how many different ways could they order themselves? Using a systematic list, students might come up with six different ways.

  Repeat with a similar-type question, asking students to look for patterns that involve the use of factorials. Have them generalize the formula to apply for n objects selected r at a time.
- Students should solve problems where arrangements are created with and without repetition. Consider the arrangement of a five-digit password if only the digits 0–9 can be used. Ask students the following:
  - Is the order of the digits in a password important? Explain.
  - How many arrangements are possible if repetition is allowed? Do the number of choices stay the same or are they reduced?
  - How many arrangements are possible if repetition is not allowed? Do the number of choices stay the same or are they reduced?
  - In which case is there a greater number of permutations possible?

- When students are comfortable determining the number of permutations, introduce the use of permutations to determine wanted probabilities.
- To explain 0!

Method 1

Think about it as determining the number of ways there are to count an empty set. Since there is nothing to count, ask students how many ways it is possible to count nothing? They are likely to understand that the answer would be 1. Using this reasoning allows students to understand why both 1! and 0! equal 1.

Method 2

Ask students how the formula changes if all of the objects are used in the arrangement. Using substitution where $n = r$, results in ${}_nP_n$ which is n!. If the number of permutations of six people arranged in a line is 6!, ask students to illustrate this using the permutation formula. They should be able to explain that this example is a permutation of a set of six objects from a set of six. Therefore, applying the formula ${}_nP_r = \frac{n!}{(n-r)!}$, ${}_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!}$; we note that ${}_6P_6 = \frac{6!}{0!}$, but we know that there are 6! ways to arrange six items in a line; therefore, ${}_6P_6 = \frac{6!}{0!} = 6!$ and we must conclude that 0! = 1.

Method 3

Using a pattern with division.

$$3! = \frac{4!}{4}$$
$$2! = \frac{3!}{3}$$
$$1! = \frac{2!}{2}$$
$$0! = \frac{1!}{1}$$

Suggested Models and Manipulatives

- coloured counters

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- factorial
- permutation

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - Section 11.1 Permutations
    - Student Book: pp. 516–527
    - Teacher Resource: pp. 280–286

SCO PCB03 Students will be expected to determine the number of combinations of *n* different elements taken *r* at a time to solve problems. [C, PS, R, V]			
[**C**] Communication [**T**] Technology	[**PS**] Problem Solving [**V**] Visualization	[**CN**] Connections [**R**] Reasoning	[**ME**] Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

PCB03.01 Explain, using examples, the difference between a permutation and a combination.
PCB03.02 Determine the number of ways that a subset of *k* elements can be selected from a set of *n* different elements.
PCB03.03 Determine the number of combinations of *n* different elements taken *r* at a time to solve a problem.
PCB03.04 Explain why *n* must be greater than or equal to *r* in the notation ${}_nC_r$ or $\binom{n}{r}$.
PCB03.05 Explain, using examples, why ${}_nC_r = {}_nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$.
PCB03.06 Solve an equation that involves ${}_nC_r$ or $\binom{n}{r}$ notation, such as ${}_nC_2 = 15$ or $\binom{n}{2} = 15$.

Scope and Sequence

Mathematics 11 / Pre-calculus 11	Pre-calculus 12
–	**PCB03** Students will be expected to determine the number of combinations of *n* different elements taken *r* at a time to solve problems.

Background

For this outcome, students will investigate combinations, where the order of the selection is not important.

It is important that students be able to identify when the order is important and when it is not important.

In this outcome, students are introduced to the formula for finding combinations and then compare this formula to the formula for finding permutations.

For example, if three people are chosen to be given door prizes and the first one is given $20, the second one chosen $15, and the third one selected gets $10, then the order in which they were chosen is important and this would be a permutation. However, if the three people chosen each got $15, then the order would not matter, and we would use a combination.

Rather than simply giving students the combination formula, have them develop it by considering how many of the permutations that are possible would be considered to be the same.

Suppose that five people are represented by the letters *A*, *B*, *C*, *D*, and *E*. If *A*, *B*, and *C* are selected to be on the committee, the number of permutations of choosing *A*, *B*, and *C* from the larger set of five letters, ${}_5P_3$, will include all 3! permutations of *ABC* (that is *ABC*, *ACB*, *BCA*, *BAC*, *CAB*, and *CBA*) . To eliminate counting each of these permutation as unique, the total number of possible permutations must be divided by 3!. It is clear to students that the number of combinations must be smaller than the number of permutations, and with a bit of explanation, they will understand that ${}_nC_r = \frac{{}_nP_r}{r!}$.

The number of combinations, denoted as ${}_nC_r$ or $\binom{n}{r}$ or *n* choose *r*, would simplify to

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}. $$

For this example ${}_5C_3$ or $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = 10$, so there are 10 possible ways to select three people from a group of five when order is not important.

Similar to permutations, teachers should provide combination problems where students may have to use the fundamental counting principle or problems that involve conditions. A good application-type problem involves asking students to form sub-committees. In the case of selecting two groups from the same larger group, the number of possible combinations will combine both groups.

For example, if a committee of four people and a committee of three people are selected from a group of 10 people, and no person is assigned to both committees, the combinations for each committee will be determined first. For the first committee of four from 10 people, ${}_{10}C_4$ or $\binom{10}{4} = \frac{10!}{4!6!} = 210$, so there are 210 ways to form the committee. There are just six people left, so for the second committee of three from six people, ${}_6C_3$ or $\binom{6}{3} = \frac{6!}{3!3!} = 20$, and there are 20 ways to form this committee. Combining the two committees, there are 210 × 20 = 4200 ways to form both committees from 10 people. Selecting the smaller committee first will yield the same result.

Combinations are sometimes used along with other counting techniques. Students should be comfortable using technology to determine combinations, permutations, and factorials.

For example, a 17-member student council at the high school consists of nine girls and eight boys, and one of the committees has four council members, and must have two girls and two boys. There are ${}_9C_2 = \frac{9!}{2!7!} = 36$ ways of selecting the two girls, and ${}_8C_2 = \frac{8!}{2!6!} = 28$ ways of selecting two boys. Because the committee must include two girls and two boys, there are 36 × 28 = 1008 ways of forming the committee. If the four committee members are selected at random, there are ${}_{17}C_4 = \frac{17!}{4!13!} = 2380$ possible combinations.

Once students have solved many problems that involve permutations and combinations independently, they should progress to a mixture of problems where they must make a decision as to which concept applies.

When reading a problem, students should be asking themselves questions, such as, Does order matter in this problem? If yes, then they know to solve using permutations; otherwise they will use combinations.

- For a permutations problem, the next question could be, Are the objects identical or distinguishable?
- For a combination problem, the next logical question to follow would be, Are there multiple tasks (calculations) required? If yes, then does the fundamental counting principle apply?

By developing a systematic approach, students should gain more confidence when faced with novel problem situations.

Students are expected to solve a variety of equations in the form ${}_nC_r = k$ for *n*. Ask students to solve an equation such as ${}_nC_2 = 15$. They should be able to easily verify that the value of *n* must be greater than or equal to *r* in ${}_nC_r$, having explained the reasoning when working with permutations. If they understand that the notation ${}_nC_r$ means choosing *r* elements from a set of *n* elements, it naturally flows that *r* cannot be larger than *n* and that *r* must be a natural number.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve: $n^2 - n = 12$
- List all the possible arrangements of *ABC*.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Why is the number of combinations of six letters less than the number of permutations of six letters?
- There are three black marbles and two white marbles in a box. Without looking in the box, choose two of the five marbles. How many ways can

(a) two marbles that are the same colour be selected

(b) each marble be a different colour

- At a local health food store, you can order a yogurt parfait with a choice of toppings. There are three different fruit toppings to choose from (blueberry, strawberry, pineapple) and four different dry toppings (peanuts, granola, raisins, coconut). When selecting one fruit and one dry topping, how many different yogurt parfaits could you order?
- A choir has 12 tenors. How many different tenor quartets are possible?
- If a committee of eight people is to be formed from a pool of 13 people, but Mitchell and Lisa must be on the committee, how many different committees can be formed?
- The student council decides to form a sub-committee of five members to plan their holiday concert. There are a total of 11 student members: five males and six females.
  (a) How many different sub-committees are possible of exactly three females?
  (b) How many different ways can the sub-committee consist of at least three females?
  (c) How many different ways can the sub-committee consist of at least one female?
- Create a display or foldable in which you list or draw all of the possible combinations for a scenario and verify the answer using a combination formula (e.g., a pizza shop offers five toppings).
- From a standard deck of 52 cards, how many four-card hands have
  (a) no spades
  (b) at least one red card
  (c) at least one face card (J, K, or Q)
  (d) at most three aces
- A volleyball coach decides to use a starting line-up of one setter, two middle hitters, two power hitters, and one right-side hitter. She chooses 14 players for the team: three setters, four middle hitters, four power hitters, three right-side hitters. How many possible starting line-ups are there?
- Stores sell “combination locks.” Is this correct mathematical terminology?
- Create a foldable or flowchart outlining a series of questions or problem-solving strategies for problems involving permutations or combinations.
- Solve for x: ${}_{x+2}C_2 = 21$
- Explain, using a specific context, why ${}_6C_2 = {}_6C_4$.

Planning for Instruction

Suggested Learning Tasks

Consider the following sample instructional strategies when planning lessons.

- To emphasize the difference between the order of selection being important and the order or selection not being important, have each student in the class write their name on a piece of paper and then place their name in a hat. Draw three names.
  - For the first draw where the order does not matter, give the first student a dark chocolate bar, the second a dark chocolate bar, and the third a dark chocolate bar (or some other item—all items must be the same).
  - For the second draw where order does matter, give the first student a pencil, the second a dark chocolate bar, and the third a loonie.

- To distinguish between a permutation and a combination, students should be given a situation for each where the number of possibilities can be determined with simple counting methods. They have been introduced to permutations as an arrangement of objects in which order matters. Combinations, however, is a grouping of objects where order does not matter. It is important for students to highlight the differences between permutations and combinations. Ask students to identify problems, such as the following, as a permutation or a combination.
  - My fruit salad is a combination of apples, grapes, and strawberries. Students should understand that whether it is a combination of “strawberries, grapes, and apples” or “grapes, apples, and strawberries,” it is the same fruit salad.
  - The code to the safe was 4-7-2. In this case, students should understand that the order 7-2-4 or 2-4-7 would not work. It has to be exactly 4-7-2.
- Starting with an arrangement involving a small number will give students a visual representation of the possibilities. For example, an assignment consists of three questions (*A*, *B*, *C*) and students are required to attempt two.
  - Calculate the number of permutations for choosing two of the three questions.
  - Write the number of arrangements to verify your answer. (*AB*, *BA*, *BC*, *CB*, *AC*, *CA*)
  - How many ways can two questions be arranged?
  - Is the order in which questions are chosen important?
  - Why is it necessary to divide ${}_3P_2$ by 2! to determine the number of combinations?

  Discuss with students that the order in which questions are chosen is not important; therefore, each group of two permutations is just one combination. The number of combinations of *n* elements taken *r* at a time is represented by ${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$.
- Group work is effective. Have groups of students use systematic lists, and look for patterns as they solve problems. Having previously worked with permutations and the formula involving factorials, the formula can now be extended. Have students work toward generalizing the formula to apply for *n* objects selected *r* at a time.
- Ask students to participate in a Quiz-Quiz-Trade activity. Provide students with cards, each having a scenario that is either a permutation or combination. Student 1 must read a card to student 2 who then decides whether it is a permutation or combination, explaining why they think so. Students then switch roles, after which the students will trade cards and find another partner.

Suggested Models and Manipulatives

- coloured counters
- playing cards

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- combination

Resources/Notes

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - Section 11.2 Combinations
    - Student Book: pp. 528–536
    - Teacher Resource: pp. 287–292

SCO PCB04 Students will be expected to expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]			
[**C**] Communication	[**PS**] Problem Solving	[**CN**] Connections	[**ME**] Mental Mathematics and Estimation
[**T**] Technology	[**V**] Visualization	[**R**] Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have met the corresponding specific curriculum outcome.

PCB04.01 Explain the patterns found in the expanded form of $(x+y)^n$, $n \leq 4$, and $n \in$ N, by multiplying n factors of $(x+y)$.
PCB04.02 Explain how to determine the subsequent row in Pascal's triangle, given any row.
PCB04.03 Relate the coefficients of the terms in the expansion of $(x+y)^n$ to the $(n+1)$ row in Pascal's triangle.
PCB04.04 Explain, using examples, how the coefficients of the terms in the expansion of $(x+y)^n$ are determined by combinations.
PCB04.05 Expand, using the binomial theorem, $(x+y)^n$.
PCB04.06 Determine a specific term in the expansion of $(x+y)^n$.

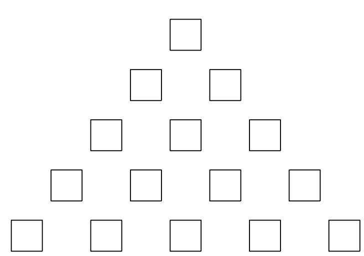
Scope and Sequence

Mathematics 11 / Pre-calculus 11	**Pre-calculus 12**
–	**PCB04** Students will be expected to expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

Background

In Mathematics 10, students multiplied polynomial expressions, including monomials, binomials, and trinomials (AN04).

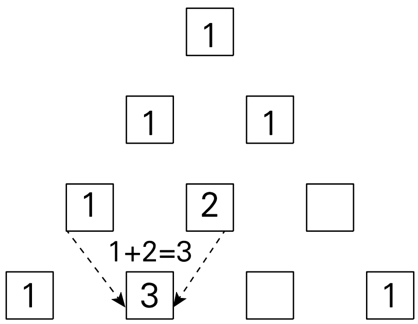
Students will perform the following binomial expansions algebraically and record the coefficients of the terms:

Expression	**Coefficients of the Terms**
$\left(x+y\right)^0 = 1$ $\left(x+y\right)^1 = x+y$ $\left(x+y\right)^2 = x^2+2xy+y^2$ $\left(x+y\right)^3 = x^3+3x^2y+3xy^2+y^3$ $\left(x+y\right)^4 = x^4+4x^3y+6x^2y^2+4xy^3=y^4$	

Students should look for patterns in the exponents of the terms for each expansion.

- How is the exponent of the binomial related to the exponents of the first and last terms in the expansions?
- Moving from left to right in each expansion, what patterns are there in the exponents for *x* and *y*?

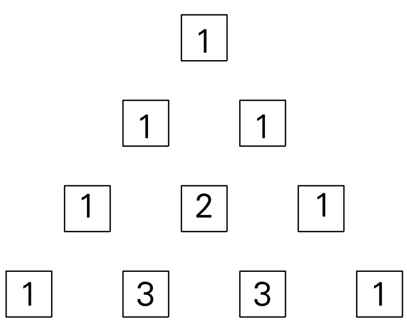
Students should also look for patterns in the coefficients of the terms. For example, each element is the sum of the two elements immediately above it.

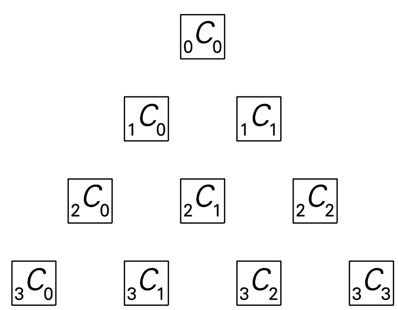


This activity develops the elements in rows 1 through 5 for Pascal's triangle, a triangular array of numbers described by Blaise Pascal in 1653. From the activity, students should be able to explain how to determine the elements in any row once the preceding row is known.

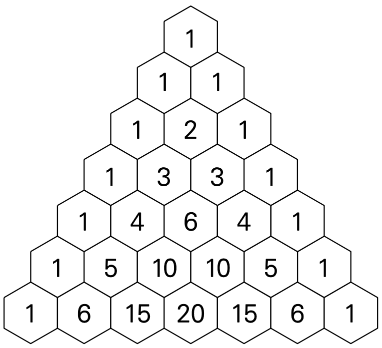
Using Pascal's triangle, students should now also be able to find the expansion for $(x + y)^n$ for any value of n where $n \le 12$.

The elements in each row of Pascal's triangle can be determined using the formula for ${}_nC_r$, as shown in the diagram.





Students are expected to expand a binomial using Pascal's triangle and the binomial theorem. They see that if Pascal's Triangle has already been completed for the desired expansion, the appropriate row can be read with no computation required.



Students are expected to recognize that a disadvantage in using Pascal's triangle is that each of the preceding rows in the display must be completed before the row needed for an expansion can be obtained. This leads to the development of the binomial theorem.

Using combinations, students should be able to predict the element or the coefficient of any element in the expansion of $(x + y)^n$. In the expansion of $(x + y)^7$, for example, the coefficient of x^3y^4 is ${}_7C_4 = 35$.

Generally, the expansion of $(x + y)^n$ is
${}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \ldots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$.

This is known as the binomial theorem.

Students should now apply the binomial theorem to expansions of other binomial expressions, such as $(a + b)^7$, $\left(\frac{x}{3}-2\right)^5$, and $\left(x+\frac{1}{x^3}\right)^{10}$.

Assessment, Teaching, and Learning

Assessment Strategies

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Expand and simplify $(x+y)^2$.
- Expand and simplify $(x+y)(x^2+2xy+y^2)$.
- Expand and simplify $(x^2-2xy+y^2)^2$.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Describe the pattern that the exponents of x and y follow when doing an expansion such as $(x + y)^4$.
- Suppose a friend has missed the class in which you learned to construct Pascal's triangle. Write a point-form description of the steps you would tell her to follow in order to construct the triangle on her own.
- Based on what was observed for expansions with powers up to $n = 4$, predict the exponents for the fifth term in the expansion of $(x + y)^7$.
- Explain how using Pascal's triangle makes expanding binomials easier when using larger exponents.
- Would there be any disadvantages to using Pascal's triangle if you wanted to find a single term in a binomial expansion with a very large exponent?
- For the expansion of $(x + y)^{12}$,
  - (a) use a combination formula to find the coefficient of the term containing x^8y^4
  - (b) which row and entry in Pascal's triangle could be used to accomplish the same task
  - (c) which term in the expansion will this give the coefficient
  - (d) what will be the exponents of the variables for the third term
  - (e) what will be the largest coefficient in the expansion of $(x + y)^{12}$
- Expand and simplify each of the following.
  - (a) $(a + 3b)^4$
  - (b) $(1 - 5y)^3$
  - (c) $\left(\frac{x}{2}+4\right)^6$
  - (d) $\left(x^2-\frac{3}{x^2}\right)^{12}$
- Determine the middle term in the expansion of $(2d - 3f)^6$.
- Find the indicated term in each of the following expansions.
  - (a) the second term of $(5 + x)^6$
  - (b) the fifth term of $(x + 7)^7$
  - (c) the third term of $(x - 2)^6$
  - (d) the third term of $(2x + 3y)^7$
  - (e) the fourth term of $(3x - 7y)^5$
- What is the coefficient of the x^4y^2 term in each of the following?
  - (a) $(x + y)^6$
  - (b) $(x - 2y)^6$
  - (c) $(2x + y)^6$
  - (d) $(3x - 2y)^6$
- When examining the terms from left to right, find the specified term in each expansion.
  - (a) tenth in $(x - y)^{12}$
  - (b) twentieth in $(2x - 1)^{19}$
  - (c) eighth in $(a + b)^{10}$
  - (d) second in $(x^3 - 5)^7$
  - (e) third in $(1 - 2x)^9$
  - (f) fifteenth in $(1 + a^2)^{24}$
- Find a decimal approximation for $(1.05)^8$ by writing it as a binomial. Do you need to write all the terms to get close to the correct answer? Explain.
- When expanding $(a^2 - 2b)^5$, Wally gets confused about the exponents in his answer. Write a paragraph to Wally to help him remember how to record the exponents on this expansion.

- Henrietta is expanding $(3a - 2b^2)^3$. Is her work correct? Explain. Make any necessary corrections.

  $${}_3C_0\left(3a\right)^0\left(-2b\right)^3 + {}_3C_1\left(3a\right)^1\left(-2b\right)^2 + {}_3C_2\left(3a\right)^2\left(-2b\right)^1 + {}_3C_3\left(3a\right)^3\left(-2b\right)^0$$
  $$1\cdot 1\cdot -8b^3 + 3\cdot 3a\cdot -4b^2 + 3\cdot 9a^2\cdot -2b + 1\cdot 27a^3\cdot 1$$
  $$-8b^3 - 36ab^2 - 54a^2b + 27a^3$$
- Betty Lou missed Pre-calculus 12 class today. Helen phoned her at night to tell her about how combinations are helpful when expanding binomials. Write a paragraph or two about what Helen would have told her.

Planning for Instruction

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Review how to multiply binomials with the class.
- Ensure that students have made the connection between Pascal's triangle and the coefficients in the expansion of $\left(a+b\right)^n$, $n \in \mathrm{W}$ (see background for specific details).
- Reinforce students' understanding of how the coefficients obtained when expanding a binomial expression are generated by discussing
  - the fact that when students expand $(x + y)^2$ or $(x + y)(x + y)$,
    - > there is only one way to select an x from each of these binomials; therefore, $1x^2$
    - > there are two ways to select one x and one y from the two binomials; therefore, $2xy$
    - > there is only one way to select a y from each of the three binomials; therefore, $1y^2$

    Thus, when you expand and collect like terms, your answer is
    $(x + y)^2 = (x + y)(x + y) = 1x^2 + 2xy + 1y^2$
  - Similarly, explain that when expanding $(x + y)^3$ or $(x + y)(x + y)(x + y)$,
    - > there is only one way to select an x from each of these three binomials; therefore, $1x^3$
    - > there are three ways to select two x's and one y from the three binomials; therefore, $3x^2y$
    - > there are three ways to select one x and two y's from the three binomials; therefore, $3xy^2$
    - > there is only one way to select a y from each of these three binomials; therefore, $1y^3$

    Thus, when you expand and collect like terms, your answer is
    $(x + y)^3 = (x + y)(x + y)\ (x + y) = 1x^3 + 3x^2y + 1y^3$
  - When expanding $(x + y)^4$ or $(x + y)(x + y)(x + y)(x + y)$,
    - > there is only one way to select an x from each of these four binomials; therefore, $1x^4$
    - > there are four ways to select three x's and one y from the four binomials; therefore, $4x^3y$
    - > there are six ways to select two x's and two y's from the four binomials; therefore, $6x^2y^2$
    - > there are four ways to select one x and three y's from the four binomials; therefore, $4xy^3$
    - > there is only one way to select a y from each of these four binomials; therefore, $1y^4$

    Thus, when you expand and collect like terms, your answer is

    $(x + y)^4 = (x + y)(x + y)(x + y)(x + y) = 1x^4 + 4x^3 + 6x^2y^2 + 4xy^3 + 1y^4$.

- By convention, the expansion of $(x + y)^n$ is written as
  $_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \ldots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$,
  where the first term is $_nC_0x^ny^0$, the second term is $_nC_1x^{n-1}y^1$, etc. While the expansion does not have to be done in this order, when referring to the k^{th} of an expansion $(x + y)^n$, the term referred to would be $_nC_{k-1}x^ny^{k-1}$.
- Combinations can be used to determine the coefficients for each term by determining the number of ways different terms occur. For example, when expanding $(x + y)^5$, the number of times x^5 or y^5 occurs can be determined as $_5C_5$ or 1. The number of times x^4 or y^4 occurs would be $_5C_4$ or 5, and the number of times x^3 or y^3 occurs would be $_5C_3$ or 10. Students should then link the row of Pascal's triangle, 1, 5, 10, 10, 5, 1 , to the binomial theorem.
- Students should examine the pattern changes in the signs between terms when $(x - y)^n$ is expanded. Because the second term in the binomial, $(x - y)$, could be considered negative as $[x + (-y)]$, then the terms in the expansion that have odd numbers of y-factors will be negative.
- Students should be aware that when x and y are often replaced with terms that have exponents or coefficients, for every x- and y-factor, and therefore, the coefficients of the expansion may not end up being a number from Pascal's triangle.

  For example, for $(2a + 3b^2)^3$, when x is replaced with $2a$ and y is replaced with $3b^2$, the expansion becomes

$$\left(2ab+3b^2\right)^3 = {}_3C_0\left(2a\right)^3\left(3b^2\right)^0 + {}_3C_1\left(2a\right)^2\left(3b^2\right)^1 + {}_3C_2\left(2a\right)^1\left(3b^2\right)^2 + {}_3C_3\left(2a\right)^0\left(3b^2\right)^3$$
$$=1\cdot 8a^3\cdot 1+3\cdot 4a^2\cdot 3b^2+3\cdot 2a\cdot 9b^4+1\cdot 1\cdot 27b^6$$
$$=8a^3+36a^2b^2+54ab^4+27b^6$$

SUGGESTED MODELS AND MANIPULATIVES

- copy of Pascal's triangle
- counters

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- binomial theorem

Resources

Print

- *Pre-Calculus 12* (McAskill et al. 2012)
  - Section 11.3 The Binomial Theorem
    - > Student Book: pp. 537–545
    - > Teacher Resource: pp. 293–297

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